Thin-film Rayleigh–Plateau instability around a cylinder – correction

ME-466 Instability

1 Problem formulation

We are interested in the stability of a layer of viscous liquid with dynamic viscosity η [Pa s] and surface tension γ [Pa m] of typical thickness H [m], coating a cylinder of radius $R \gg H$ [m] (figure 1). We neglect inertial effects and gravity. We further assume axisymmetry and no azimuthal velocity.

The system is governed by the incompressible Stokes equations with no-penetration and no-slip conditions at the solid surface, no tangential stress and Laplace jump in the normal stress dynamic conditions on the liquid–air interface.

$$\frac{1}{r^*}\frac{\partial(r^* u_r^*)}{\partial r^*} + \frac{\partial u_z^*}{\partial z^*} = 0, \quad (1)$$

$$-\frac{\partial p^*}{\partial z^*} + \eta\left(\frac{\partial^2 u_z^*}{\partial r^{*2}} + \frac{1}{r^*}\frac{\partial u_z^*}{\partial r^*} + \frac{\partial^2 u_z^*}{\partial z^{*2}}\right) = 0, \quad (2)$$

$$-\frac{\partial p^*}{\partial r^*} + \eta\left(\frac{\partial^2 u_r^*}{\partial r^{*2}} + \frac{1}{r^*}\frac{\partial u_r^*}{\partial r^*} + \frac{\partial^2 u_r^*}{\partial z^{*2}} - \frac{u_r^*}{r^{*2}}\right) = 0, \quad (3)$$

$$u_r^*(r^* = R) = u_z^*(r^* = R) = 0, \quad (4)$$

$$\left(1 - \left(\frac{\partial h^*}{\partial z^*}\right)^2\right)\left(\frac{\partial u_z^*}{\partial r^*} + \frac{\partial u_r^*}{\partial z^*}\right) + 2\frac{\partial h^*}{\partial z^*}\left(\frac{\partial u_r^*}{\partial r^*} - \frac{\partial u_z^*}{\partial z^*}\right)\Bigg|_{r^*=R+h^*} = 0, \quad (5)$$

$$-p^* + \frac{2\eta}{1+\left(\frac{\partial h^*}{\partial z^*}\right)^2}\left(\left(\frac{\partial h^*}{\partial z^*}\right)^2\frac{\partial u_z^*}{\partial z^*} - \frac{\partial h^*}{\partial z^*}\left(\frac{\partial u_z^*}{\partial r^*} + \frac{\partial u_r^*}{\partial z^*}\right) + \frac{\partial u_r^*}{\partial r^*}\right)\Bigg|_{r^*=R+h^*} = -\gamma\mathcal{C}^*, \quad (6)$$

where the asterisks $*$ indicate dimensional variables, and the total curvature of the free surface reads

$$\mathcal{C}^* = \frac{1}{(R+h^*)\sqrt{1+\left(\frac{\partial h^*}{\partial z^*}\right)^2}} - \frac{\frac{\partial^2 h^*}{\partial z^{*2}}}{\left(1+\left(\frac{\partial h^*}{\partial z^*}\right)^2\right)^{3/2}}. \quad (7)$$

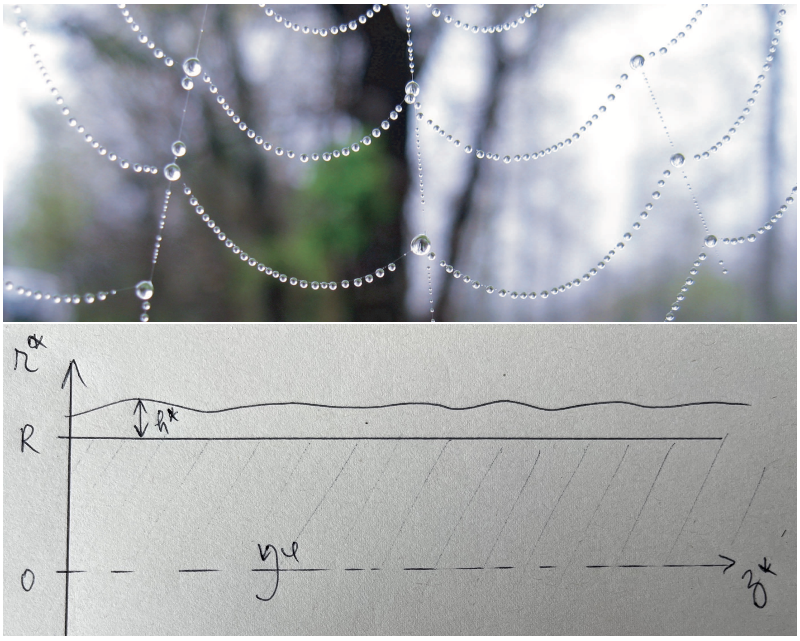

Figure 1: (top panel) Spider web with dew pearls (reprinted from Duprat [2009]). (bottom panel) Cartoon of the problem at hand.

2 Thin-film (lubrication) equation

We start by non-dimensionalising the system of equations,

$$
\begin{aligned}
z^* = Rz\,, \quad r^* = Rr\,, \quad \mathrm{d}r^* = H\mathrm{d}r, \\
u_z^* = Uu_z\,, \quad u_r^* = U_r u_r\,, \quad p^* = Pp\,, \quad h^* = Hh.
\end{aligned} \tag{8}
$$

To make use of the separation of scales between the film thickness and the cylinder's radius, we introduce two length scales, R and $H = \delta R$, with $\delta \ll 1$, and two velocity scales, U and U_r. It is crucial to notice that while r^* scales as R, the variations along r^* happen on an interval of the scale of H.

Notice that the axial dimension is also rescaled with R. This is because of the anticipation that the characteristic length of the Raleigh–Plateau instability is set by the liquid column's radius.

We introduce the rescaled variables in the full governing equations. The continuity equation (1) becomes

$$
\frac{U_r}{H}\frac{1}{r}\frac{\partial (r u_r)}{\partial r} + \frac{U}{R}\frac{\partial u_z}{\partial z} = 0 \Rightarrow U_r = \frac{HU}{R} = \delta U. \tag{9}
$$

The axial momentum equation (2) becomes

$$
-\frac{P}{R}\frac{\partial p}{\partial z} + \frac{\eta U}{H^2}\left(\frac{\partial^2 u_z}{\partial r^2} + \cancel{\delta\frac{1}{r}\frac{\partial u_z}{\partial r}} + \cancel{\delta^2\frac{\partial^2 u_z}{\partial z^2}}\right) = 0 \Rightarrow U = \frac{\delta P H}{\eta}. \tag{10}
$$

The radial momentum equation (3) becomes

$$
-\frac{P}{H}\frac{\partial p}{\partial r} + \cancel{\delta^2\frac{P}{H}\left(\frac{\partial^2 u_r}{\partial r^2} + \delta\frac{1}{r}\frac{\partial u_r}{\partial r} + \delta^2\frac{\partial^2 u_r}{\partial z^2} - \delta^2\frac{u_r}{r^2}\right)} = 0 \Rightarrow \frac{\partial p}{\partial r} = 0. \tag{11}
$$

The pressure is constant throughout the liquid layer and set by the curvature-induced Laplace pressure from the normal dynamic boundary condition (6) becomes

$$-Pp + \cancel{\delta^2 P(\cdots)} = -\frac{\gamma}{R}\left(\frac{1}{(1+\delta h)\sqrt{1+\delta^2\left(\frac{\partial h}{\partial z}\right)^2}} - \frac{\delta\frac{\partial^2 h}{\partial z^2}}{\left(1+\delta^2\left(\frac{\partial h}{\partial z}\right)^2\right)^{3/2}}\right), \tag{12}$$

which by expanding in δ (remember that $(1+\epsilon)^\alpha = 1 + \alpha\epsilon + \cdots$)

$$-Pp = -\frac{\delta\gamma}{R}\left(\delta^{-1} - h - \frac{\partial^2 h}{\partial z^2}\right) \Rightarrow P = \frac{\delta\gamma}{R}, \tag{13}$$

where we have chosen to rescale the pressure with the non-constant term of interest.

Now, going back to equation (10) and knowing that the pressure is a constant of r, we are able to integrate the axial velocity's wall-normal profile $u_z(r)$,

$$\frac{\partial^2 u_z}{\partial r^2} = \frac{\partial p}{\partial z} \Rightarrow u_z = \frac{\partial p}{\partial z}\frac{r^2}{2} + ar + b, \tag{14}$$

which is a parabola. The constants of integration are set by the no-slip (4) and free-shear (5) boundary conditions. The latter reads

$$\frac{U}{H}\left(1 - \cancel{\delta^2\left(\frac{\partial h}{\partial z}\right)^2}\right)\left(\frac{\partial u_z}{\partial r} + \cancel{\delta^2\frac{\partial u_r}{\partial z}}\right) + \cancel{\frac{2\delta^2 U}{H}\frac{\partial h}{\partial z}\left(\frac{\partial u_r}{\partial r} - \frac{\partial u_z}{\partial z}\right)}\Bigg|_{r=1+h} = 0, \tag{15}$$

thus,

$$u_z = \frac{1}{2}\frac{\partial p}{\partial z}(r-1)(r-1-2h). \tag{16}$$

We have now solved all fields of the thin-film flow. This allows us to write a unique evolution equation for the film thickness h. We can either integrate u_r from the continuity equation (9) and plug it into the kinematic boundary condition, or equivalently (and more directly) we can perform the following mass balance on an elementary interval $[z, z + \mathrm{d}z]$: the change of volume in a time step Δt must be compensated by the difference between the inflow and outflow,

$$\frac{\partial h}{\partial t}\mathrm{d}z\Delta t = \left(\int_1^{1+h(z)} u_z(z)\mathrm{d}r - \int_1^{1+h(z+\mathrm{d}z)} u_z(z+\mathrm{d}z)\mathrm{d}r\right)\Delta t = -\frac{\partial q}{\partial z}\mathrm{d}z\Delta t, \tag{17}$$

with the flow rate per azimuthal length

$$q = \int_1^{1+h(z)} u_z(z)\mathrm{d}r = -\frac{\partial p}{\partial z}\frac{h^3}{3}. \tag{18}$$

We obtain the, so-called, thin-film or lubrication equation in the absence of external body forces,

$$\frac{\partial h}{\partial t} = -\frac{\partial}{\partial z}\left[\frac{h^3}{3}\left(\frac{\partial h}{\partial z} + \frac{\partial^3 h}{\partial z^3}\right)\right]. \tag{19}$$

It is at this point fully non-linear.

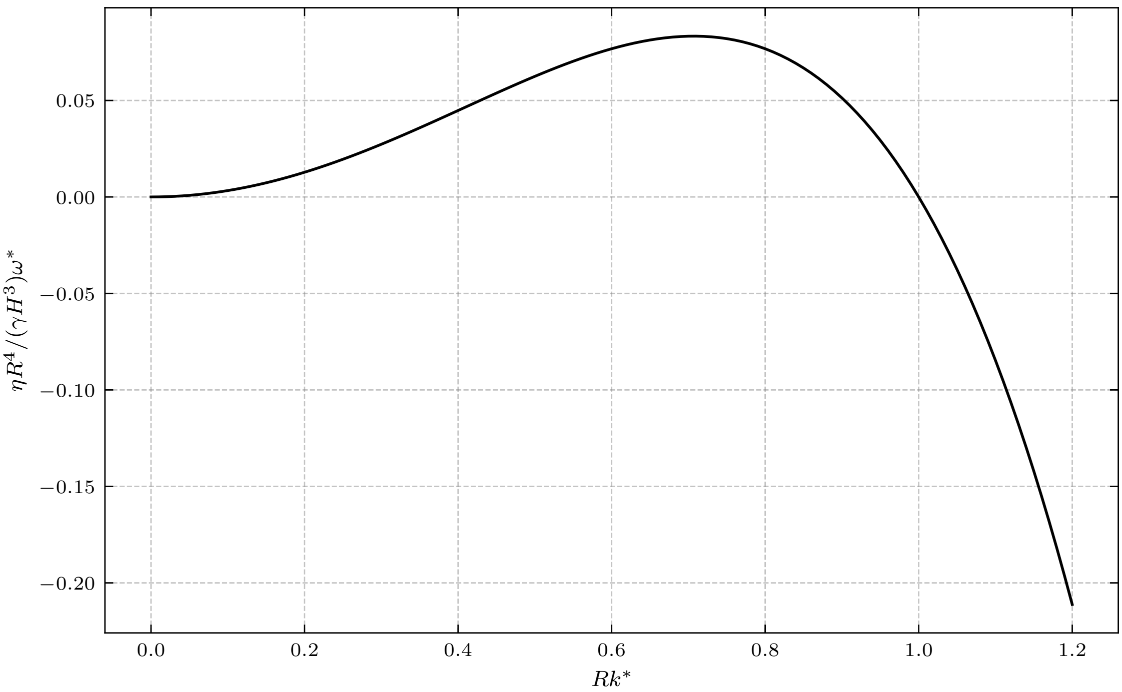

Figure 2: Dispersion relation (22).

3 Linear instability analysis

The basic flow, the linear stability of which we will study, is the trivial one $h_0 = 1$. We add an infinitesimal perturbation, which we decompose in normal modes,

$$h(z,t) = 1 + \epsilon \hat{h}_1 \exp\left[\mathrm{i}(kz - \omega z)\right] + \text{c.c.} \tag{20}$$

and linearise the governing equation (19),

$$-\mathrm{i}\omega\hat{h}_1 = (k^2 - k^4)\hat{h}_1/3 \Rightarrow \omega(k) = \mathrm{i}\frac{k^2(1-k^2)}{3}. \tag{21}$$

We have obtained the dispersion relation, that we can re-dimensionalise; remember that k is the axial wave number, so $k = Rk^*$, and $-\mathrm{i}\omega\hat{h}$ is the velocity of the interface, which is radial and thus, was rescaled with U_r, meaning that $\omega = (H/U_r)\omega^*$. This results in

$$\omega^*(k^*) = \mathrm{i}\frac{\gamma H^3 k^{*2}(1 - R^2 k^{*2})}{3\eta R^2}. \tag{22}$$

It is presented in figure 2. Similarly to the classical Rayleigh–Plateau instability of a free liquid column, wavelengths larger than $2\pi R$ are linearly unstable and the most unstable wave number is $k_{\text{opt}} = 1/\sqrt{2}$, corresponding to $2\pi\sqrt{2}R$.

4 Discussion

This dispersion relation (22) strikingly resembles the one from the previous week of the Rayleigh–Taylor instability of a thin film suspended on the underside of a plate,

$$\omega^*_{\mathrm{RT}}(k^*) = \mathrm{i}\frac{\rho g H^3 k^{*2}(1 - l_c^2 k^{*2})}{3\eta}, \tag{23}$$

with the capillary length $l_c = \sqrt{\gamma/\rho g}$.

It is true that for very thin layers, the bottom of the horizontal cylinder which they coat, looks like a horizontal plate; the effects of the curvature are negligible. One can reasonably ask when the Rayleigh–Plateau mechanism of this exercise will be dominant, and when the Rayleigh–Taylor one. (Remember that here we have completely neglected gravity, while in the previous week's exercise, we hate not included any possible curvature effects.) The two result in distinct lenght scales.

The ratio of the highest growth rates of the two dispersion relations above is a classical dimensionless group in interfacial fluid mechanics, named the Bond number,

$$\mathrm{Bo} = \frac{R^2}{l_c^2} = \frac{\rho g R^2}{\gamma}, \tag{24}$$

and it will set the transition between the two instability mechanisms: when Bo is very small, fibre's radius much inferiour to the capillary length (think spider webs), gravity has almost no effect, the thin film destabilises into pearls, centred on the fibre; contrarily, when Bo is very large, $R \gg l_c$ (think blue beams below lecture hall's ceiling), surface tension is very weak and drops remain suspended on or drip from the underside of the cylinder.

References

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