ME-467: Turbulence 14 March 2025
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Exercise Set 2: Deterministic Chaos and Statistics – Solutions

The **Lorenz System** is a dynamical system of three time-dependent coordinates $x(t)$, $y(t)$, $z(t)$ described by the differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sigma(y-x), \tag{1}$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = x(\rho-z)-y, \tag{2}$$
$$\frac{\mathrm{d}z}{\mathrm{d}t} = xy-\beta z. \tag{3}$$

The coefficients σ, ρ and β are positive, real numbers. Set $\sigma = 10$, $\rho = 28$ and $\beta = 8/3$. The Matlab script `plotlorenz.m` solves the Lorenz system up to a time $t_{\max}$, which is passed as a parameter `tmax`.

1 Sensitivity on Initial Conditions

Imagine you want to reproduce a measurement of the time series $z(t)$, which has been produced from the initial condition $\boldsymbol{x}_0 = (x_0, y_0, z_0) = (-8, 8, 27)$. However, in setting up your experiment, your initial condition $\boldsymbol{x}_0^{exp}$ deviates from the original initial condition by

$$\varepsilon_0 := |\boldsymbol{x}_0^{exp} - \boldsymbol{x}_0|. \tag{4}$$

You can decrease this error (increase precision) by being more careful, but can never decrease it to zero. The error in the initial condition causes an error in the time series $z^{exp}(t)$, with

$$\varepsilon_z(t) := |z^{exp}(t) - z(t)|. \tag{5}$$

a) Find the ε_0 required so that ...
   - $\varepsilon_z(t) < 10^{-3}$ for all $t < 10$.
   - $\varepsilon_z(t) < 10^{-8}$ for all $t < 10$.
   - $\varepsilon_z(t) < 10^{-3}$ for all $t < 20$.

   How are your chances of predicting $z(t)$ for even longer times?

The values are approximately 10^{-8}, 10^{-12} and 10^{-11}, respectively. To decrease the error at $t = 10$ by 5 orders of magnitude, the error of the initial condition must also be about 5 orders of magnitude smaller (linear growth). However, a similar increase in *accuracy of the initial condition is only enough to increase the time from 10 to 20, far less than even one order of magnitude: In the chaotic system, the error grows so fast (exponentially!) that even the smallest of errors in the initial conditions will become significant after some time.*

b) Find the smallest value for ε_0 that gives meaningful results. Why are results for smaller ε_0 not meaningful?

Below $\varepsilon_0 \sim 10^{-15}$*, the two curves look identical. This is by no means a property of the Lorenz attractor, but rather an issue of the numerical representation on the computer: A double-precision floating point number is only exact up to an error of around* 10^{-16}*. Below that, the initial conditions look the same to the computer, so it also runs exactly the same simulation twice and gets the same results twice. A system in nature usually has a continuous state space, so two initial conditions are never identical.*

c) What is the maximum error $\varepsilon_z(t)$ you observe for $t < 100$? Why does it not depend on the initial error ε_0?

The maximum error is around 50: For the initial conditions given, the z-coordinate of the solution oscillates in a range between 0 and 50, so it is impossible for two z-coordinates to ever have a distance larger than 50.

d) Compare $\varepsilon_z(t)$ to the exponential function

$$f_\lambda(t) = \varepsilon_0 \cdot \exp(\lambda t) \tag{6}$$

(with some constant λ).

- For $\varepsilon_0 = 10^{-12}$ and $t < 30$, find a good value for λ so that the functions match.
- Do $\varepsilon_z(t)$ and $f_\lambda(t)$ also match for other ε_0?
- If you could use smaller ε_0 than the limit you found in b): What ε_0 would be required so that $\varepsilon_z(t) < 10^{-3}$ for all $t < 100$?

A good value for λ *(the* Lyapunov exponent*) is* $\lambda = 0.9$*. For the given system parameters* $(\sigma,\ \rho,\ \beta)$*, this exponent will be constant and will not depend on the initial error* ε_0*. For a precision* $\varepsilon_1 = 10^{-3}$ *at* $t_1 = 100$*,*

$$\varepsilon_0 \;=\; \varepsilon_1 \cdot \exp(-\lambda t_1) \;\approx\; 10^{-53},$$

a value far below the limits of double precision.

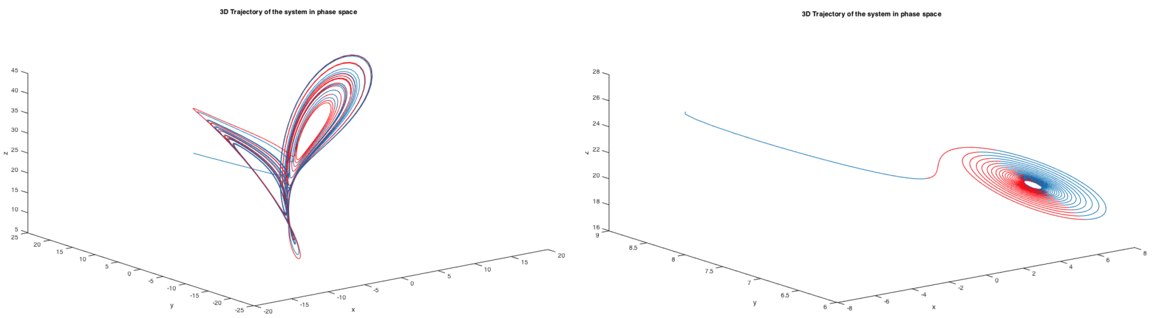
2 Chaotic vs. Non-Chaotic Behavior

We can change the behavior of the Lorenz system by changing the coefficients σ, ρ and β. There is a qualitative change in the behavior if

$$\rho < \sigma \cdot \frac{\sigma + \beta + 3}{\sigma - \beta - 1} \quad (\approx 24.7 \text{ for } \sigma = 10,\ \beta = 8/3).$$

In `plotlorenz.m`, set $\rho = 20$.

a) Describe the change.



While at $\rho = 28$ (left) *the system follows a chaotic trajectory through phase space, at* $\rho = 20$ (right) *the trajectory spirals inward. The two trajectories from different initial conditions follow the same path.*

b) Find the ε_0 required so that $\varepsilon_z(t) < 10^{-8}$ for all $t < 100$.

The error between the two trajectories decreases, so that $\varepsilon_0 = 10^{-8}$ *is enough to make sure that the error at later times is always below that value.*

c) Again, compare $\varepsilon_z(t)$ to $f_\lambda(t)$:

- Find a good λ for $\varepsilon_0 = 10^{-1}$ and $t < 20$.
- What is the qualitative difference between this λ and the value for $\rho = 28$?

The error decreases, which is described by a negative Lyapunov exponent, here $\lambda = -0.15$.