**ME-467: Turbulence** 07 March 2025
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# Exercise Set 2: Deterministic Chaos and Statistics

The **Lorenz System** is a dynamical system of three time-dependent coordinates $x(t)$, $y(t)$, $z(t)$ described by the differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sigma(y - x), \tag{1}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = x(\rho - z) - y, \tag{2}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = xy - \beta z. \tag{3}$$

The coefficients $\sigma$, $\rho$ and $\beta$ are positive, real numbers. Set $\sigma = 10$, $\rho = 28$ and $\beta = 8/3$. The Matlab script `plotlorenz.m` solves the Lorenz system up to a time $t_{\max}$, which is passed as a parameter `tmax`.

## 1 Sensitivity on Initial Conditions

Imagine you want to reproduce a measurement of the time series $z(t)$, which has been produced from the initial condition $\boldsymbol{x}_0 = (x_0, y_0, z_0) = (-8, 8, 27)$. However, in setting up your experiment, your initial condition $\boldsymbol{x}_0^{exp}$ deviates from the original initial condition by

$$\varepsilon_0 := |\boldsymbol{x}_0^{exp} - \boldsymbol{x}_0|. \tag{4}$$

You can decrease this error (increase precision) by being more careful, but can never decrease it to zero. The error in the initial condition causes an error in the time series $z^{exp}(t)$, with

$$\varepsilon_z(t) := |z^{exp}(t) - z(t)|. \tag{5}$$

a) Find the $\varepsilon_0$ required so that ...
   - $\varepsilon_z(t) < 10^{-3}$ for all $t < 10$.
   - $\varepsilon_z(t) < 10^{-8}$ for all $t < 10$.
   - $\varepsilon_z(t) < 10^{-3}$ for all $t < 20$.

   How are your chances of predicting $z(t)$ for even longer times?

b) Find the smallest value for $\varepsilon_0$ that gives meaningful results. Why are results for smaller $\varepsilon_0$ not meaningful?

c) What is the maximum error $\varepsilon_z(t)$ you observe for $t < 100$? Why does it not depend on the initial error $\varepsilon_0$?

d) Compare $\varepsilon_z(t)$ to the exponential function

$$f_\lambda(t) = \varepsilon_0 \cdot \exp(\lambda t) \tag{6}$$

(with some constant $\lambda$).

- For $\varepsilon_0 = 10^{-12}$ and $t < 30$, find a good value for $\lambda$ so that the functions match.
- Do $\varepsilon_z(t)$ and $f_\lambda(t)$ also match for other $\varepsilon_0$?
- If you could use smaller $\varepsilon_0$ than the limit you found in b): What $\varepsilon_0$ would be required so that $\varepsilon_z(t) < 10^{-3}$ for all $t < 100$?

## 2 Chaotic vs. Non-Chaotic Behavior

We can change the behavior of the Lorenz system by changing the coefficients $\sigma$, $\rho$ and $\beta$. There is a qualitative change in the behavior if

$$\rho < \sigma \cdot \frac{\sigma + \beta + 3}{\sigma - \beta - 1} \quad (\approx 24.7 \text{ for } \sigma = 10,\ \beta = 8/3).$$

In `plotlorenz.m`, set $\rho = 20$.

a) Describe the change.

b) Find the $\varepsilon_0$ required so that $\varepsilon_z(t) < 10^{-8}$ for all $t < 100$.

c) Again, compare $\varepsilon_z(t)$ to $f_\lambda(t)$:

- Find a good $\lambda$ for $\varepsilon_0 = 10^{-1}$ and $t < 20$.
- What is the qualitative difference between this $\lambda$ and the value for $\rho = 28$?