Exercise Set 4: Transport and Mixing

1 Mixing of a Passive Scalar

Consider the concentration field $C(\boldsymbol{x}, t)$ of a *passive scalar* in a periodic box. It is described by the **advection-diffusion equation**

$$\frac{\partial C}{\partial t} + (\boldsymbol{u} \cdot \nabla)C \;=\; \alpha \nabla^2 C, \tag{1}$$

where $\boldsymbol{u}$ is the flow velocity of an incompressible Newtonian fluid, and $\alpha > 0$ some diffusion constant. The scalar quantity C is called *passive* because it does not have any influence on the flow, but is merely advected by it.

The concentration field C is inhomogeneous, i.e. it has regions of high and low concentration. One can describe the distribution in terms of its mean $\langle C \rangle$ and variance $\langle C^2 \rangle$, where $\langle \ldots \rangle$ denotes a spatial average over the box. Assume an initial concentration field $C(\boldsymbol{x}, t = 0)$ with

$$\langle C \rangle_{t=0} \;\equiv\; 0. \tag{2}$$

a) Show that $\langle C \rangle = 0$ at all times t.

b) Show that the variance evolves as

$$\frac{\mathrm{d}\langle C^2 \rangle}{\mathrm{d}t} \;=\; -\,2\alpha \langle (\nabla C)^2 \rangle. \tag{3}$$

(Use $\frac{\partial}{\partial t}(C^2) = 2C\frac{\partial C}{\partial t}$ and partial integration in the periodic box.)

c) Does the variance increase, decrease or stay the same? Consequently, what concentration field $C(\boldsymbol{x})$ do you expect after a very long time?

d) The turbulent flow field $\boldsymbol{u}$ does not appear in equation (3). Still we claim that turbulence *enhances* mixing, i.e. the variance decreases faster in a turbulent flow than in a fluid at rest. Explain.