Stratified shear flow : from Rayleigh's theorem to Richardson's criterion

Q1

The Squire theorem, although one should be careful using it. Indeed, it stipulates that conclusions of the simplified analysis , *without* the transverse direction, applies to a full analysis *with* it, **but for different non-dimensional numbers.**

Q2

The proposed expansions are to be injected in Euler equations plus the temperature equation and collecting terms at order ϵ.

Q3

It is the continuity equation, and, as written in (8), expresses the divergence of the perturbation field in velocity to be equal to zero. The physical counterpart is the incompressibility of the fluid (everything that arrives in a fluid control volume must go out : no fluid accumulation is possible).

Q4

The Boussinesq hypothesis is used. Namely, the density is assumed constant (of value ρ_0) in the continuity and momentum equations, excepted for the buoyancy term.

Q5

The flow is indeed said to be stably stratified as the heavy (cold) fluid is *below* the light (hot) fluid. No Rayleigh-Bénard instability is to be expected (which does not imply the flow to be stable)

Q6

We differentiate the Eq.(5) with respect to z :

$$(-i\omega + ikU)Du + ikU'u + U''v + U'v' = -ikDp/\rho_0$$

then multiply Eq.(6) by ik :

$$(-i\omega + ikU)(ikv) = -ikDp/\rho_0 + ik\frac{\beta}{\rho_0}g\theta$$

then subtract the former to the latter :

$$(-i\omega + ikU)(Du - ikv) + ikU'u + U''v + U'v' = -ik\frac{\beta}{\rho_0}g\theta$$

In terms of **streamfunction** : $v = -ik\psi$, $u = D\psi$. We inject :

$$(-i\omega + ikU)(D^2 - k^2)\psi + ikU'D\psi + U''(-ik\psi) + U'(-ikD\psi) = -ik\frac{\beta}{\rho_0}g\theta \Leftrightarrow$$
$$(-i\omega + ikU)(D^2 - k^2)\psi - ikU''\psi = -ik\frac{\beta}{\rho_0}g\theta$$

but $\theta = -T'(z)v/(-i\omega + ikU)$. Thereby :

$$(-i\omega + ikU)(D^2 - k^2)\psi - ikU''\psi = -ik\frac{\beta}{\rho_0}g\left(\frac{T'(z)ik\psi}{-i\omega + ikU}\right) \Leftrightarrow$$
$$(-i\omega + ikU)(D^2 - k^2)\psi - ikU''\psi = \frac{\beta}{\rho_0}g\left(\frac{k^2T'(z)\psi}{-i\omega + ikU}\right) \Leftrightarrow$$
$$(-i\omega + ikU)(D^2 - k^2)\psi - ikU''\psi = \left(\frac{k^2N^2}{-i\omega + ikU}\right)\psi$$

with :

$$N^2 = \frac{\beta g T'(z)}{\rho_0}$$

Q7

$[\beta] = [\rho_0]/[T]$, so

$$[N^2] = \frac{[\rho_0]}{[T]} * \frac{m}{s^2} * \frac{[T]}{m} * \frac{1}{[\rho_0]} = \frac{1}{s^2}$$

N being in s^{-1} (Hertz), it is indeed a frequency. Without stratification, the Taylor-Goldstein equation reduces to the Rayleigh equation.

Q8

It is a polynomial eigenvalue problem in a temporal sense ($\omega(k)$) (and also in a spatial ($k(\omega)$) sense). Indeed there are terms proportional to ω and ω^2.

Q9

With the chosen form for the Fourier mode : $\propto e^{i(kx-\omega t)} = e^{i(kx-\omega_r t - i\omega_i t)} = e^{\omega_i t}e^{i(kx-\omega_r t)}$. Thus instability happens whenever $\omega_i > 0$.

Q10

Let's first notice that $-i\omega$ becomes $-i(\omega_r + i\omega_i) = \omega_i - i\omega_r$. The Taylor-Goldstein becomes in turns :

$$
\begin{aligned}
[\omega_i + i(kU - \omega_r)](D^2 - k^2)\psi - ikU''\psi &= \left(\frac{k^2N^2}{[\omega_i + i(kU - \omega_r)]}\right)\psi \Leftrightarrow \\
[\omega_i + i(kU - \omega_r)](D^2 - k^2)\psi - ikU''\psi &= \left(\frac{k^2N^2[\omega_i - i(kU - \omega_r)]}{[\omega_i^2 + (kU - \omega_r)^2]}\right)\psi \Leftrightarrow
\end{aligned}
$$

We multiply by ψ^* then integrates over y.

$$
\begin{aligned}
[\omega_i + i(kU - \omega_r)]\int(D^2 - k^2)\psi\psi^* dy - ik\int U''\psi\psi^* dy &= \int\left(\frac{k^2N^2[\omega_i - i(kU - \omega_r)]}{[\omega_i^2 + (kU - \omega_r)^2]}\right)\psi\psi^* dy \Leftrightarrow \\
[\omega_i + i(kU - \omega_r)]\int(-|D\psi|^2 - k^2|\psi|^2)dy - ik\int U''|\psi|^2 dy &= \int\left(\frac{k^2N^2[\omega_i - i(kU - \omega_r)]}{[\omega_i^2 + (kU - \omega_r)^2]}\right)|\psi|^2 dy \Leftrightarrow \\
\int(-|D\psi|^2 - k^2|\psi|^2)dy - \frac{ik[\omega_i - i(kU - \omega_r)]}{[\omega_i^2 + (kU - \omega_r)^2]}\int U''|\psi|^2 dy &= \int\left(\frac{k^2N^2[\omega_i - i(kU - \omega_r)]^2}{[\omega_i^2 + (kU - \omega_r)^2]^2}\right)|\psi|^2 dy \Leftrightarrow
\end{aligned}
$$

Where boundary terms have been put to 0 thanks to the boundary conditions. We use $[\omega_i - i(kU - \omega_r)]^2 = \omega_i^2 - 2i\omega_i(kU - \omega_r) - (kU - \omega_r)^2$, and take the imaginary part of this integral equation :

$$
\begin{aligned}
\frac{-k\omega_i}{[\omega_i^2 + (kU - \omega_r)^2]}\int U''|\psi|^2 dy &= \int\left(\frac{-k^2N^2 2\omega_i(kU - \omega_r)}{[\omega_i^2 + (kU - \omega_r)^2]^2}\right)|\psi|^2 dy \Leftrightarrow \\
\int U''|\psi|^2 dy &= \int\left(\frac{2kN^2(kU - \omega_r)}{[\omega_i^2 + (kU - \omega_r)^2]}\right)|\psi|^2 dy \Leftrightarrow \\
U'' &= \left(\frac{2kN^2(kU - \omega_r)}{[\omega_i^2 + (kU - \omega_r)^2]}\right)
\end{aligned}
$$

It is not a useful explicit condition, because it contains the unknown frequency.

Q11

Key is to go by step :

$$
D\chi = -\frac{ikU'}{2}(-i\omega + ikU)^{-3/2}\psi + \frac{D\psi}{\sqrt{-i\omega + ikU}}
$$

so

$$(-i\omega + ikU)D\chi = -\frac{ikU'}{2\sqrt{-i\omega+ikU}}\psi + D\psi\sqrt{-i\omega+ikU}$$

so

$$\begin{aligned}
D[(-i\omega+ikU)D\chi] = &-\frac{ik}{2}\left(\frac{U''}{\sqrt{-i\omega+ikU}} - \frac{ikU'^2}{2}(-i\omega+ikU)^{-3/2}\right)\psi - \frac{ikU'}{2\sqrt{-i\omega+ikU}}D\psi + \\
&D^2\psi\sqrt{-i\omega+ikU} + \frac{ikU'}{2\sqrt{-i\omega+ikU}}D\psi \\
&= -\frac{ik}{2}\left(\frac{U''}{\sqrt{-i\omega+ikU}} - \frac{ikU'^2}{2}(-i\omega+ikU)^{-3/2}\right)\psi + D^2\psi\sqrt{-i\omega+ikU} \\
&= -\frac{ik}{2}\frac{U''\psi}{\sqrt{-i\omega+ikU}} - \frac{k^2U'^2}{4}(-i\omega+ikU)^{-3/2}\psi + D^2\psi\sqrt{-i\omega+ikU}
\end{aligned}$$

If we multiply by $\sqrt{-i\omega+ikU}$ we obtain :

$$\sqrt{-i\omega+ikU}D[(-i\omega+ikU)D\chi] = -\frac{ik}{2}U''\psi - \frac{k^2U'^2}{4(-i\omega+ikU)}\psi + D^2\psi(-i\omega+ikU)$$

Now we deal with the second part, also multiplied by $\sqrt{-i\omega+ikU}$. it is equal to :

$$k^2\frac{U'^2}{4(-i\omega+ikU)}\psi - k^2\frac{N^2}{(-i\omega+ikU)}\psi - \frac{ikU''}{2}\psi - k^2(-i\omega+ikU)\psi$$

The sum of the two parts gives effectively :

$$(D^2-k^2)(-i\omega+ikU)\psi + D^2\psi(-i\omega+ikU) - ikU''\psi - k^2\frac{N^2}{(-i\omega+ikU)} = 0$$

Q12

Again step by step :

$$\int D[(-i\omega+ikU)D\chi]\chi^* = [BT] - (-i\omega+ikU)\int D\chi D\chi^* = -[\omega_i + i(kU-\omega_r)]\int |D\chi|^2$$

This becomes :

$$- [\omega_i + i(kU - \omega_r)] \int |D\chi|^2 \, dy +$$
$$\int \left(k^2(U'^2/4 - N^2) \frac{[\omega_i - i(kU - \omega_r)]}{[\omega_i^2 + (kU - \omega_r)^2]} - ikU''/2 - k^2[\omega_i + i(kU - \omega_r)] \right) |\chi|^2 \, dy = 0$$

If we take the real part of this equation:

$$-\omega_i \int |D\chi|^2 \, dy + \int \left(k^2(U'^2/4 - N^2) \frac{\omega_i}{[\omega_i^2 + (kU - \omega_r)^2]} - k^2\omega_i \right) |\chi|^2 \, dy = 0 \Leftrightarrow$$
$$\int \left(k^2(U'^2/4 - N^2) \frac{\omega_i}{[\omega_i^2 + (kU - \omega_r)^2]} \right) |\chi|^2 \, dy = \omega_i \left(\int |D\chi|^2 + k^2 |\chi|^2 \, dy \right)$$

but $[\omega_i^2 + (kU - \omega_r)^2] > 0$. If $U'^2/4 - N^2 < 0$ everywhere in the flow domain, then we equate a positive and a negative number and the only solution is $\omega_i = 0$, so no instability. **the necessary condition for an instability to occur is that there exist a pocket in y where $U'^2/4 + N^2 > 0$.**

Q13

A necessary condition for the instability to occur is :

$$\frac{N^2(z)}{U'(z)^2} = \frac{\beta g T'(z)}{\rho_0 U'(z)^2} < \frac{1}{4}$$

to occur somewhere in the flow. It is clear that a higher β and or $T'(z)$ makes it less likely to occur (everything else being fixed), thus increasing the strength of stratification stabilizes the flow. Notice that the dimensionless number Ri compare the potential energy of the flow $\beta g T'(z) L^2$ over the shear-induced kinetic one $\rho_0 U'(z)^2 L^2$ (where L is the length scale of the problem). As $Ri \to 0$, the density field is easily disturbed by the velocity one, stratification not being strong enough to resist the shear

Q14

It is sufficient to integrate :

$$N^2(z) = J(1 - \tanh^2(Rz)) = \frac{\beta g T'(z)}{\rho_0}$$

which gives $T(z) = T_a + \frac{\rho_0 J}{\beta g} \left(\frac{\tanh(Rz)}{R} - \frac{\tanh(Rz_a)}{R} \right)$. This is an increasing function of z : the hotter fluid is above thus the flow is stably stratified.

Q15

$N-1$ interior points and $2(N-1)$ eigenvalues.

Q16

For $R=1$:

$$Ri(z) = \frac{N(z)^2}{U'(z)^2} = \frac{J}{(1-\tanh^2(z))}$$

Its minimum value is simply equal to J since the denominator is at most equal to one (in $z=0$).

Q17

For $R=1$, the necessary condition for the instability to occur is $J<1/4$. In other terms, no instability can occur if $J>1/4$, which seems to be the case on figure $3a$.

Q18

The instability *necessary* (not sufficient) condition predicts that the flow *could be* unstable for all possible J. Figure 4 shows that unstable modes indeed exist for all the J considered.

Q19

Increasing the stratification suppresses the Kelvin-Helmoltz instability for that the entertainment of the heavy fluid in the Kelvin-Helmotz billows cost too much potential energy.

Increasing the stratification creates the Holmboe instability, briefly make its growth rate higher then suppresses it for high J . The most unstable wavenumber seems to monotonously increase with J. Everything else being fixed, a sharp density (temperature) interface (strong R) seems necessary to create the Holmboe instability.

Q20

For Kelvin-Helmoltz instability : the dispersion relation is said to be non-dispersive, as the frequency does not change with the wavenumber. Thus $dw/dk = 0 \ \forall k$ meaning that the wavepacket is static (it growth on place) : the instability is certainly absolute.

For Holmboe instability, it is impossible to deduce if the instability will be convective or absolute be visual inspection only.