ME-467: Turbulence 28 March 2025
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Exercise Set 4: Transport and Mixing – Solutions

1 Mixing of a Passive Scalar

Consider the concentration field $C(\boldsymbol{x}, t)$ of a *passive scalar* in a periodic box. It is described by the **advection-diffusion equation**

$$\frac{\partial C}{\partial t} + (\boldsymbol{u} \cdot \nabla)C \;=\; \alpha \nabla^2 C, \tag{1}$$

where $\boldsymbol{u}$ is the flow velocity of an incompressible Newtonian fluid, and $\alpha > 0$ some diffusion constant. The scalar quantity C is called *passive* because it does not have any influence on the flow, but is merely advected by it.

The concentration field C is inhomogeneous, i.e. it has regions of high and low concentration. One can describe the distribution in terms of its mean $\langle C \rangle$ and variance $\langle C^2 \rangle$, where $\langle \ldots \rangle$ denotes a spatial average over the box. Assume an initial concentration field $C(\boldsymbol{x}, t = 0)$ with

$$\langle C \rangle_{t=0} \;\equiv\; 0. \tag{2}$$

a) Show that $\langle C \rangle = 0$ at all times t.

b) Show that the variance evolves as

$$\frac{\mathrm{d}\langle C^2 \rangle}{\mathrm{d}t} \;=\; -\,2\alpha \langle (\nabla C)^2 \rangle. \tag{3}$$

(Use $\frac{\partial}{\partial t}(C^2) = 2C\frac{\partial C}{\partial t}$ and partial integration in the periodic box.)

c) Does the variance increase, decrease or stay the same? Consequently, what concentration field $C(\boldsymbol{x})$ do you expect after a very long time?

d) The turbulent flow field $\boldsymbol{u}$ does not appear in equation (3). Still we claim that turbulence *enhances* mixing, i.e. the variance decreases faster in a turbulent flow than in a fluid at rest. Explain.

Solution:

a) The advection-diffusion equation has no source term. Therefore, $\langle C\rangle$ does not change with time:

$$\begin{aligned}\frac{\mathrm{d}\langle C\rangle}{\mathrm{d}t} &= \left\langle \frac{\partial C}{\partial t}\right\rangle \\ &= \langle \alpha\nabla^2 C\rangle - \langle(\boldsymbol{u}\cdot\nabla)C\rangle \\ &= -\langle(\nabla\alpha)\cdot(\nabla C)\rangle - \langle u_x\partial_x C\rangle - \langle u_y\partial_y C\rangle - \langle u_z\partial_z C\rangle \\ &= -\langle\mathbf{0}\cdot(\nabla C)\rangle + \langle C\partial_x u_x\rangle + \langle C\partial_y u_y\rangle + \langle C\partial_z u_z\rangle \\ &= 0 + \langle C(\nabla\cdot\boldsymbol{u})\rangle \;=\; 0.\end{aligned}$$

Then, $\langle C\rangle(t) = \langle C\rangle_{t=0} = 0$.

b)

$$\begin{aligned}\frac{\mathrm{d}\langle C^2\rangle}{\mathrm{d}t} &= \left\langle \frac{\partial}{\partial t}(C^2)\right\rangle \\ &= \left\langle 2C\frac{\partial C}{\partial t}\right\rangle \\ &= \left\langle 2C[\alpha\nabla^2 C - (\boldsymbol{u}\cdot\nabla)C]\right\rangle \\ &= 2\alpha\langle C\nabla^2 C\rangle - \langle 2C(\boldsymbol{u}\cdot\nabla)C\rangle \\ &= \underbrace{-2\alpha\langle(\nabla C)^2\rangle}_{\text{via } \langle(\Delta f)g\rangle = -\langle(\nabla f)\cdot(\nabla g)\rangle} - \underbrace{\langle(\boldsymbol{u}\cdot\nabla)C^2\rangle}_{\text{via } \nabla(f^2)=2f\nabla f} \\ &= -2\alpha\langle(\nabla C)^2\rangle,\end{aligned}$$

since $\langle(\boldsymbol{u}\cdot\nabla)C^2\rangle = 0$ by the same argument as in a).

c) The term $(\nabla C)^2$ is always positive, so the time derivative of $\langle C^2\rangle$ is always negative: Regions of very high or very low concentration get less and less pronounced. As the magnitude of C decreases, the gradients ∇C are more shallow, slowing down the process. By definition, $\langle C^2\rangle \geq 0$: For long times, the system approaches the completely mixed state $C(\boldsymbol{x}) = 0$ (but never reaches it).

d) The equality is valid at any given time. However, the decay in the variance depends on the concentration gradients ∇C, which are influenced by the advection. If a certain flow pattern enhances the mixing, that means it brings regions of high and low concentration closer together, so that steep concentration gradients appear that are equilibrated by diffusion.