ME-467: Turbulence 07 March 2025
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Exercise Set 1: Symmetries – Solutions

Consider the diffusion equation

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + s(x,t) \tag{1}$$

on the interval $x \in [-L/2, L/2)$ with periodic boundary conditions and some source function $s(x,t)$. The temperature $T(x,t)$ has initial conditions

$$T(x, t=0) = T_0(x). \tag{2}$$

1 Symmetries of the System

The symmetries of the system depend on the function $s(x,t)$.

a) Assume s is of the form $s(t)$, i.e. it does not depend on x. What are the symmetries of the system?

b) Under which condition (on s) is the system symmetric under reflection?

Solution:

a) The most basic symmetries of the system are a continuous translation symmetry σ_t and a reflection symmetry σ_r around an arbitrary point (which we can choose to lie at $x = 0$).

Translation symmetry
Let σ_t:

$$t, x, T \longmapsto \hat{t} = t,\ \hat{x} = x + a,\ \hat{T} = T; \quad a \in \mathbb{R}.$$

We find that

$$\begin{aligned} \tfrac{\partial}{\partial \hat{t}} &= \tfrac{\partial}{\partial t} \\ \tfrac{\partial}{\partial \hat{x}} &= \tfrac{\partial x}{\partial \hat{x}} \cdot \tfrac{\partial}{\partial x} = \tfrac{\partial}{\partial x} \\ \hat{T}(\hat{x}, \hat{t}) &= T(\hat{x} - a, \hat{t}) = T(x,t) \end{aligned}$$

Therefore, the left hand side of (1) in the transformed system reads

$$\tfrac{\partial}{\partial \hat{t}} \hat{T}(\hat{x}, \hat{t}) = \tfrac{\partial}{\partial \hat{t}} T(\hat{x} - a, \hat{t}) = \tfrac{\partial}{\partial t} T(x,t)$$

and the right hand side reads

$$\frac{\partial^2}{\partial \hat{x}^2}\hat{T}(\hat{x},\hat{t}) + s(\hat{t}) \;=\; \frac{\partial^2}{\partial \hat{x}^2}T(\hat{x}-a,\hat{t}) + s(\hat{t}) \;=\; \frac{\partial^2}{\partial x^2}T(x,t) + s(t).$$

We see that the form of the PDE is invariant under σ_t. The boundary conditions are also invariant under σ_t, because they are periodic. Therefore, the time evolution f^t does not change under translation; $f^t = \sigma_t^{-1} \circ f^t \circ \sigma_t$. Since we did not make assumptions about T, this result is valid for all initial conditions T_0.

Reflection symmetry

We first only consider reflection around the point $x = 0$. Let σ_r:

$$t, x, T \longmapsto \hat{t} = t,\ \hat{x} = -x,\ \hat{T} = T.$$

We find that

$$\begin{aligned}
\frac{\partial}{\partial \hat{t}} &= \frac{\partial}{\partial t} \\
\frac{\partial}{\partial \hat{x}} &= \frac{\partial x}{\partial \hat{x}} \cdot \frac{\partial}{\partial x} = -\frac{\partial}{\partial x} \\
\hat{T}(\hat{x},\hat{t}) &= T(-\hat{x},\hat{t}) = T(x,t)
\end{aligned}$$

Therefore, the left hand side of (1) in the transformed system reads

$$\frac{\partial}{\partial \hat{t}}\hat{T}(\hat{x},\hat{t}) \;=\; \frac{\partial}{\partial \hat{t}}T(-\hat{x},\hat{t}) \;=\; \frac{\partial}{\partial t}T(x,t)$$

and the right hand side reads

$$\begin{aligned}
\frac{\partial^2}{\partial \hat{x}^2}\hat{T}(\hat{x},\hat{t}) + s(\hat{t}) &= \frac{\partial^2}{\partial \hat{x}^2}T(-\hat{x},\hat{t}) + s(\hat{t}) \\
&= (-1)^2\frac{\partial^2}{\partial x^2}T(x,t) + s(t) \\
&= \frac{\partial^2}{\partial x^2}T(x,t) + s(t).
\end{aligned}$$

Again, the boundary conditions also fulfill the symmetry, so that the time evolution remains unchanged and σ_r is an equivariance. Note that the system is also reflection symmetric around any other point $b \in [-L/2, L/2)$. Since the symmetries form a group, we can construct this symmetry from translations σ_t and a reflection σ_r at the origin.

b) The function s does not have to be a constant (in x) in order for σ_r to be a symmetry of the equation: From a), we can confirm that, if

$$s(x,t) \;=\; s(-x,t),$$

the argument still holds. By translation, if s is symmetric around any point in the x-range, that point is a center of reflection symmetry.

2 Symmetries of the Solution

a) Let $s(x,t) = s(t)$ and $T_0(x) = const$. Use a symmetry to show that $T(x,t)$ is constant with respect to x for all times.

b) Let $s(x,t) = 0$ and $T_0(x) = \cos\left(\frac{4\pi x}{L}\right)$. Without solving the diffusion equation, what can you say about the shape of $T(t)$ at $t > 0$?

c) Let $s(x,t) = 0.01 \cdot \cos\left(\frac{2\pi x}{L}\right)$ and $T_0(x) = \cos\left(\frac{4\pi x}{L}\right)$. How does this change your answer from b)? Solve the equation to validate your predictions.

Solution:

a) We use translation symmetry σ_t, which is a symmetry of the system and the initial condition: We know that $\sigma_t \circ T_0 = T_0$ (from $T_0(x+a) = T_0(x)$). Therefore, the solution $T(x,t)$ at a later time $t > 0$ fulfills

$$T(x-a,t) \;=\; \sigma_t \circ T(x,t) \;=\; \sigma_t \circ f^t(T_0) \;=\; f^t(\sigma_t \circ T_0) \;=\; f^t(T_0) \;=\; T(x,t)$$
$$\implies T(x-a,t) = T(x,t) \;\; \forall a \in \mathbb{R}.$$

The only form of T which is translation invariant for any a is a constant function.

b) The function $s(x,t) = 0$ meets the criterion $s(x,t) = s(t)$. Therefore the system is translation invariant by any displacement and reflection invariant around any point. The initial condition T_0 however has far less symmetries; it is translation invariant by $a = k\frac{L}{2}, k \in \mathbb{Z}$ and reflection invariant at $b \in \{-L/2, -L/4, 0, L/4\}$. Because the symmetries of the initial condition are symmetries of the system, the solution $T(x,t)$ must have the same symmetries.

Remark: A reflection plus flipping the sign of T is another symmetry that one could consider here.

c) The system now only has discrete symmetries; a translation invariance σ_t with $a = kL, k \in \mathbb{Z}$ and reflection at points $b \in \{-L/2, 0\}$. They are also symmetries of the initial condition, and therefore of the solution; but the other symmetries of the initial condition are not symmetries of the solution anymore.

The equation can be solved like the pressure field in the Navier-Stokes-equation: Let

$$T(x) \;=\; \sum_{k\in\mathbb{Z}} \hat{T}_k \exp\left(i\frac{2\pi k}{L}x\right)$$

With $s(x) \;=\; \sum_{k\in\mathbb{Z}} \hat{s}_k \exp\left(i\frac{2\pi k}{L}x\right)$ and the definition of $\hat{s}_k$, we get

$$\hat{s}_k = \frac{1}{L}\int_{-L/2}^{L/2} \exp\left(-i\frac{2\pi k}{L}x\right) s(x)\mathrm{d}x \implies \hat{s}_k \;=\; \begin{cases} 0.005 & k = \pm 1 \\ 0 & \text{otherwise.} \end{cases}$$

The k-th component of the diffusion equation reads

$$\partial_t \hat{T}_k = -\frac{4\pi^2 k^2}{L^2}\hat{T}_k + \hat{s}_k$$

with initial conditions

$$\hat{T}_{0,k} = \begin{cases} 0.5 & k = \pm 2 \\ 0 & \text{otherwise.} \end{cases}$$

Solving these equations for each k separately gives

$$\hat{T}_k(t) = \begin{cases} \frac{0.005L^2}{4\pi^2}\left[1 - \exp\left(-\frac{4\pi^2}{L^2}t\right)\right] & k = \pm 1 \\ 0.5\exp\left(-\frac{16\pi^2}{L^2}t\right) & k = \pm 2 \\ 0 & \text{otherwise,} \end{cases}$$

or, after evaluating the sum over the $\hat{T}_k$,

$$T(x,t) = \frac{0.01L^2}{4\pi^2}\left[1 - \exp\left(-\frac{4\pi^2}{L^2}t\right)\right]\cos\Big(\frac{2\pi}{L}x\Big) + \exp\left(-\frac{16\pi^2}{L^2}t\right)\cos\Big(\frac{4\pi}{L}x\Big).$$

In this solution, we find again the reflection and translation symmetries that we predicted.

Alternative approach: We can also solve the equation via separation of variables. We first solve the homogeneous equation $\partial_t T_h = \partial_x^2 T_h$ by assuming $T_h(x,t) = \tau(t)X(x)$. After rearranging, we get

$$\frac{\tau'}{\tau} = \frac{X''}{X}$$

The left-hand side purely depends on t, while the right-hand side only on x. This is only possible if both sides are constant

$$\frac{\tau'}{\tau} = \frac{X''}{X} = -\alpha$$

Solving the equation for τ, we get $\tau(t) = Ae^{-\alpha t}$. From the periodic boundary conditions we can deduce that $\alpha > 0$, so we can write $\alpha = \lambda^2$, and solve for $X(x)$

$$\begin{aligned} X'' &= -\lambda^2 X \\ \implies X(x) &= B\cos(\lambda x) + C\sin(\lambda x) \end{aligned}$$

for $\lambda \neq 0$. If $\lambda = 0$ then $X'' = 0 \implies X = B_0 x + C_0$. Since we need $X(-L/2) = X(L/2)$, we need $B_0 = 0$ and $\sin(\lambda L/2) = 0$, and thus $\lambda = \frac{2\pi k}{L}$, with $k \in \mathbb{Z}$. Thus

$$X(x) = C_0 + \sum_{n=1}^{\infty}\Big\{B_n\cos\Big(\frac{2\pi n}{L}x\Big) + C_n\sin\Big(\frac{2\pi n}{L}x\Big)\Big\}$$

For the particular solution, we guess from the form of $s(x,t)$ that we might want to try $T_p = D\cos(2\pi x/L)$. Plugging this into the full equation, we find $D = 0.01(L/2\pi)^2$. By combining all of these results and after relabelling we have

$$\begin{aligned} T(x,t) = \; & 0.01\Big(\frac{L}{2\pi}\Big)^2 \cos\Big(\frac{2\pi x}{L}\Big) \\ & + \exp\Big(-\frac{4\pi^2 n^2}{L^2}t\Big)\Big(c_0 + \sum_{n=1}^{\infty}\Big\{b_n\cos\Big(\frac{2\pi n}{L}x\Big) + c_n \sin\Big(\frac{2\pi n}{L}x\Big)\Big\}\Big) \end{aligned}$$

Matching with the initial condition $T_0(x) = \cos(4\pi x/L)$ we get $b_1 = -0.01(L/2\pi)^2, b_2 = 1$ and $c_n = b_n = 0$ otherwise, giving the same solution as before

$$T(x,t) \;=\; \frac{0.01L^2}{4\pi^2}\left[1 - \exp\left(-\frac{4\pi^2}{L^2}t\right)\right]\cos\Big(\frac{2\pi}{L}x\Big) + \exp\left(-\frac{16\pi^2}{L^2}t\right)\cos\Big(\frac{4\pi}{L}x\Big).$$