Dispersion relation for flowing cylindrical thread held between two nozzles

We are considering a viscous liquid thread (viscosity is μ, surface tension γ) injected at velocity U by a nozzle of inner radius h_0 and collected at a distance L by a similar nozzle of same radius. We introduce the capillary number $Ca = \mu U/\gamma$ (check the dimensions) as rescaled velocity and rescale the time with $\tau = \mu h_0/\gamma$ We will assume the equation governing the evolution of the radius perturbation $R = R_0 + \epsilon h(z,t)$ is governed by

$$\frac{\partial h}{\partial t} = -Ca\frac{\partial h}{\partial z} + \frac{h}{6} + \frac{1}{6}\frac{\partial^2 h}{\partial z^2} \tag{1}$$

with boundary conditions

$$h(0,t) = h(l,t) = 0 \tag{2}$$

where $l = L/R$.

Dispersion relation

Ignoring for now the boundary conditions, use a normal mode expansion $h(z,t) = H\exp(i(kz - \omega t))$ to obtain the following dispersion relation

$$\omega = Cak + i\frac{1}{6}(1-k^2), \tag{3}$$

as seen during the class. Read the slide which shows that the transition from convective to absolute happens at $Ca = 1/3$ as the absolute frequency writes $\omega_0 = i(-\frac{3}{2}Ca^2 + 1/6)$.

Global modes

The normal mode assumption does not satisfy the boundary conditions. To do so, a more general expansion should be used

$$h(z,t) = \eta(z)\exp(\lambda t) \tag{4}$$

where the λ is the global eigenvalue. With this definition, global instability will happen if there exists an eigenvalue with positive real part.

1. Show that under these conditions the PDE becomes an ODE

$$(\lambda - \frac{1}{6})\eta = -Ca\frac{d\eta}{dz} + \frac{1}{6}\frac{d^2\eta}{dz^2} \tag{5}$$

$$\eta(0) = \eta(l) = 0 \tag{6}$$

2. We look for solution under the form

$$h(z) = \exp(\alpha z). \tag{7}$$

Show that the roots write

$$\alpha_{\pm} = 3Ca \pm \frac{1}{2}\sqrt{36Ca^2 + 24\lambda - 4} \tag{8}$$

3. We look for a combination of these two fundamental solutions $h = h^+ \exp(\alpha_+ z) + h^- \exp(\alpha_- z)$. By imposing the boundary conditions, show that

$$\begin{pmatrix} 1 & \exp(\alpha_+ l) \\ 1 & \exp(\alpha_+ l) \end{pmatrix} \begin{pmatrix} h^+ \\ h^- \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{9}$$

and therefore that

$$\sqrt{36Ca^2 + 24\lambda - 4} = i\frac{2\pi m}{l} \tag{10}$$

where m is an integer.

In other words, find the following expression for the eigenvalues

$$\lambda = -\frac{3}{2}Ca^2 + \frac{1}{6} - \frac{m^2\pi^2}{6l^2} \tag{11}$$

4. Deduce that **local absolute instability is a necessary condition for global instability**. While this is not always true (it is also dependent on the boundary conditions), this condition is often accepted.