Exercise Set 3: Conservation Laws – Solutions

Consider the flow of an incompressible Newtonian fluid in a periodic box. It is described by the Navier-Stokes equations

$$\partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} = -\nabla p + \nu \nabla^2 \boldsymbol{v}, \quad (1)$$

$$\nabla \cdot \boldsymbol{v} = 0. \quad (2)$$

Such a flow satisfies several conservation laws. One quantity of interest is the helicity

$$H = \left\langle \tfrac{1}{2}\, \boldsymbol{v} \cdot \boldsymbol{\omega} \right\rangle, \quad (3)$$

where $\langle \ldots \rangle$ denotes an average over the box. Helicity is an invariant of the Euler equations (i.e. in flow at zero viscosity, $\nu = 0$).

1 Deriving the Vorticity Equation

The vorticity $\boldsymbol{\omega} = \nabla \times \boldsymbol{v}$ evolves according to the **vorticity equation**

$$\partial_t \boldsymbol{\omega} = \nabla \times (\boldsymbol{v} \times \boldsymbol{\omega}) + \nu \nabla^2 \boldsymbol{\omega}. \quad (4)$$

a) Derive the vorticity equation from the Navier-Stokes equations, using the vector identity $\nabla |\boldsymbol{v}|^2 = 2\boldsymbol{v} \cdot \nabla \boldsymbol{v} + 2\boldsymbol{v} \times (\nabla \times \boldsymbol{v})$ (which you do not need to prove).

Take the rotation of Navier-Stokes:

$$\nabla \times \partial_t \boldsymbol{v} + \underbrace{\nabla \times (\boldsymbol{v} \cdot \nabla \boldsymbol{v})}_{\textit{nonlinear term}} = - \underbrace{\nabla \times \nabla p}_{\textit{curl of gradient}=0} + \nu \nabla \times \nabla^2 \boldsymbol{v}.$$

For the nonlinear term, insert the provided vector identity into the brackets:

$$\begin{aligned} \nabla \times (\boldsymbol{v} \cdot \nabla \boldsymbol{v}) &= \nabla \times \left[\tfrac{1}{2} \nabla |\boldsymbol{v}|^2 - \boldsymbol{v} \times (\underbrace{\nabla \times \boldsymbol{v}}_{\boldsymbol{\omega}}) \right] \\ &= \underbrace{\tfrac{1}{2} \nabla \times \nabla |\boldsymbol{v}|^2}_{\textit{curl of gradient}=0} - \nabla \times (\boldsymbol{v} \times \boldsymbol{\omega}) \\ &= -\nabla \times (\boldsymbol{v} \times \boldsymbol{\omega}) \end{aligned}$$

Return to the original equation:

$$\Leftrightarrow \quad \partial_t \underbrace{\nabla \times \boldsymbol{v}}_{\boldsymbol{\omega}} = \nabla \times (\boldsymbol{v} \times \boldsymbol{\omega}) + \nu \nabla^2 \underbrace{(\nabla \times \boldsymbol{v})}_{\boldsymbol{\omega}}$$

b) What might be an advantage of solving the vorticity equation if one can also solve Navier-Stokes directly?

Taking the curl has eliminated the pressure term, so that the equation now only contains three components. The incompressibility condition is fulfilled implicitly (and thus exactly, up to numerical precision, in a simulation). For a two-dimensional flow (where all motion occurs in two Cartesian directions only), vorticity only has one nonzero component, which further simplifies the equation.
Note however that the equation is still nonlinear, and the velocity $\boldsymbol{v}$ even makes the equation nonlocal.

2 Helicity of Box Turbulence

In a periodic box, the time derivative of the helicity is

$$\frac{\mathrm{d}}{\mathrm{d}t} \left\langle \tfrac{1}{2}\, \boldsymbol{v} \cdot \boldsymbol{\omega} \right\rangle \;=\; -\,\nu \left\langle \boldsymbol{\omega} \cdot (\nabla \times \boldsymbol{\omega}) \right\rangle . \tag{5}$$

a) Derive this relation from the vorticity equation, using the fact that partial integration over the periodic box does not generate boundary terms. (Use the identities for periodic box integrals that were given in class.)

Start with

$$\frac{d}{dt} \left\langle \tfrac{1}{2} \boldsymbol{v} \cdot \boldsymbol{\omega} \right\rangle \;=\; \tfrac{1}{2} \left\langle \tfrac{\partial \boldsymbol{v}}{\partial t} \cdot \boldsymbol{\omega} \right\rangle + \tfrac{1}{2} \left\langle \boldsymbol{v} \cdot \tfrac{\partial \boldsymbol{\omega}}{\partial t} \right\rangle .$$

For the first term, use $\langle \boldsymbol{u} \cdot (\nabla \times \boldsymbol{v}) \rangle = \langle (\nabla \times \boldsymbol{u}) \cdot \boldsymbol{v} \rangle$*:*

$$\begin{aligned} \left\langle \tfrac{\partial \boldsymbol{v}}{\partial t} \cdot \boldsymbol{\omega} \right\rangle &= \left\langle (\nabla \times \tfrac{\partial \boldsymbol{v}}{\partial t}) \cdot \boldsymbol{v} \right\rangle \\ &= \left\langle \tfrac{\partial}{\partial t} (\nabla \times \boldsymbol{v}) \cdot \boldsymbol{v} \right\rangle \\ &= \left\langle \tfrac{\partial \boldsymbol{\omega}}{\partial t} \cdot \boldsymbol{v} \right\rangle \end{aligned}$$

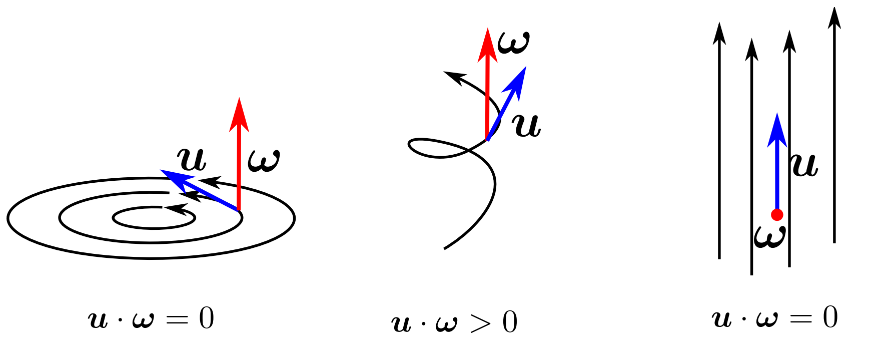
Then continue by entering the vorticity equation:

$$\begin{aligned} \frac{d}{dt} \left\langle \tfrac{1}{2} \boldsymbol{v} \cdot \boldsymbol{\omega} \right\rangle &= \left\langle \boldsymbol{v} \cdot \tfrac{\partial \boldsymbol{\omega}}{\partial t} \right\rangle \\ &= \left\langle \boldsymbol{v} \cdot (\nabla \times (\boldsymbol{v} \times \boldsymbol{\omega}) + \nu \nabla^2 \boldsymbol{\omega}) \right\rangle \\ &= \underbrace{\left\langle (\nabla \times \boldsymbol{v}) \cdot (\boldsymbol{v} \times \boldsymbol{\omega}) \right\rangle}_{via\ \langle \boldsymbol{u} \cdot (\nabla \times \boldsymbol{v}) \rangle = \langle (\nabla \times \boldsymbol{u}) \cdot \boldsymbol{v} \rangle} \;+\; \nu \langle \boldsymbol{v} \cdot \nabla^2 \boldsymbol{\omega} \rangle \\ &= \underbrace{\left\langle \boldsymbol{\omega} \cdot (\boldsymbol{v} \times \boldsymbol{\omega}) \right\rangle}_{=0} - \nu \underbrace{\left\langle (\nabla \times \boldsymbol{v}) \cdot (\nabla \times \boldsymbol{\omega}) \right\rangle}_{via\ \langle \boldsymbol{u} \cdot (\nabla^2 \boldsymbol{v}) \rangle = -\langle (\nabla \times \boldsymbol{u}) \cdot (\nabla \times \boldsymbol{v}) \rangle} \\ &= -\,\nu \left\langle \boldsymbol{\omega} \cdot (\nabla \times \boldsymbol{\omega}) \right\rangle \end{aligned}$$

b) Why is helicity an invariant of the Euler equations?

For $\nu = 0$, $\frac{d}{dt}H = 0$: In a periodic box, helicity cannot change with time; it is conserved (time invariant).

c) Draw some example flows with large or small helicity.



These are three typical flows, two with small and one with large vorticity: The vortex flow on the left has large vorticity in the z-direction, but the motion is in the x-y-plane so that the helicity vanishes. The flow on the right is a parallel flow without vorticity, hence no helicity. The flow in the center is a mixture of the two, in that it has a vortex-like structure in the x-y-plane, but also an advection component in z. The resulting, corkscrew-like motion has a strong helicity.