

Most flows are unstable...

1. Intro-definitions
2. Rayleigh-Taylor
3. Rayleigh Plateau (destabilization through surface tension)
4. Rayleigh-Benard (convection)
5. Benard-Marangoni
6. Taylor Couette-Centrifugal instability
7. Kelvin-Helmholtz
8. Inflection point theorem Rayleigh
9. Orr sommerfeld, transient growth
10. Spatial growth
11. Spatio-temporal growth

Which of these waves are unstable?

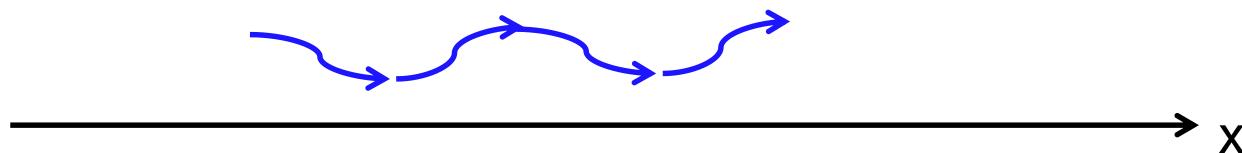
$\text{Im}(k) < 0$?

$\text{Im}(k) > 0$?

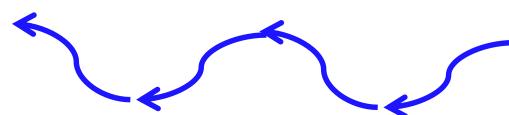
Recall : $\exp(i(kx - \omega t))$

The stability of a spatial wave can be only determined if one knows in which direction it propagates!

k^+ waves propagate towards positive x



k^- waves propagate towards negative x

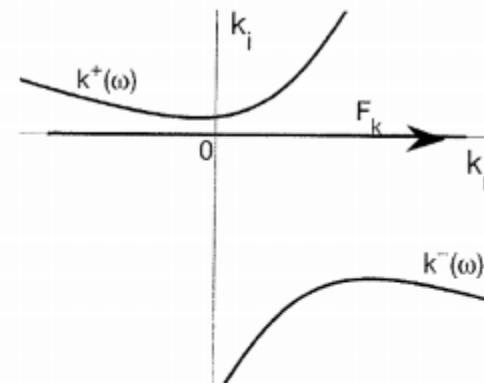
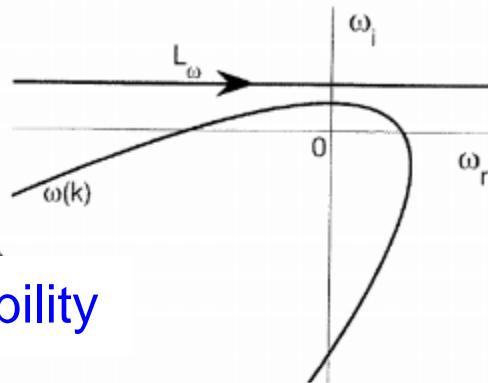


However, determining this direction of propagation is particularly difficult, except in the case of a temporally stable flow.

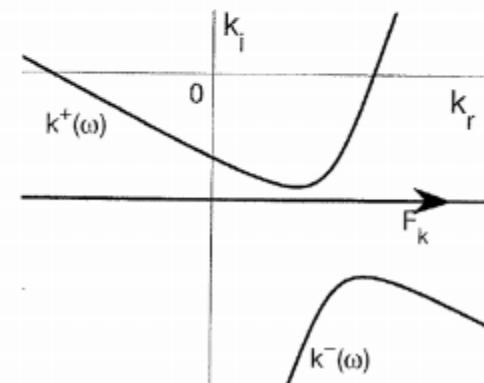
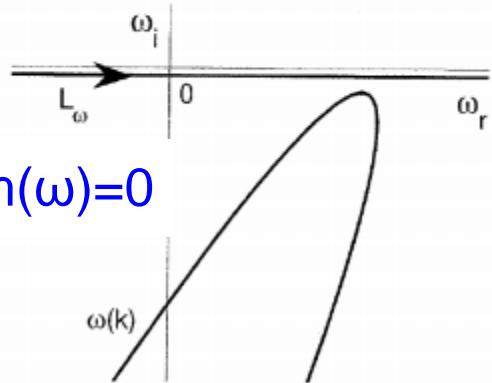
The addition of a positive imaginary offset to the frequency makes the temporal problem stable!

This separates the spatial waves into $k+$ and $k-$ waves.

offset spatial stability



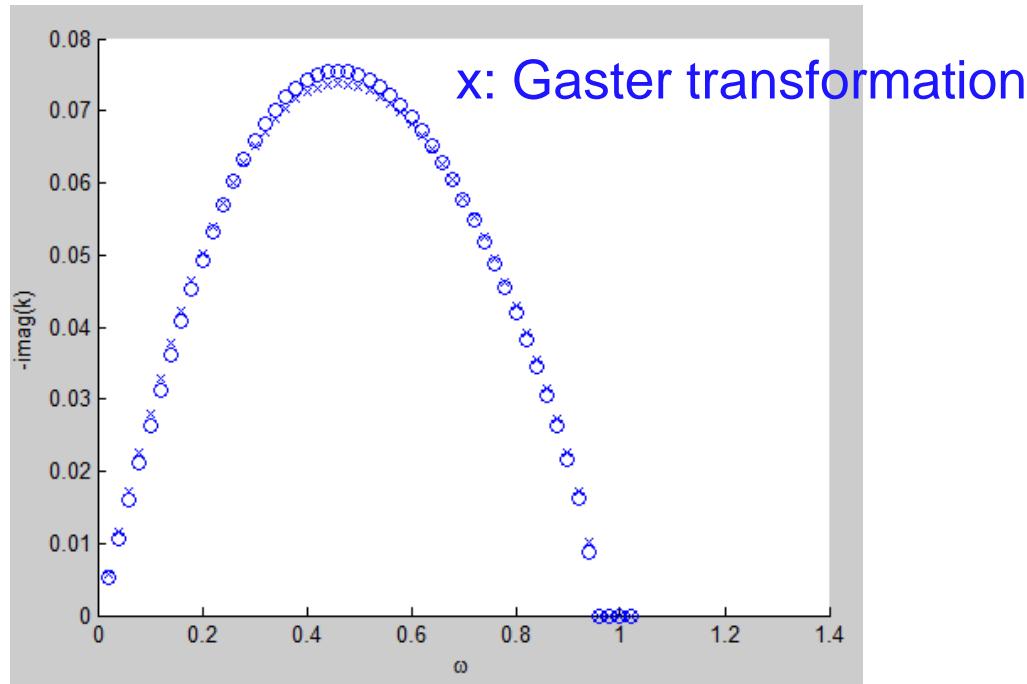
spatial stability: $\text{Im}(\omega)=0$



The branches are then followed by continuity

Validity of Gaster transformation?

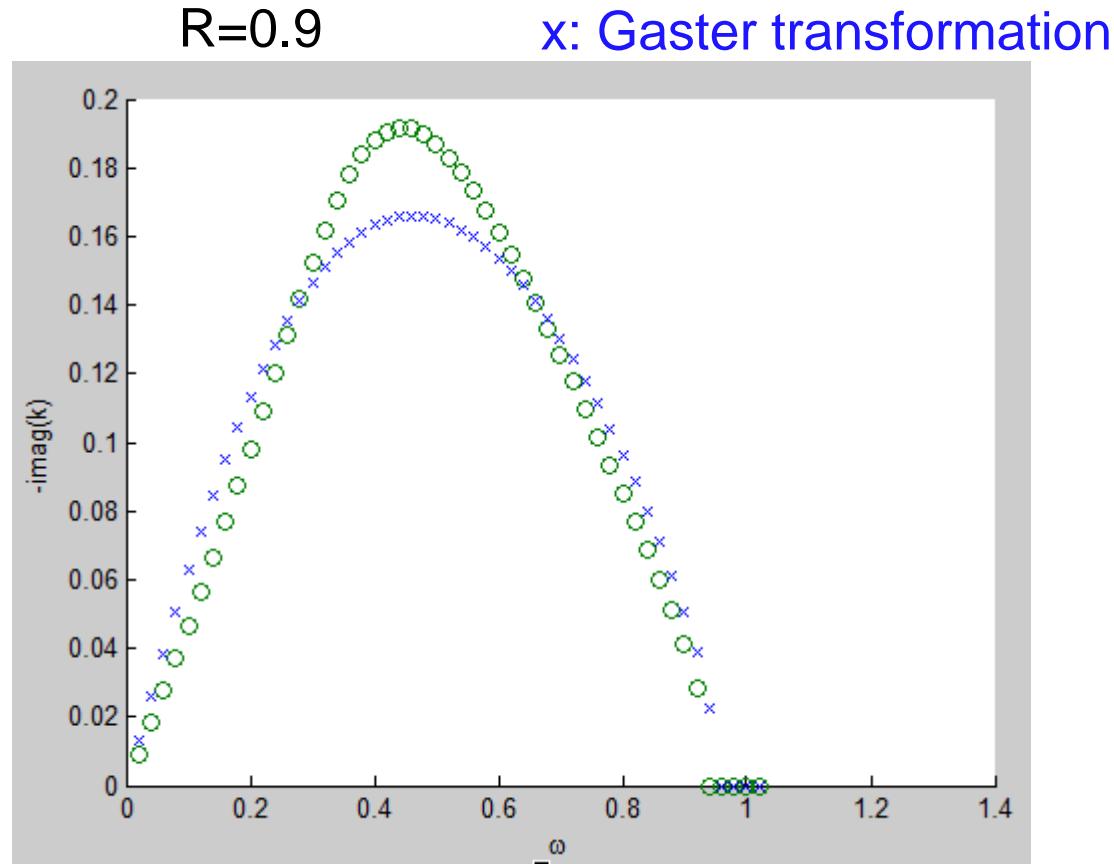
$R=0.4$



$R \ll 1$

$$-k_i(\omega, R) \sim R \omega_{1,i}(\omega)$$

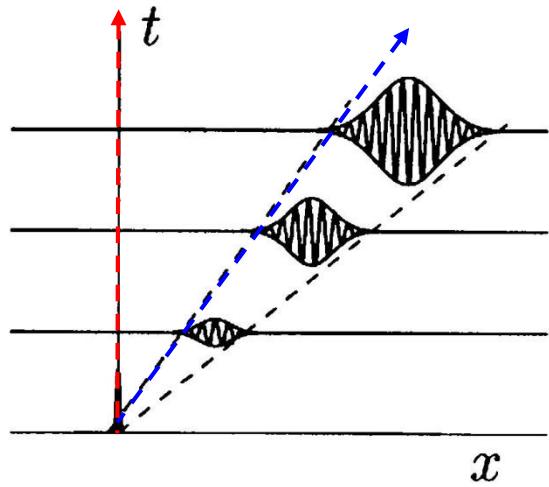
Validity of Gaster transformation?



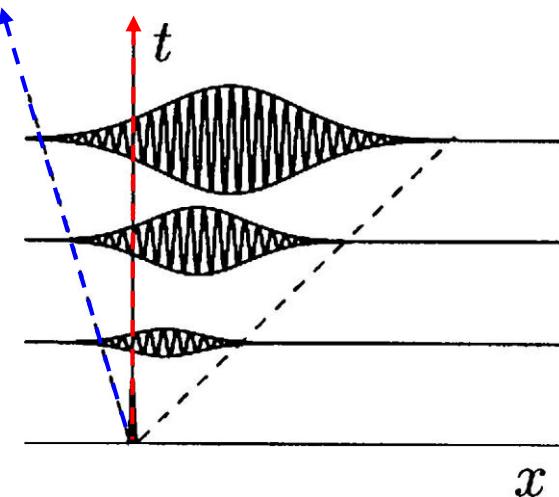
$$-k_i(\omega, R) \sim R \omega_{1,i}(\omega)$$

Spatio-temporal instability theory

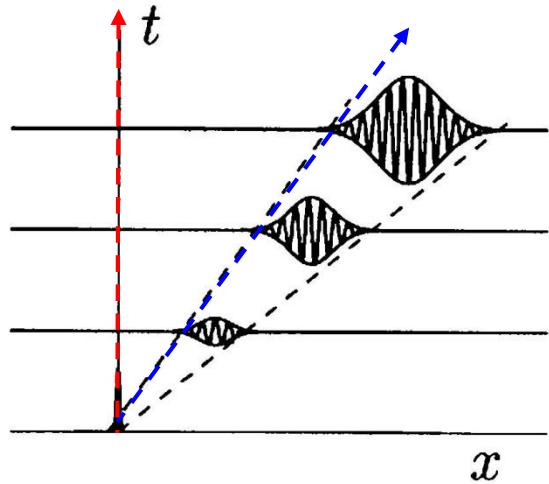
👉 Convective instability



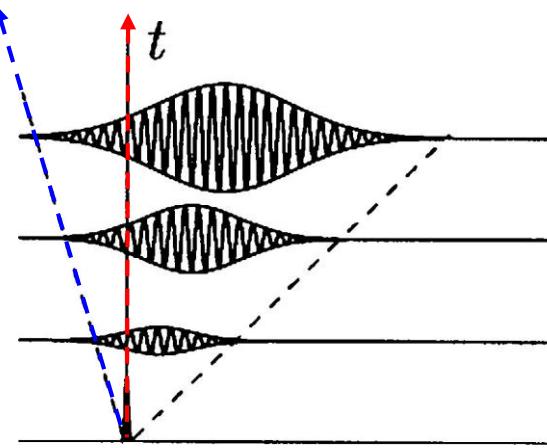
👉 Absolute instability



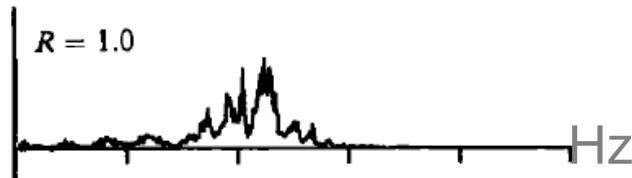
👉 Convective instability



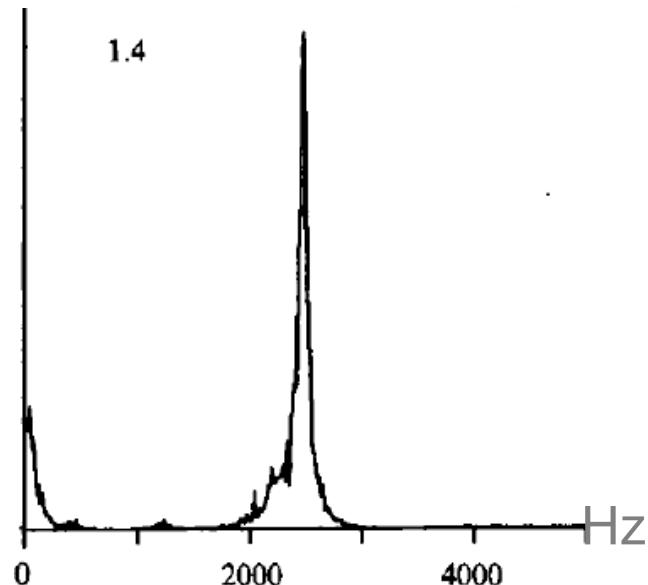
👉 Absolute instability



Spectrum



👉 **amplifier**

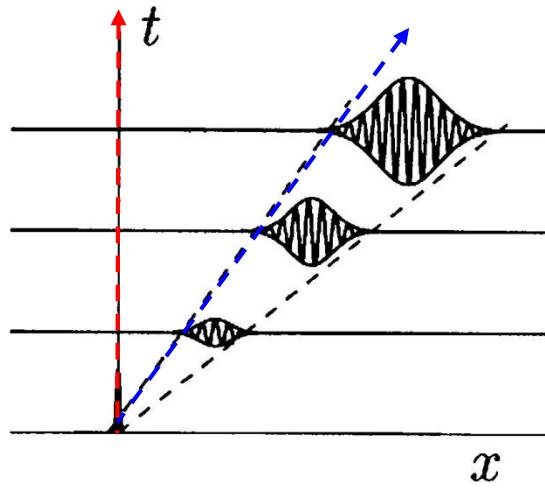


👉 **oscillator**

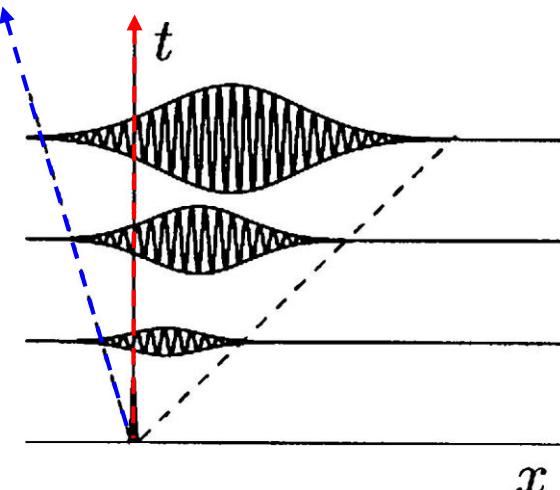
Mixing layer experiments by Strikowsky and Niccum (1991)

Spatio-temporal instability theory

👉 Convective instability



👉 Absolute instability



We need to generalize the concept of group velocity since ω (and why not k) is complex

For neutral waves, the group velocity is $d\omega/dk$

Here this quantity is the derivative of a complex function with respect to a complex variable. Cauchy-Rieman conditions apply.

Spatio-temporal spectral analysis

Inverse Fourier Transform

$$u(x,t) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \hat{u}(k, \omega) e^{i(kx - \omega t)} dk d\omega$$

$$\hat{u}(k, \omega) = (2\pi)^2 \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} u(x, t) e^{-i(kx - \omega t)} dx dt$$

Direct Fourier Transform

Spatio-temporal spectral analysis

Inverse Fourier Transform

$$u(x,t) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \hat{u}(k, \omega) e^{i(kx - \omega t)} dk d\omega$$

Use dispersion relation $\omega(k)$!

Spectral analysis

Fourier transform:

$$u(x, t) = \frac{1}{2} \int_0^{+\infty} \hat{u}(k) e^{i(kx - \omega(k)t)} dk + c.c.$$

Carrier/enveloppe

Carrier/enveloppe :

$$u(x, t) = \frac{1}{2} A(x, t) e^{i(k_0 x - \omega_0 t)} + c.c.$$

Spectral analysis

Fourier transform:

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Carrier/enveloppe :

$$u(x, t) = \frac{1}{2} A(x, t) e^{i(k_0 x - \omega_0 t)} + c.c.$$

Enveloppe :

$$A(x, t) = \int_0^{\infty} \hat{u}(k) e^{i(k - k_0)x - i(\omega - \omega_0)t} dk.$$

Spectral analysis at time=0

Fourier transform:

$$u(x, 0) = \frac{1}{2} \int_0^{+\infty} \hat{u}(k) e^{i(kx)} dk + \text{c.c.}$$

$\hat{u}(k)$ is given by Fourier transform at time $t=0$

Enveloppe :

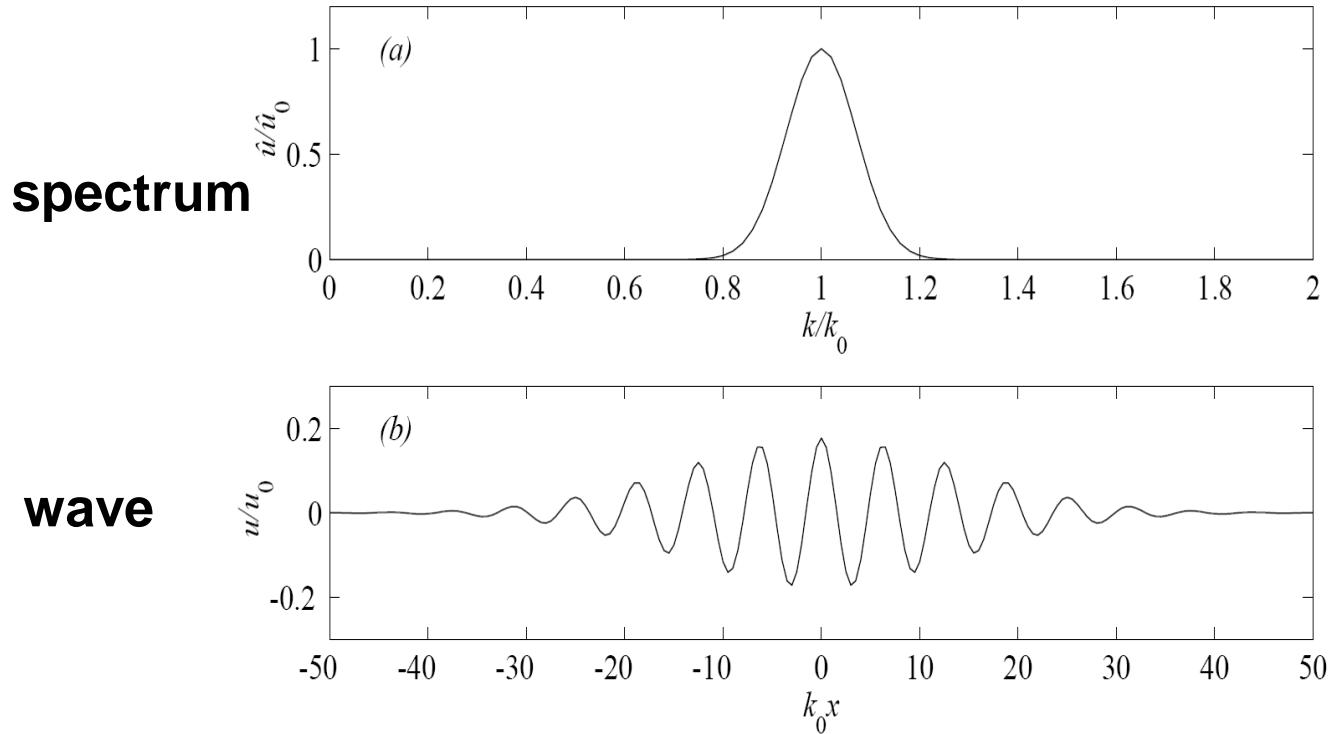
$$A(x, 0) = \int_0^{\infty} \hat{u}(k) e^{i(k-k_0)x} dk + \text{c.c.}$$

Spectral analysis

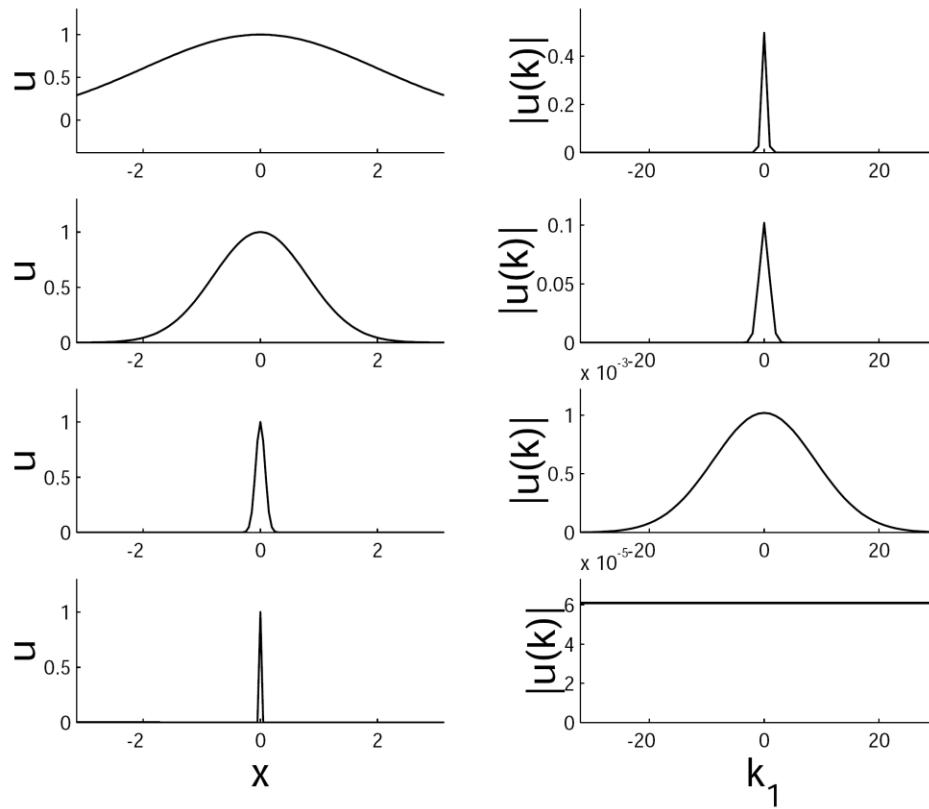
Gaussian spectrum: $\hat{u}(k) = u_0 e^{-\sigma^2(k-k_0)^2}$

Initial enveloppe : $A(x, 0) = \frac{u_0 \sqrt{\pi}}{2\sigma} e^{-\frac{x^2}{4\sigma^2}}$

Gaussian spectrum



Gaussian wavepackets



$u_0(x)$	$\hat{u}_0(k_1)$
$\exp\left(-\frac{x^2}{2\sigma^2}\right)$	$\frac{\sigma}{\sqrt{2\pi}} \exp\left(-\frac{\sigma^2 k^2}{2}\right)$
$\delta(x)$	$\frac{1}{2\pi}$

Spectral analysis

Initial enveloppe :
$$A(x, 0) = \frac{u_0 \sqrt{\pi}}{2\sigma} e^{-\frac{x^2}{4\sigma^2}}$$

Gaussian spectrum:
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Spectral analysis

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Evolution of enveloppe :
$$A(x, t) = \int_0^\infty \hat{u}(k) e^{i(k-k_0)x - i(\omega - \omega_0)t} dk.$$

Spectral analysis

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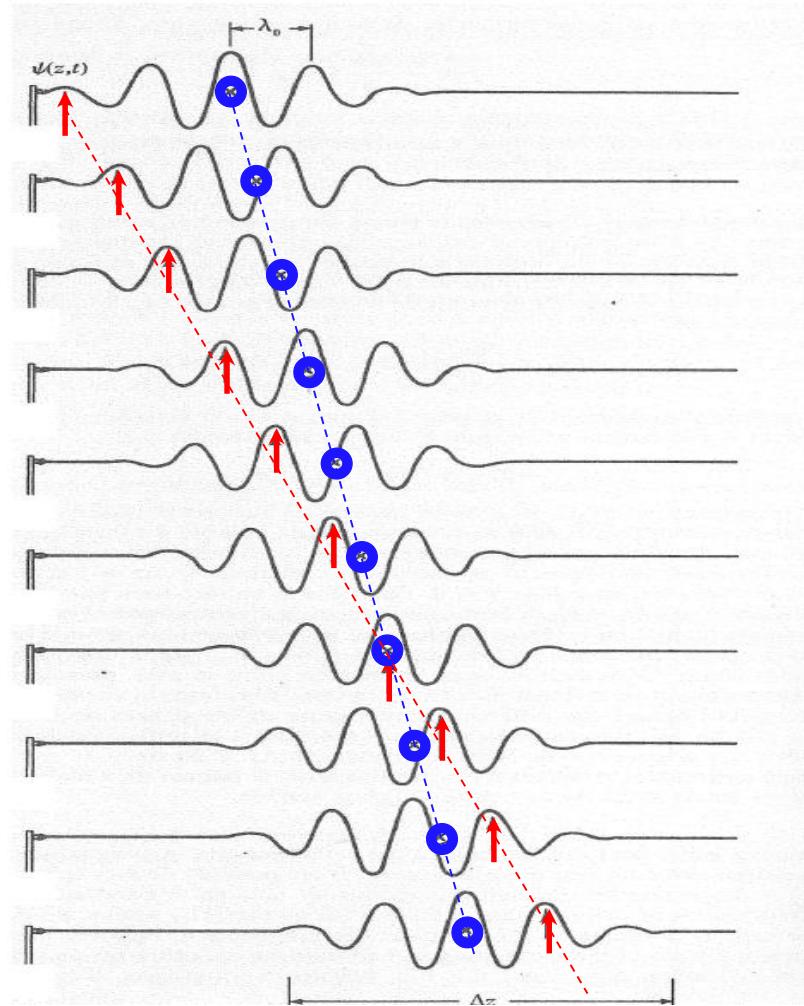
Definition group velocity $\omega - \omega_0 = c_g(k - k_0), \quad c_g = \frac{\partial \omega}{\partial k}(k_0)$

Spectral analysis

Definition of group velocity $\omega - \omega_0 = c_g(k - k_0)$, $c_g = \frac{\partial \omega}{\partial k}(k_0)$

$$A(x, t) = \frac{u_0 \sqrt{\pi}}{2\sigma} e^{-\frac{(x - c_g t)^2}{4\sigma^2}}$$

Group velocity



Wavepacket

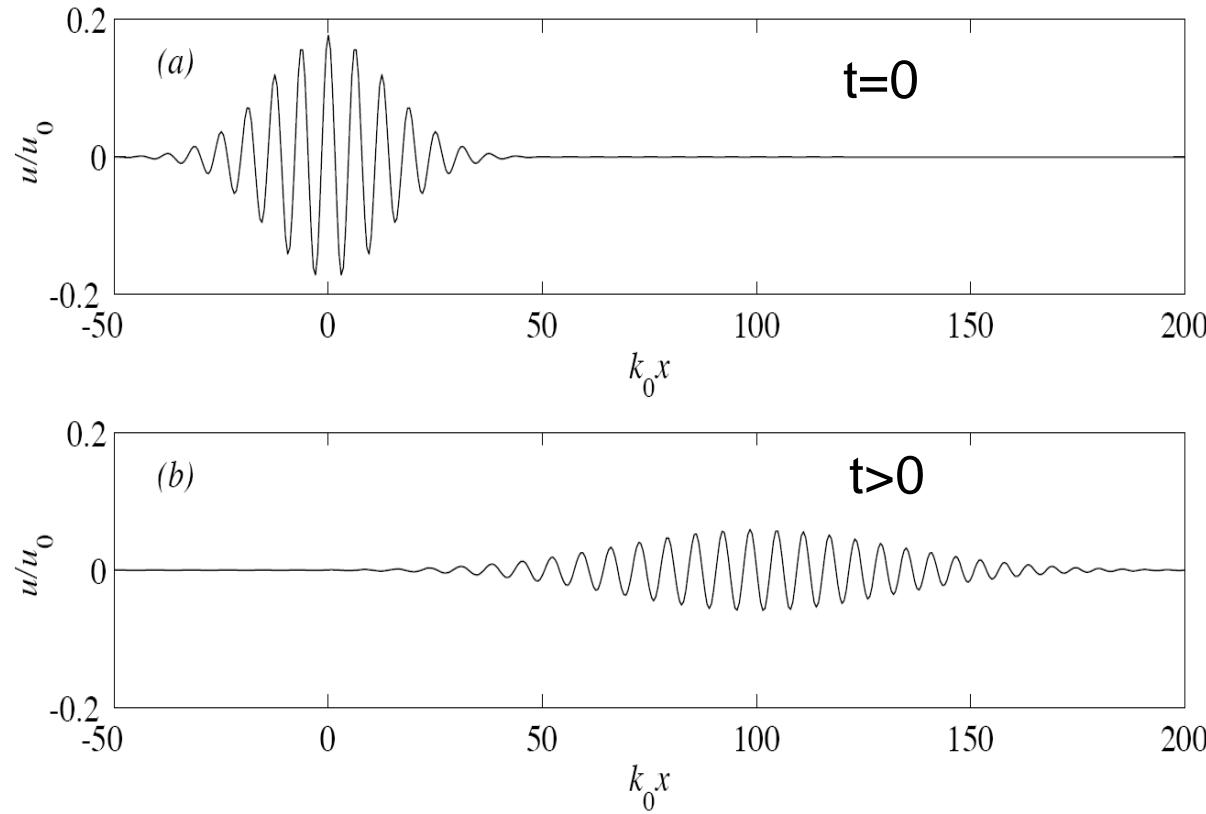
Spectral analysis

Higher order
development

$$\omega - \omega_0 = c_g(k - k_0) + \frac{\omega_0''}{2}(k - k_0)^2$$
$$c_g = \frac{\partial \omega}{\partial k}(k_0), \quad \omega_0'' = \frac{\partial^2 \omega}{\partial k^2}(k_0)$$

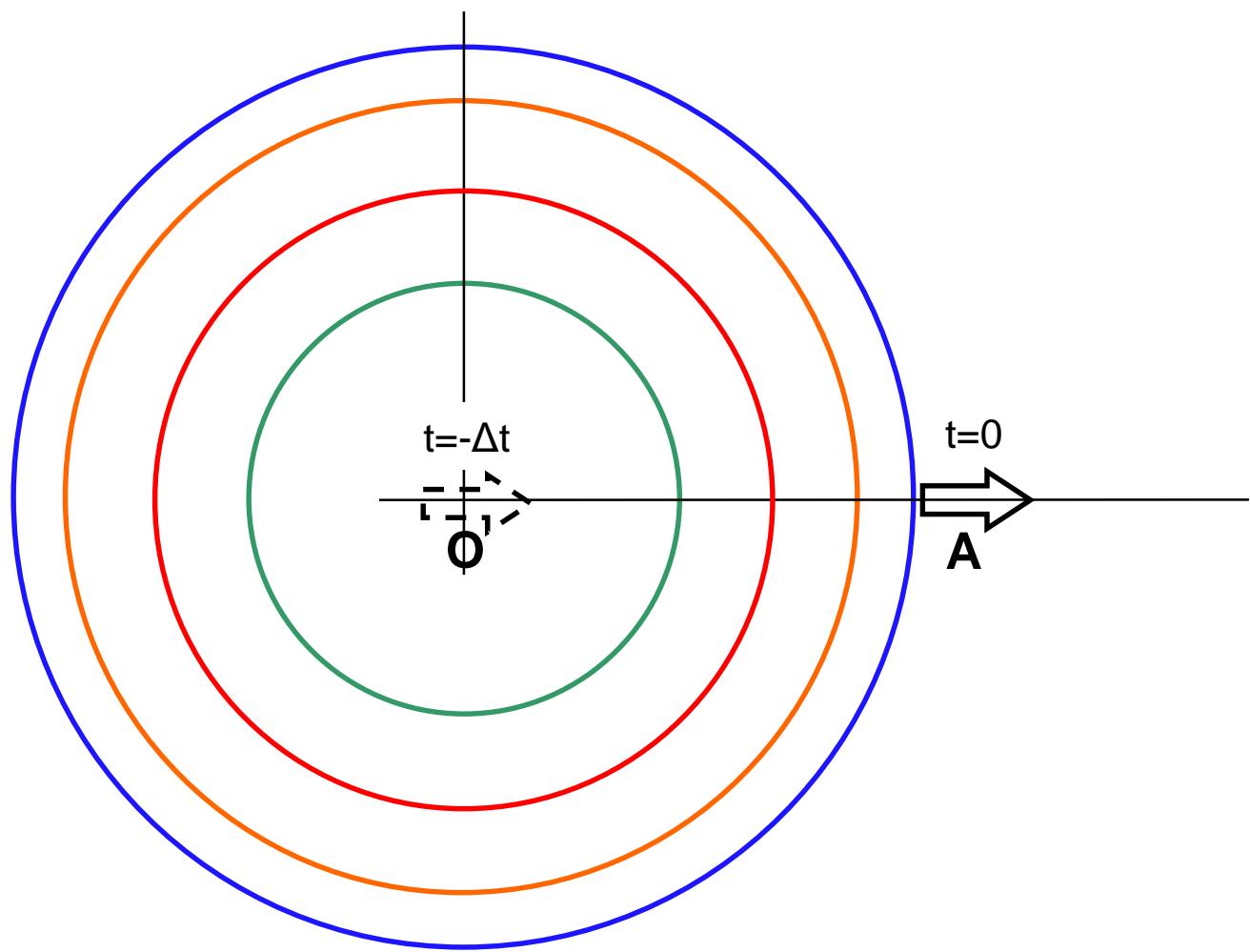
$$A(x, t) = \frac{u_0}{2} \sqrt{\frac{\pi}{\sigma^2 + \frac{1}{2}i\omega_0''t}} \exp\left(-\frac{(x - c_g t)^2}{4(\sigma^2 + \frac{1}{2}i\omega_0''t)}\right)$$

Wave packet dispersion

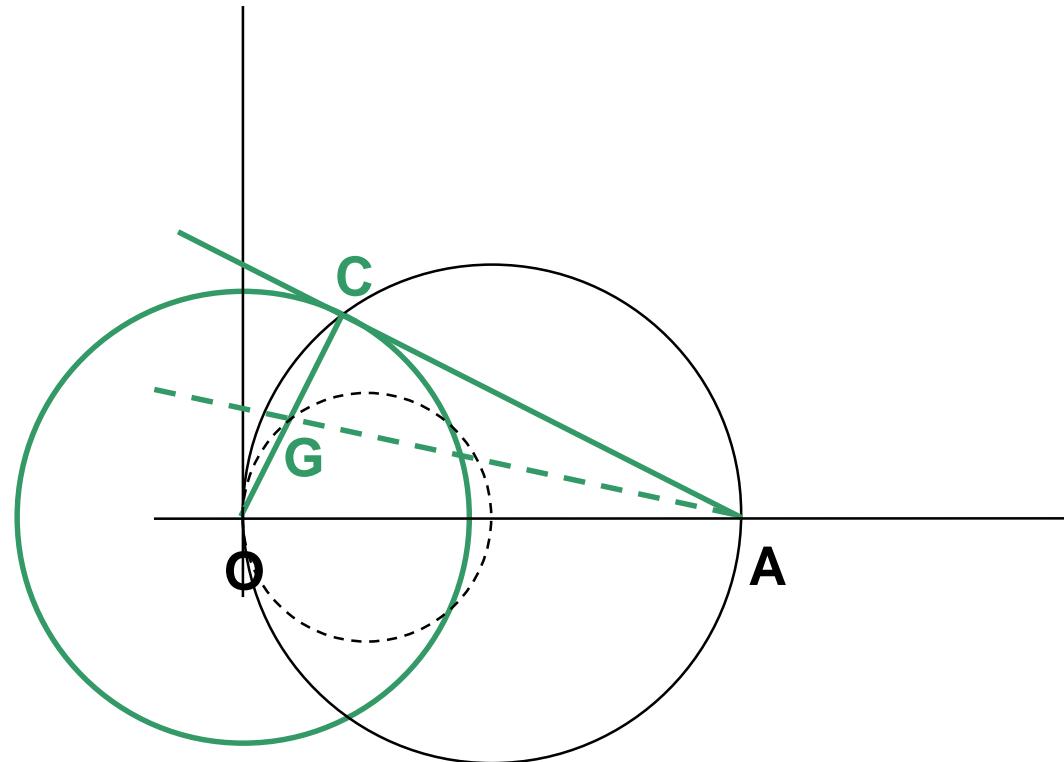


Onde correspondant à l'enveloppe pour $\sigma^{-1}k_0 = 0,1$ et $\omega_0'' = 4c_g/k_0$: (a), instant initial $t = 0$; (b), $c_g t = 100/k_0$.

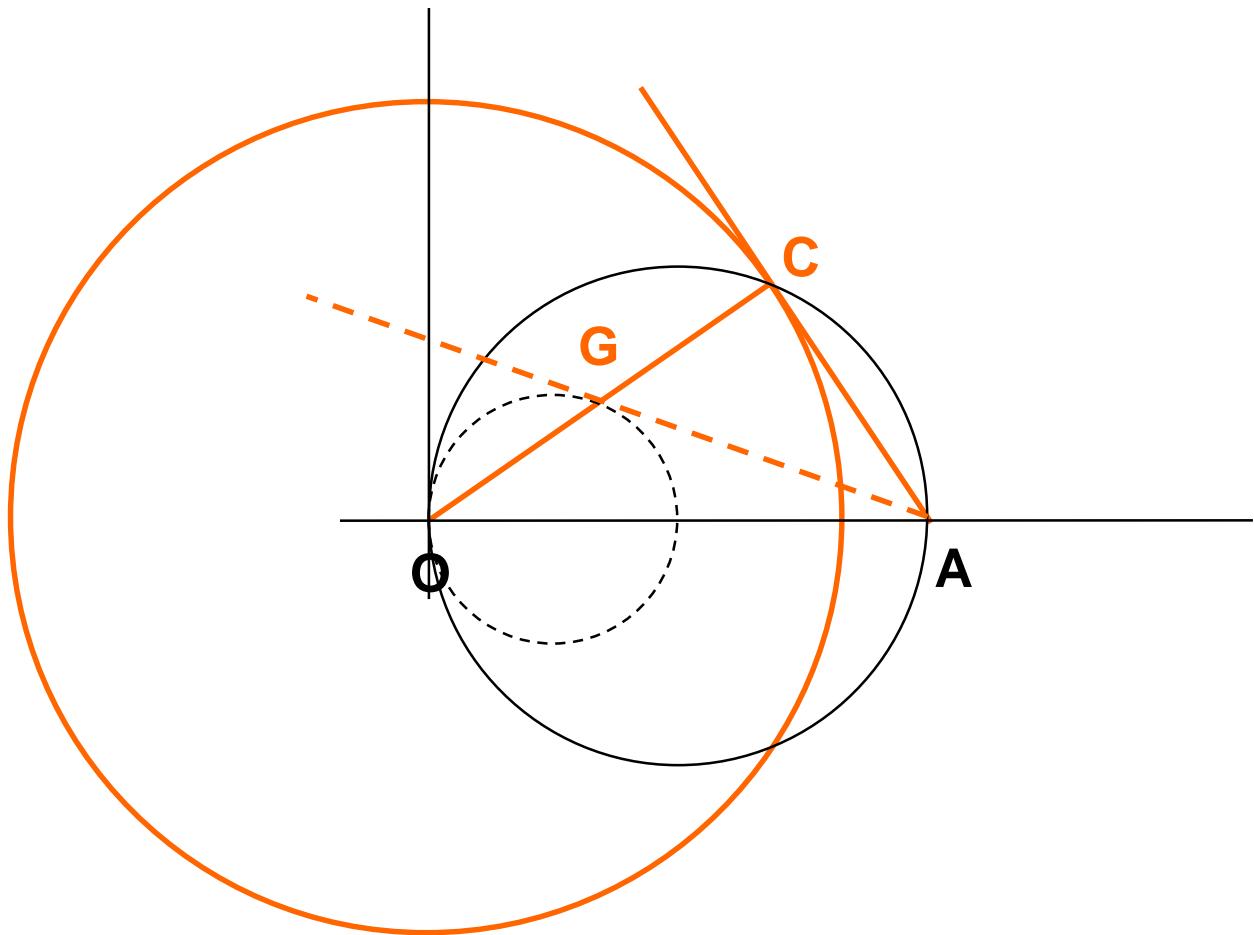
Kelvin's wake



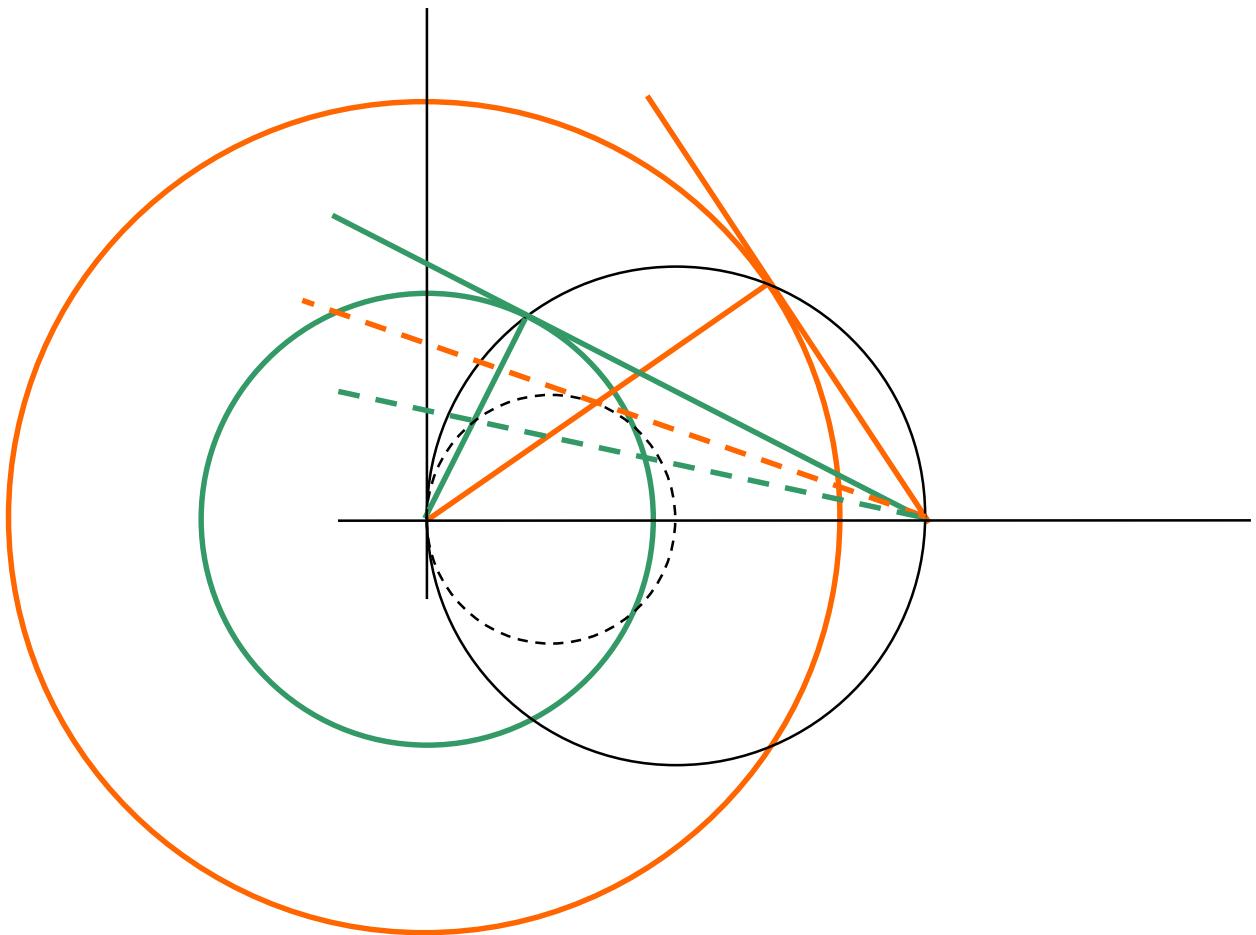
Gravity waves created by a ship



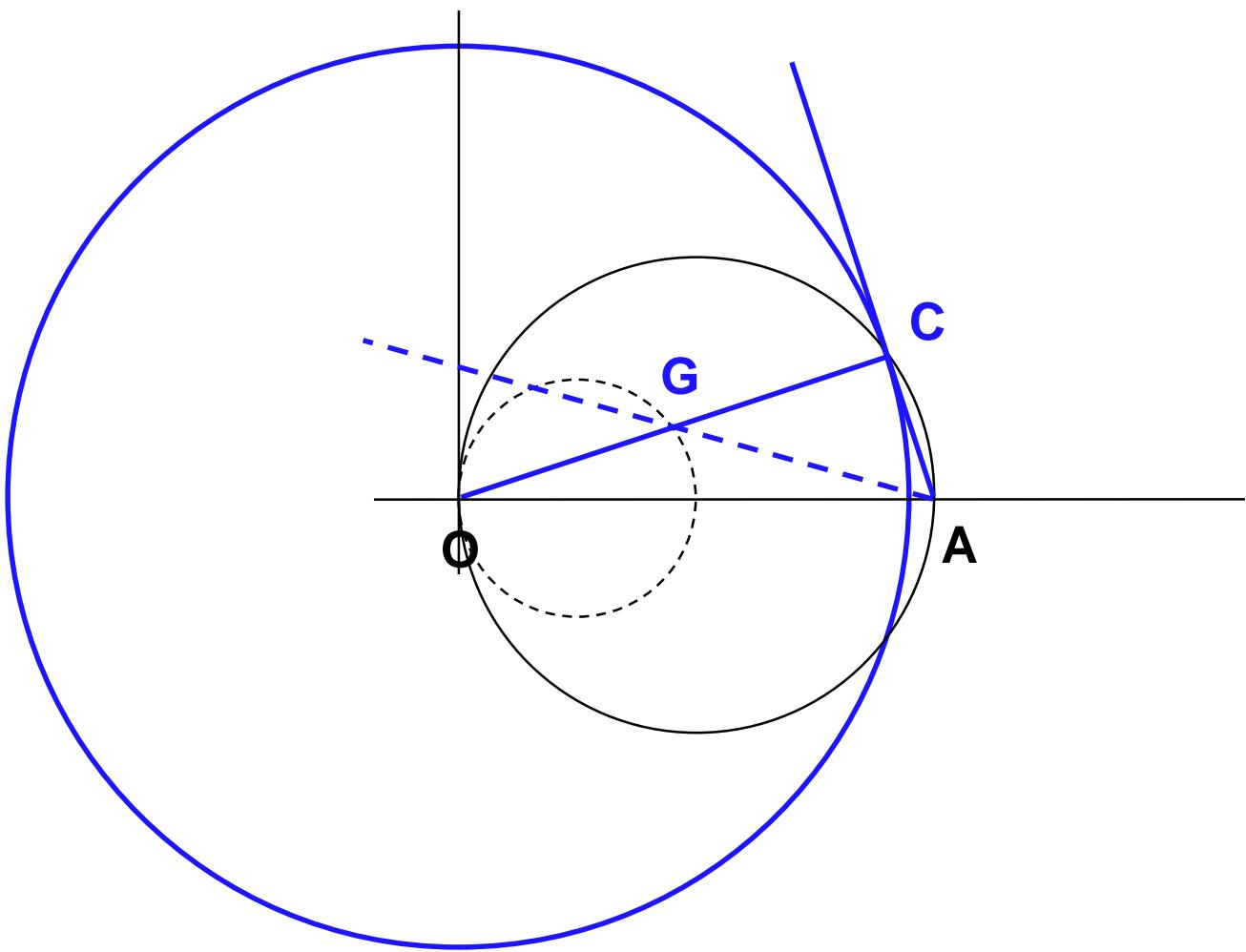
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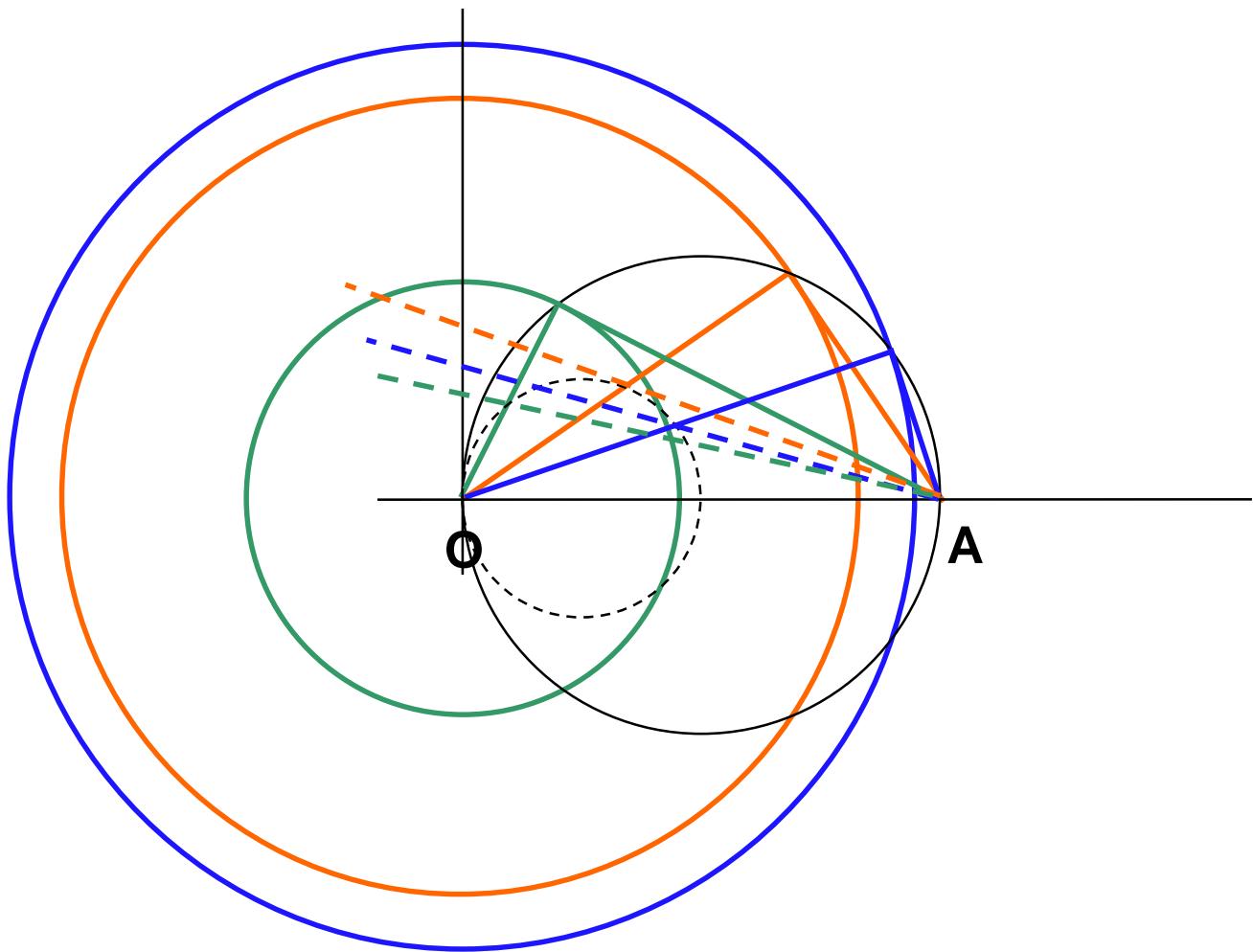
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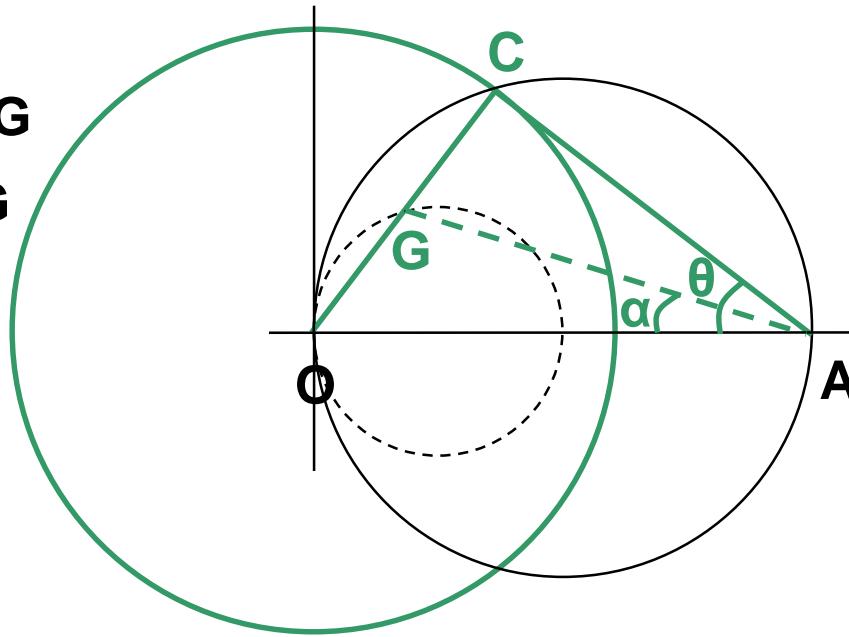


Gravity waves created by a ship



Gravity waves created by a ship

$$\begin{aligned}\sin(\alpha)/OG &= \cos(\theta)/AG \\ \sin(\theta-\alpha)AG &= GC = OG\end{aligned}$$

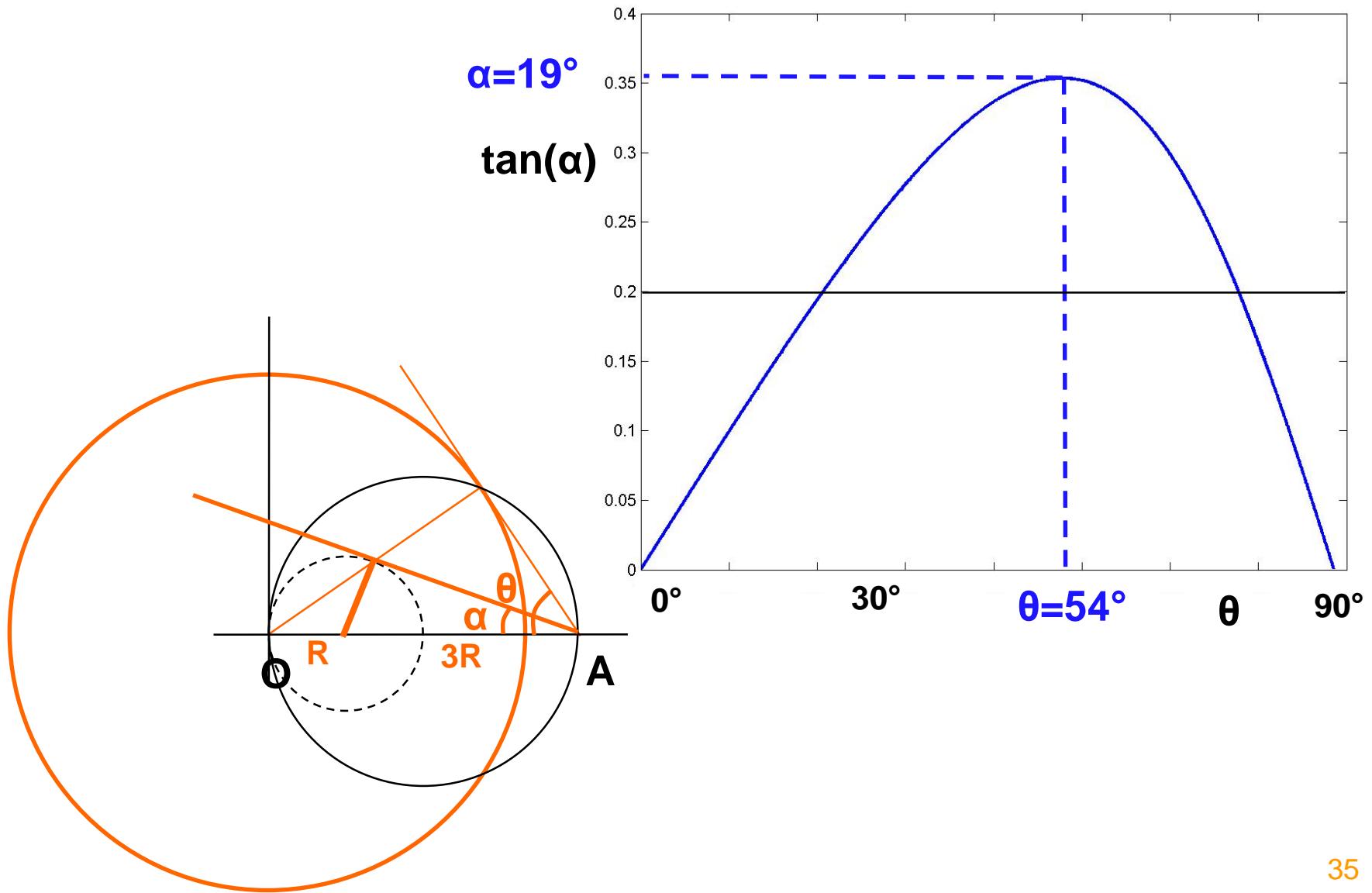


$$\Rightarrow \sin(\alpha) = \cos(\theta) \sin(\theta - \alpha)$$

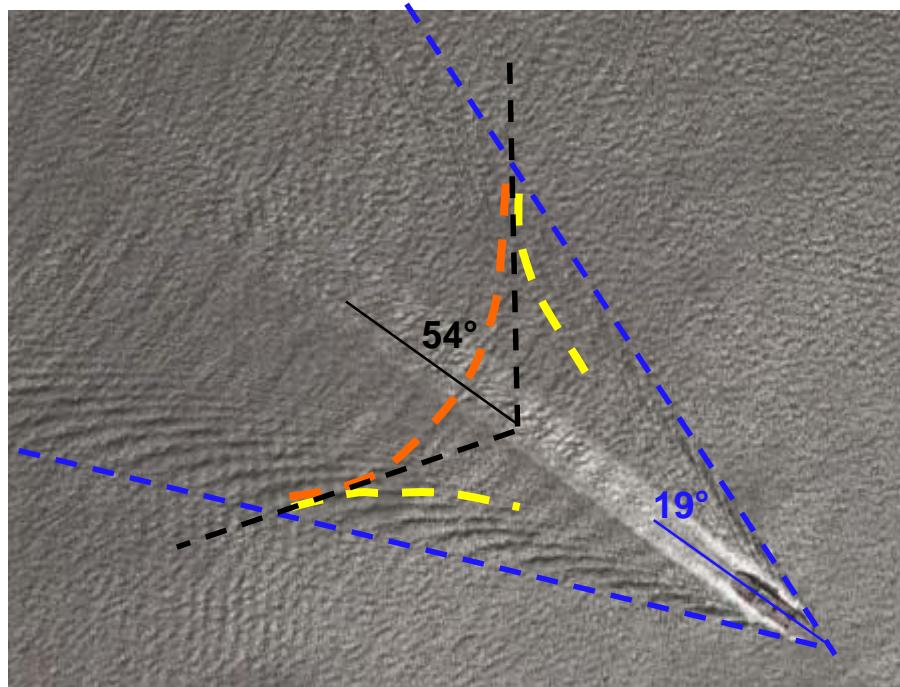
$$\Rightarrow \sin(\alpha) = \cos(\theta) (\sin(\theta) \cos(\alpha) + \cos(\theta) \sin(\alpha))$$

$$\Rightarrow \tan(\alpha) = \cos(\theta) \sin(\theta) / (1 + \cos^2(\theta))$$

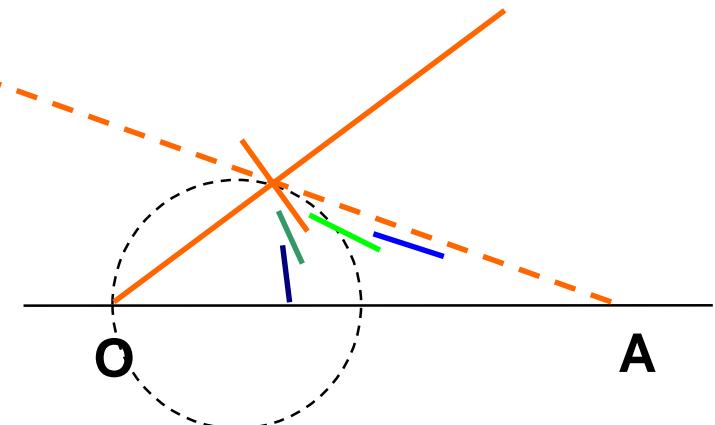
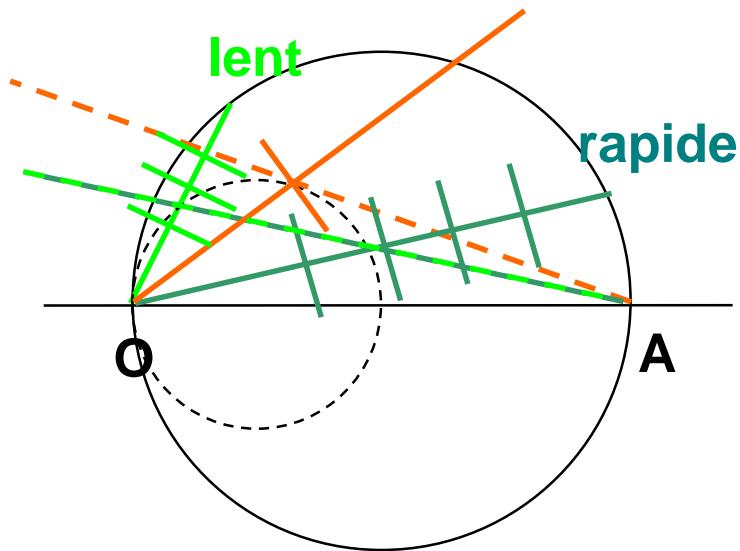
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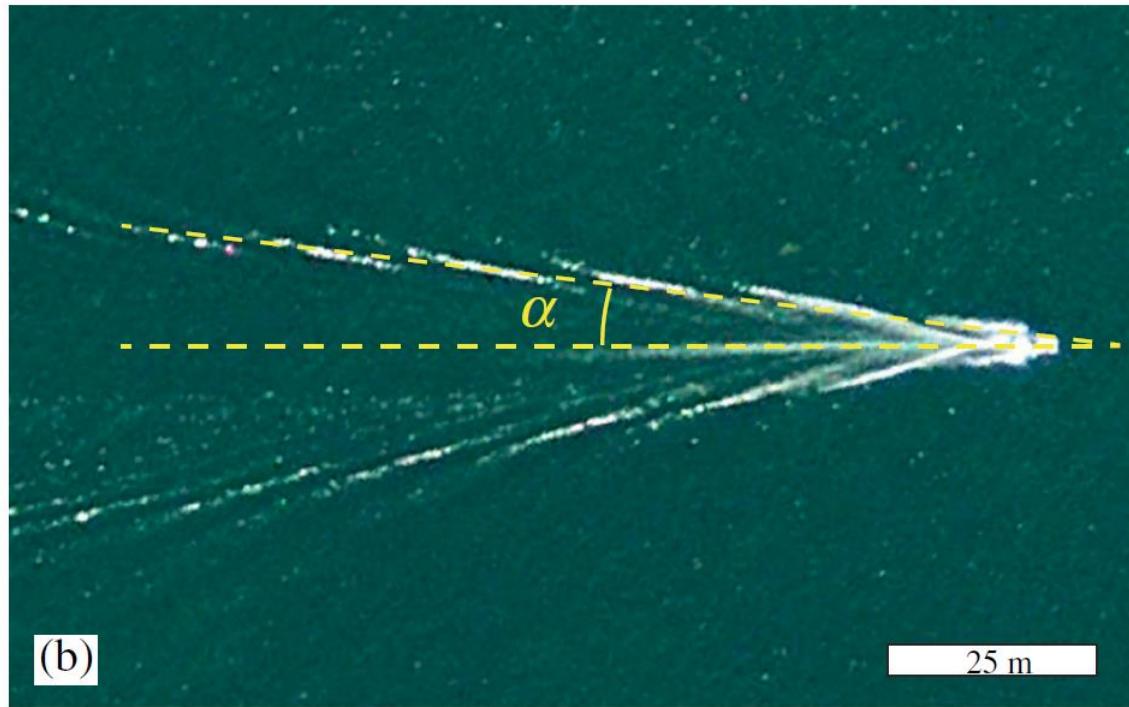
Gravity waves created by a ship



Gravity waves created by a ship

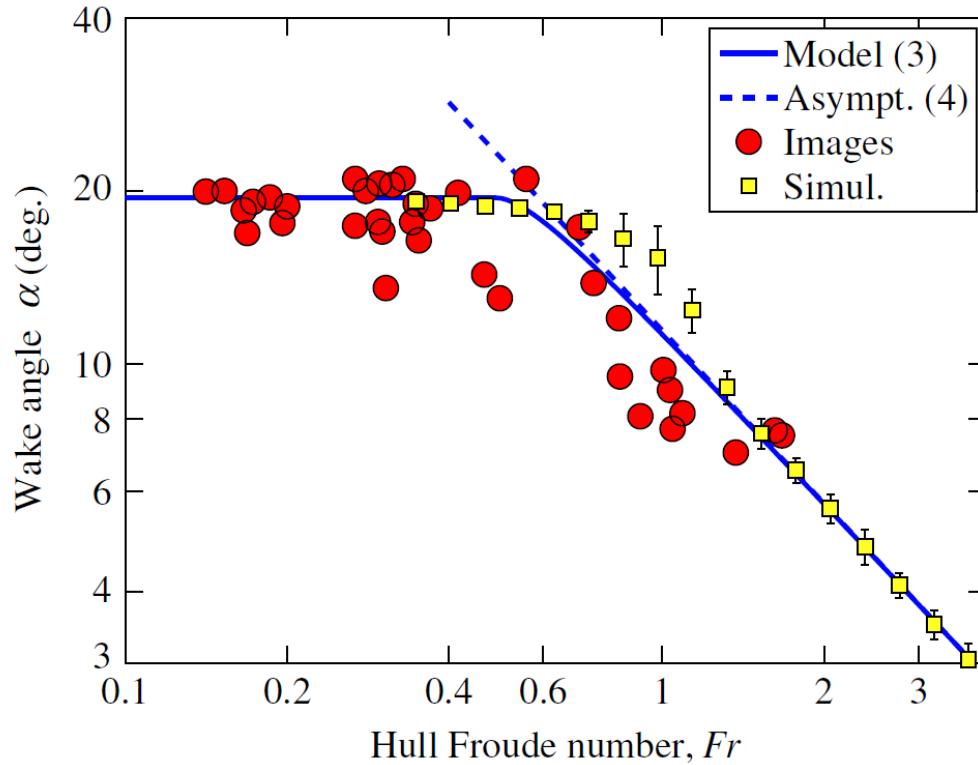


But observations show



Moisy and Rabaud 2013

But observations show

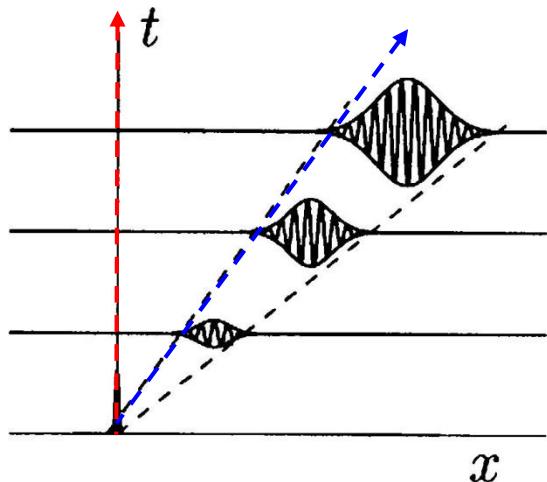


Moisy and Rabaud 2013

Generalization: Spatio-temporal instability theory

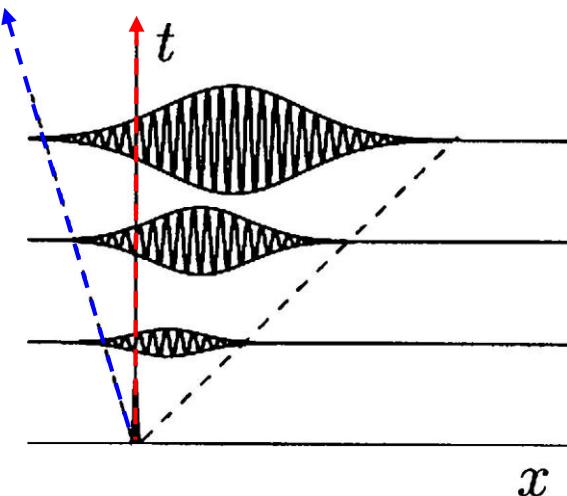
First find the zero group velocity wave: $d\omega/dk=0 \Rightarrow (k_0, \omega_0)$ and consider the sign of $\text{Im}(\omega_0)$

👉 Convective instability



$$\text{Im}(\omega_0) < 0$$

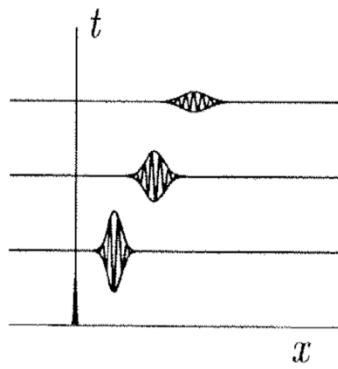
👉 Absolute instability



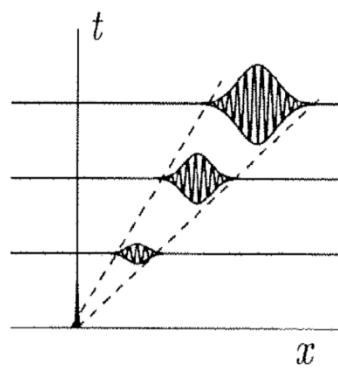
$$\text{Im}(\omega_0) > 0$$

LINEAR IMPULSE RESPONSE: ABSOLUTE/CONVECTIVE INSTABILITY

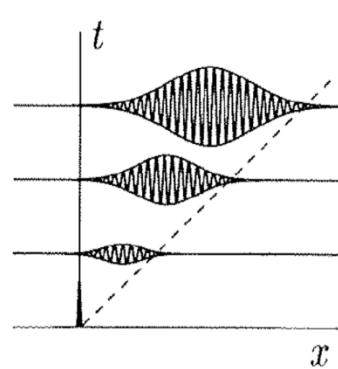
Green's function or impulse response



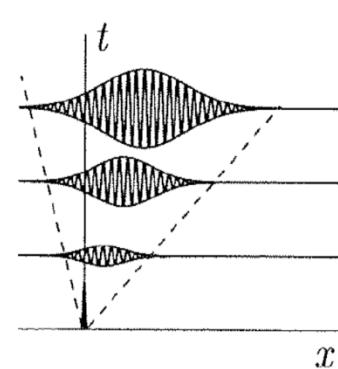
Stable



Unstable



Unstable



Unstable

Briggs (1964) Bers (1983)
Huerre and Monkewitz (1985)

LINEAR IMPULSE RESPONSE: ABSOLUTE/CONVECTIVE INSTABILITY

Linearly stable flow

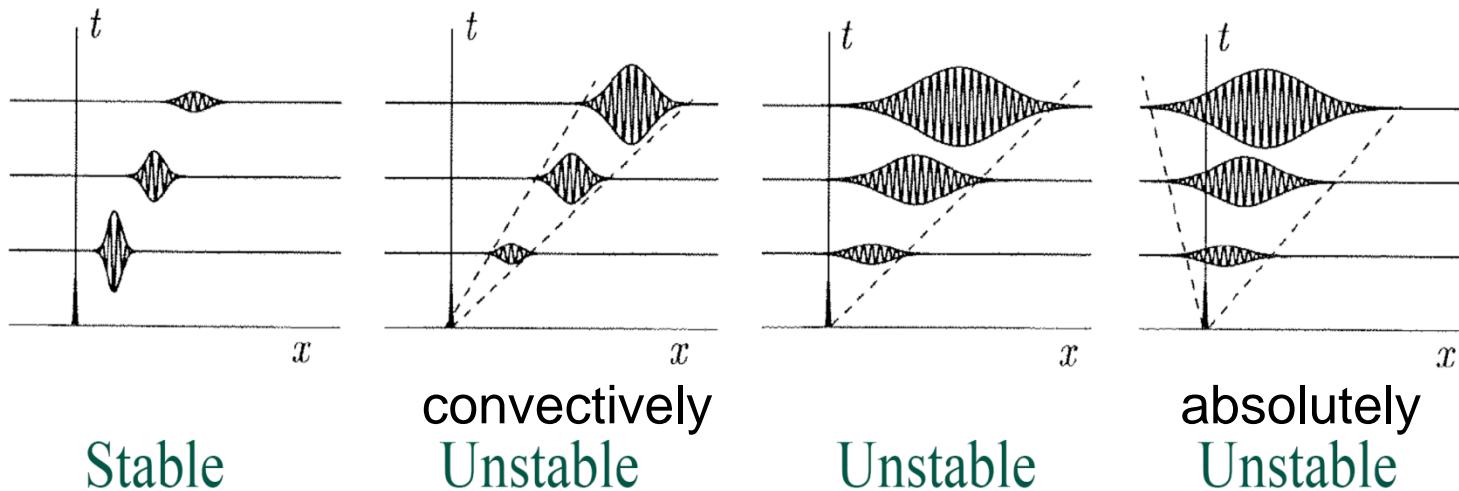
$$\lim_{t \rightarrow \infty} G(x, t) = 0 \quad \text{along all rays } x/t = \text{const.}$$

Linearly unstable flow

$$\lim_{t \rightarrow \infty} G(x, t) = \infty \quad \text{along at least one ray } x/t = \text{const.}$$

LINEAR IMPULSE RESPONSE: ABSOLUTE/CONVECTIVE INSTABILITY

Green's function or impulse response



Briggs (1964) Bers (1983)

Huerre and Monkewitz (1985)

LINEAR IMPULSE RESPONSE: ABSOLUTE/CONVECTIVE INSTABILITY

Convectively unstable flow

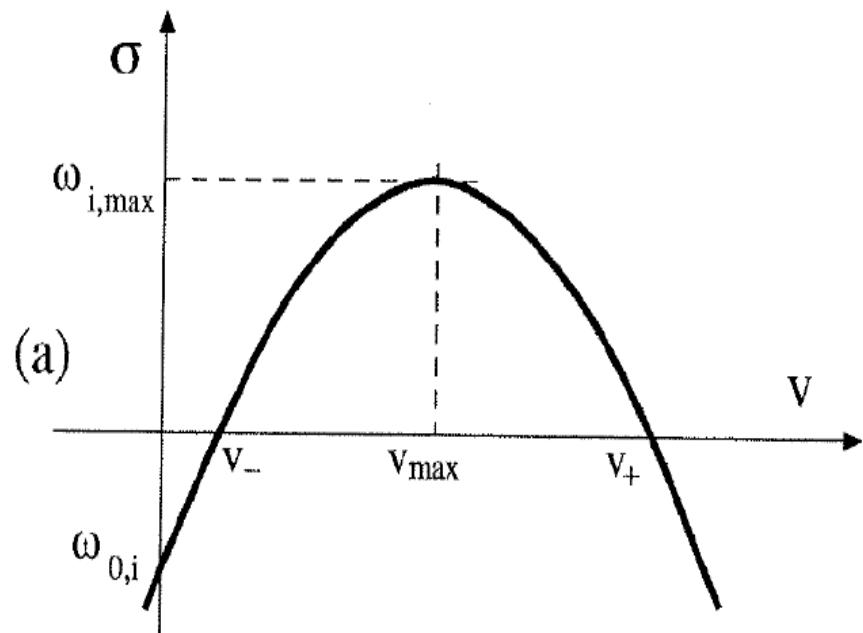
$$\lim_{t \rightarrow \infty} G(x, t) = 0 \quad \text{along the ray } x/t = 0$$

Absolutely unstable flow

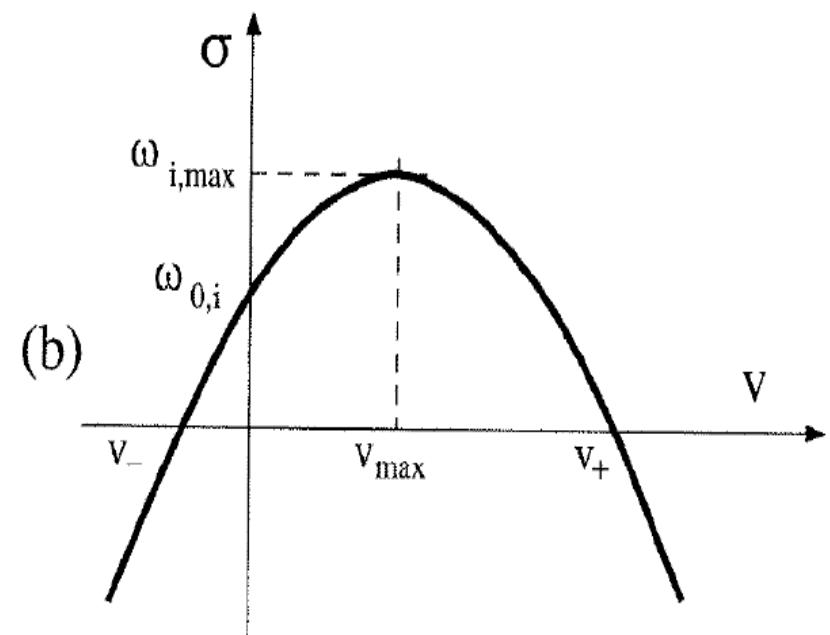
$$\lim_{t \rightarrow \infty} G(x, t) = \infty \quad \text{along the ray } x/t = 0$$

ANALYSIS IN COMPLEX FOURIER SPACE: AU/CU CRITERION

Temporal growth rate « at velocity v »



Convective instability



Absolute instability

ANALYSIS IN COMPLEX FOURIER SPACE: AU/CU CRITERION

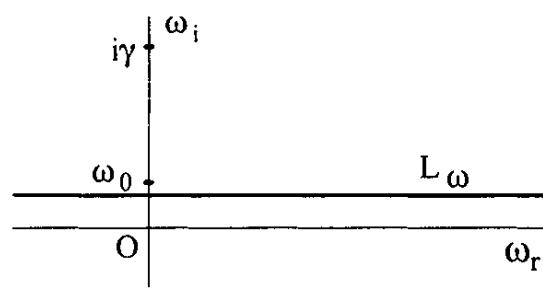
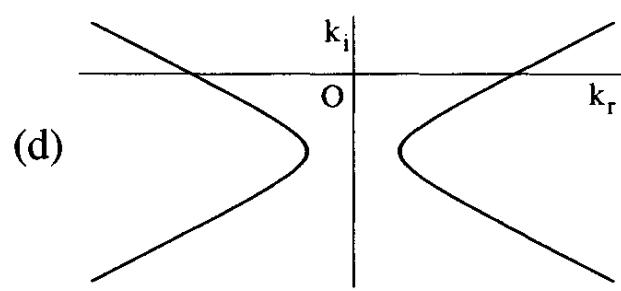
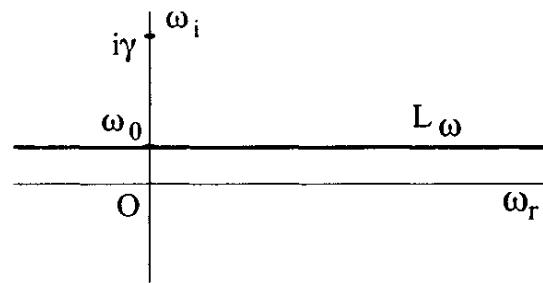
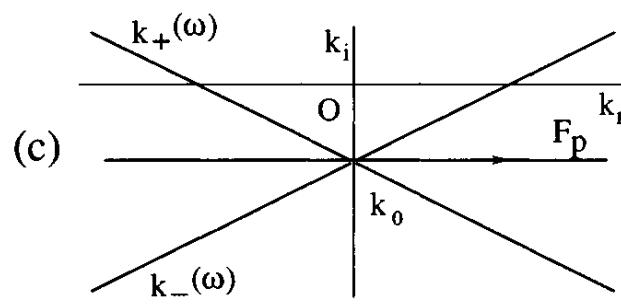
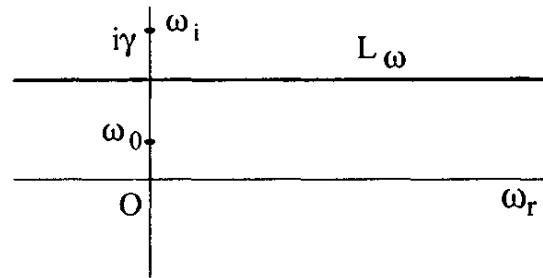
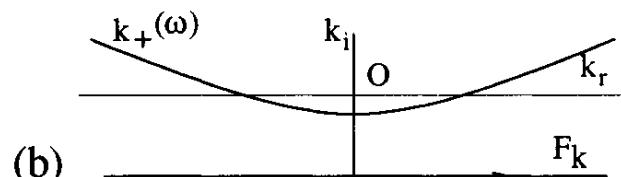
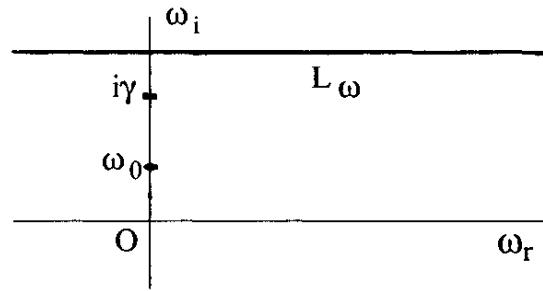
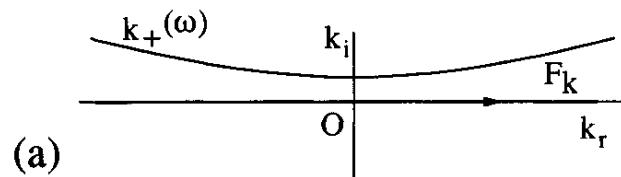
Important notions

Absolute wavenumber k_0 and frequency $\omega_0 = \omega(k_0)$ observed along ray $v = 0$, i.e. for a stationary observer, defined by

$$\frac{\partial \omega}{\partial k}(k_0) = 0$$

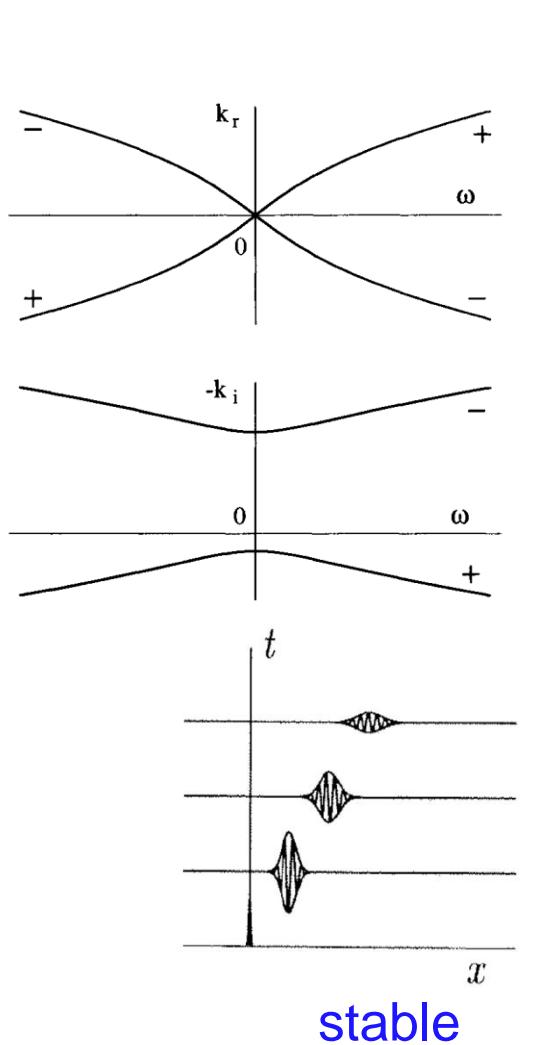
Absolute growth rate is

$$\sigma(0) = \omega_{0,i}$$



Spatial analysis

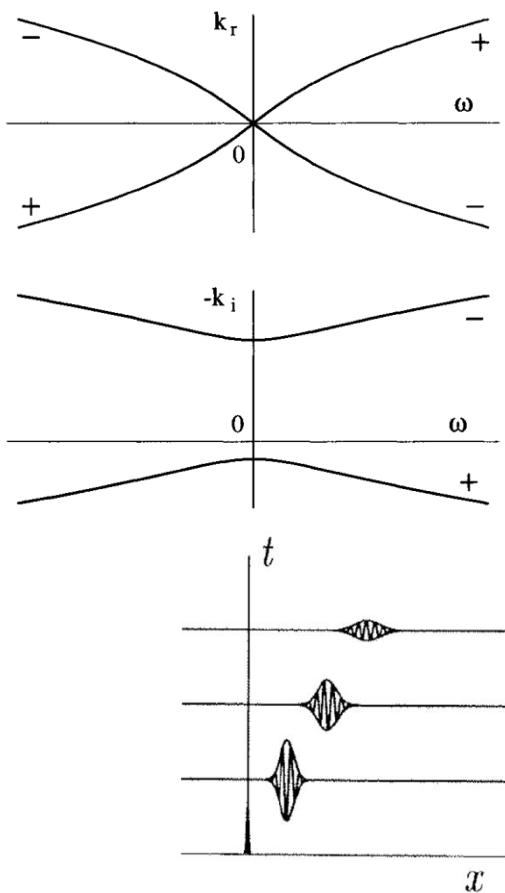
ω is real, k complex, $k+$ and $k-$ waves, plotting $kr(\omega)$ and $-ki(\omega)$:



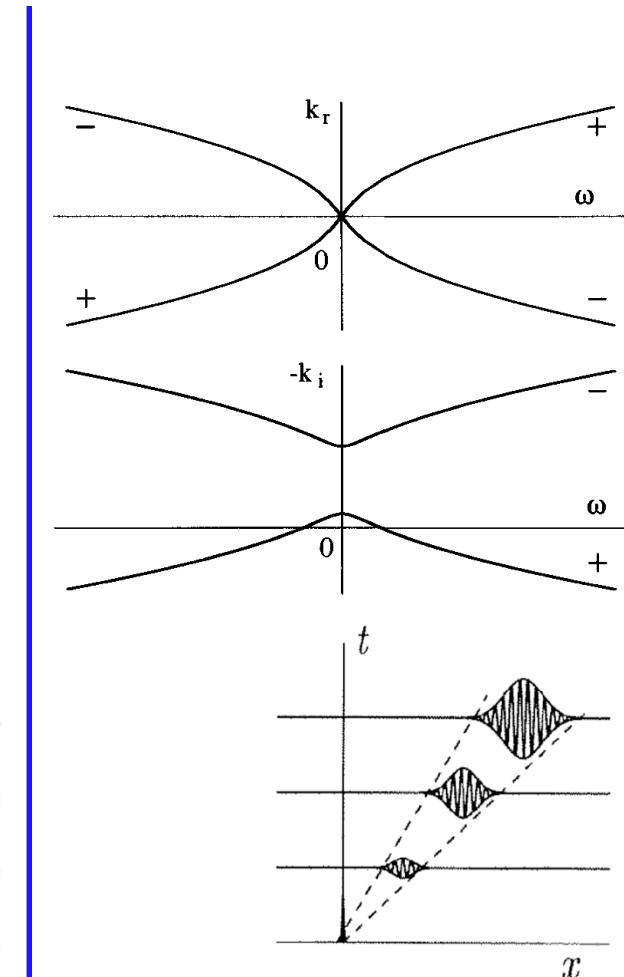
stable

Spatial analysis

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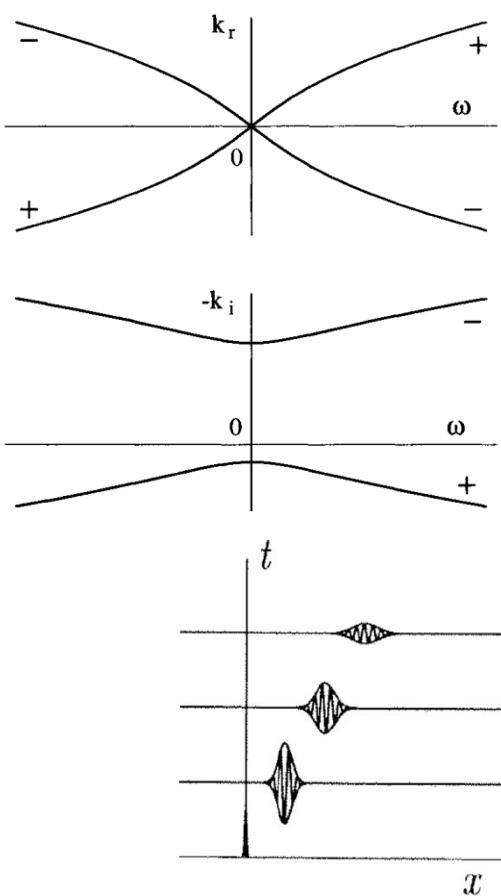
stable



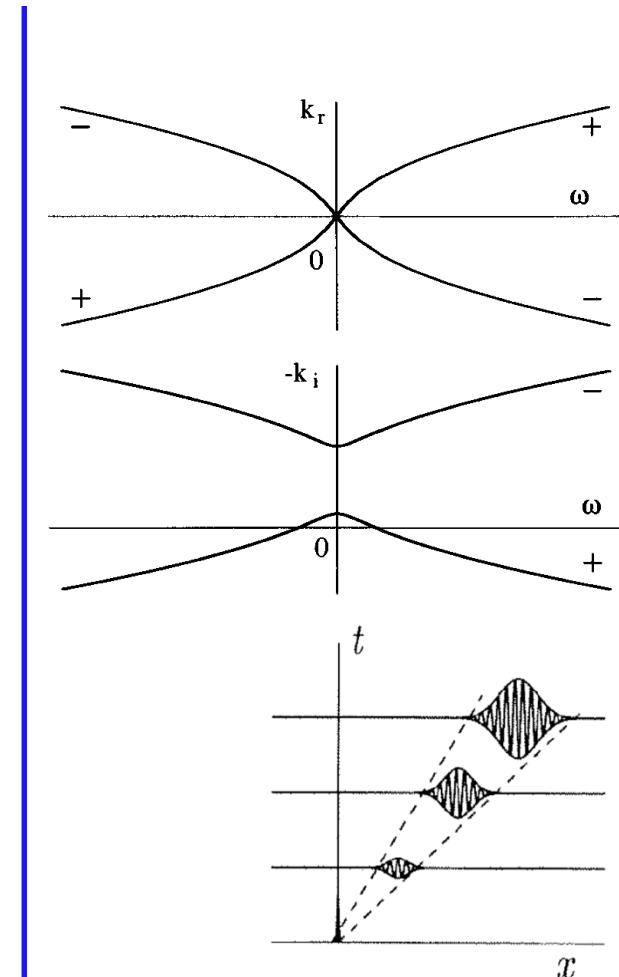
convectively unstable

Spatial analysis

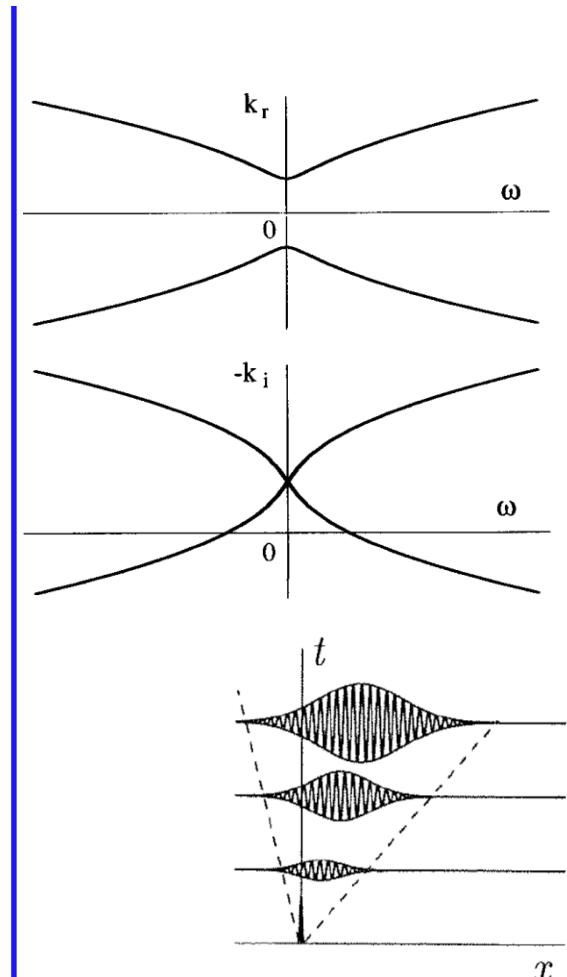
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stable



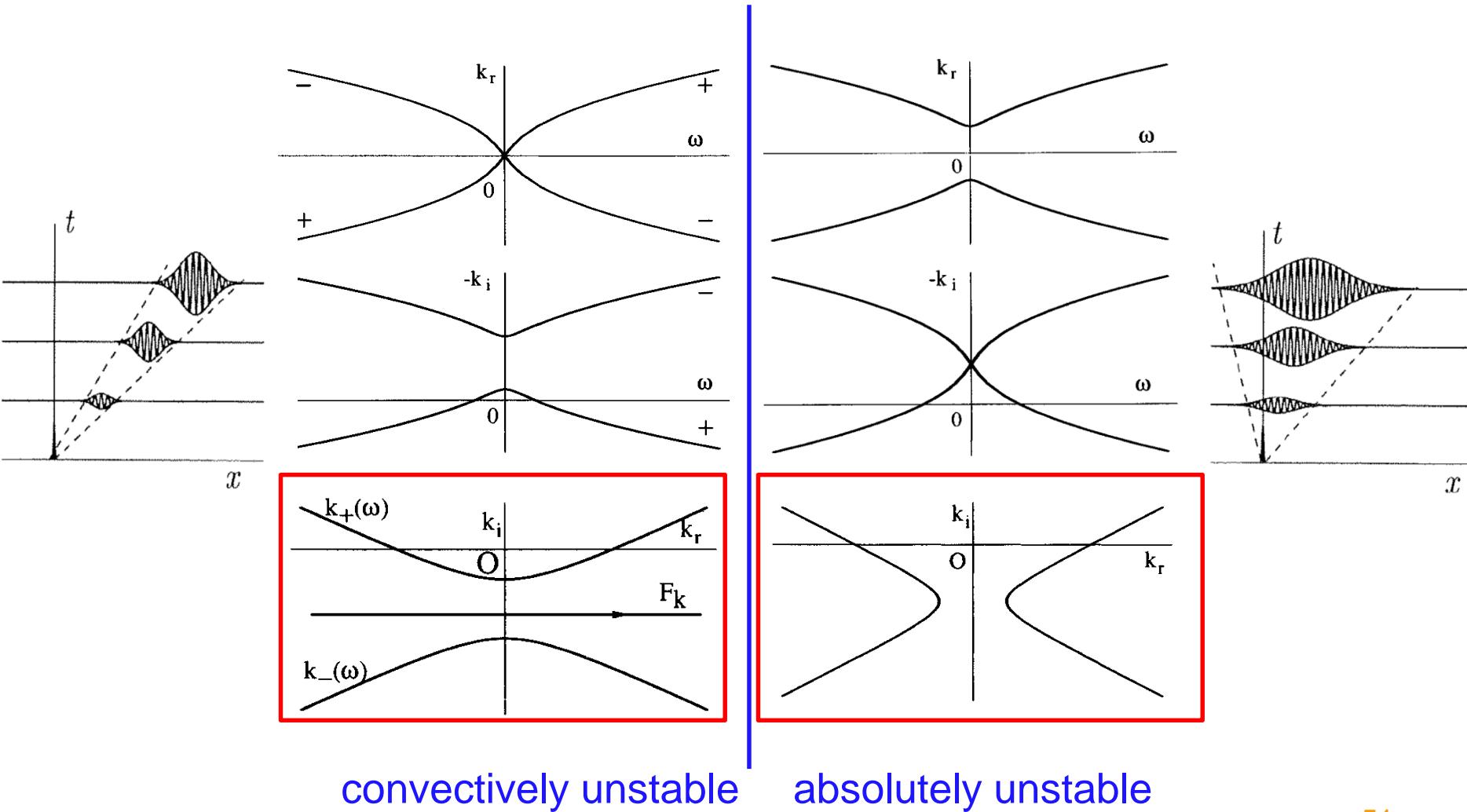
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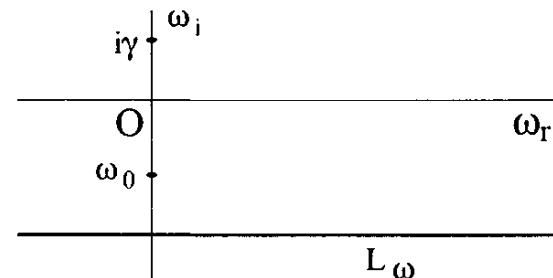
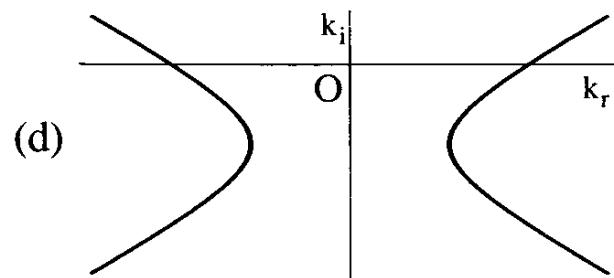
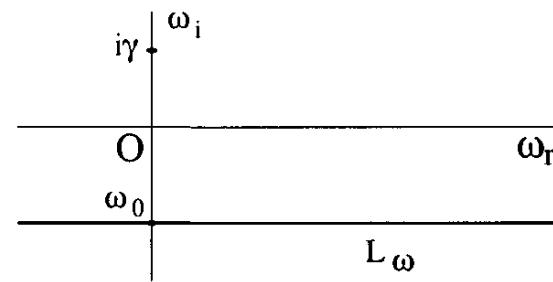
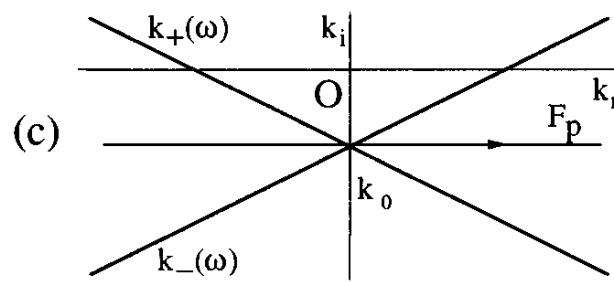
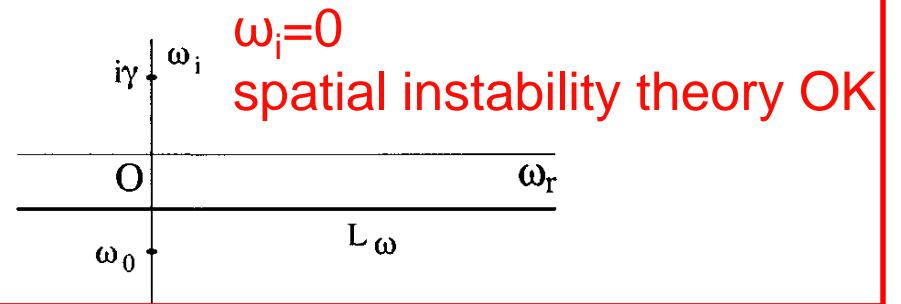
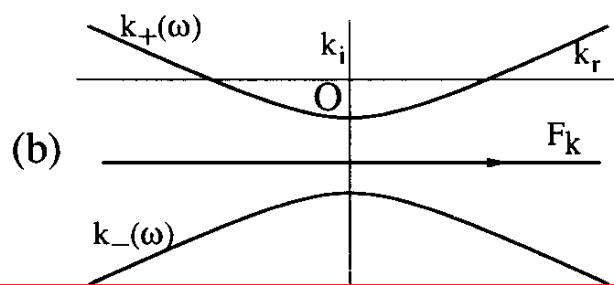
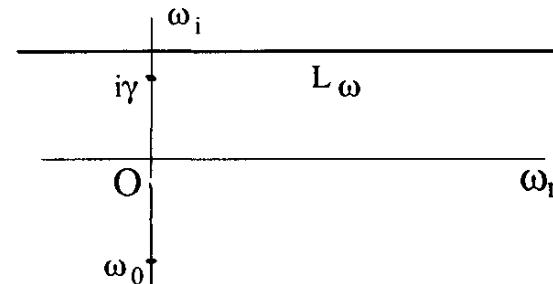
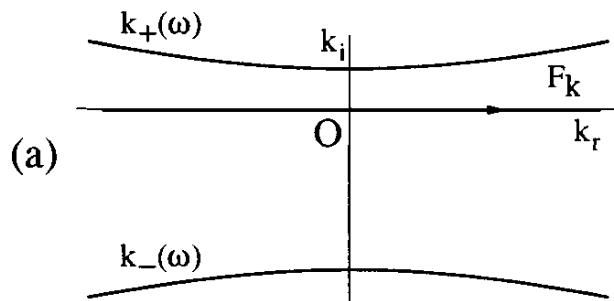
absolutely unstable

Spatial analysis

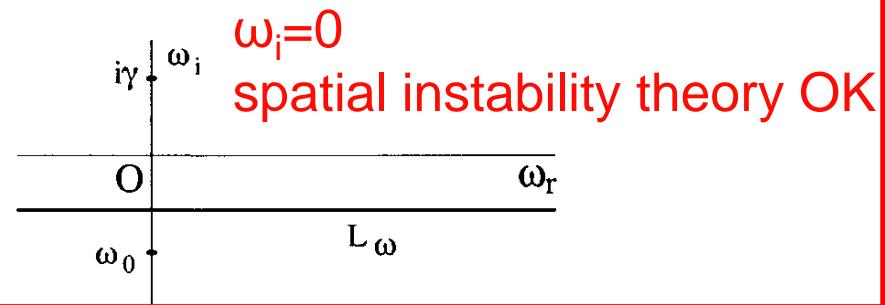
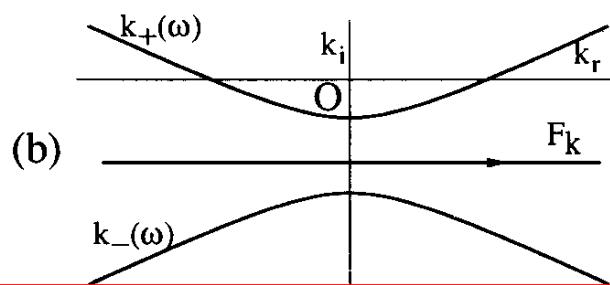
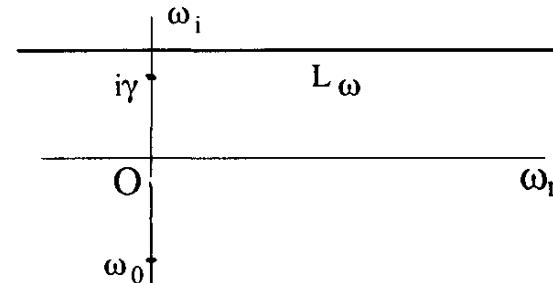
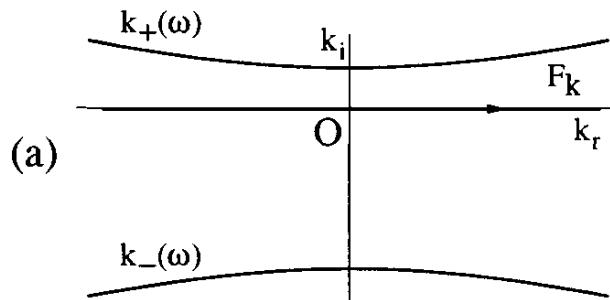
ω is real, k complex, k_+ and k_- waves, plotting $kr(\omega)$ and $-ki(\omega)$ and $ki(kr)$



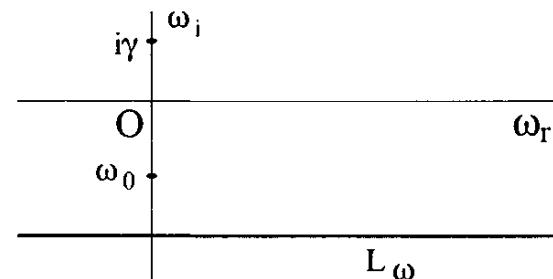
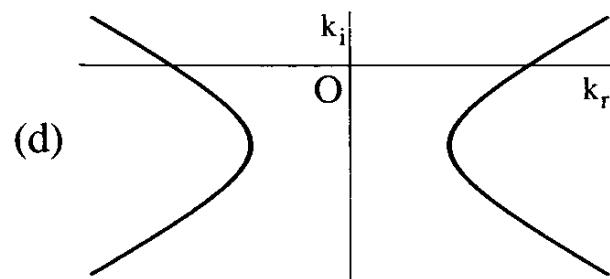
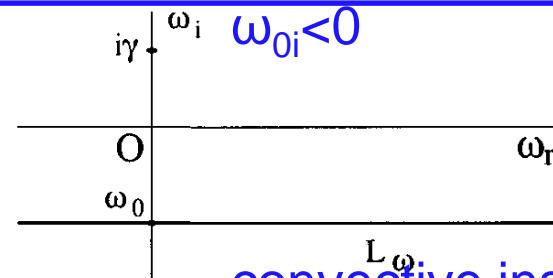
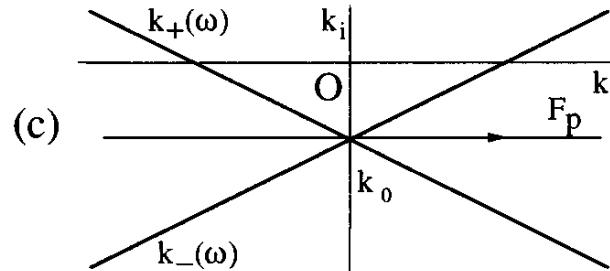
Add ω_i offset and vary ω_r



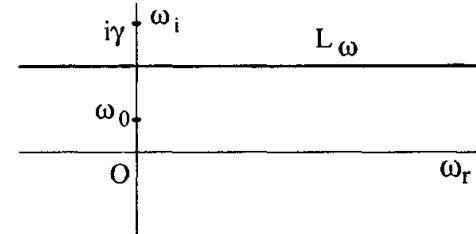
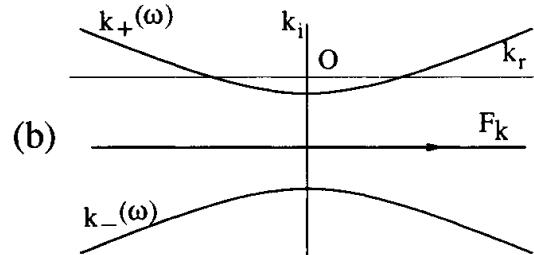
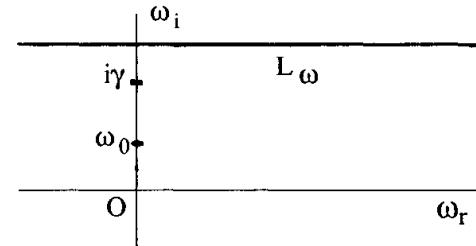
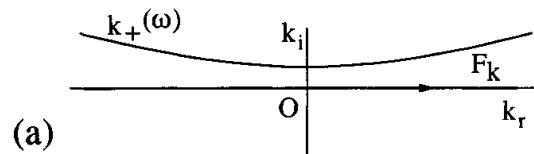
Add ω_i offset and vary ω_r



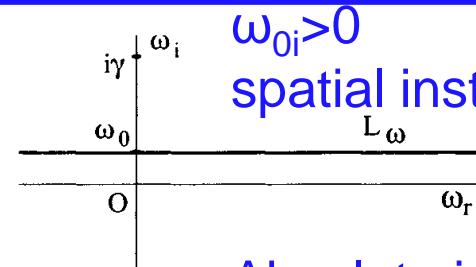
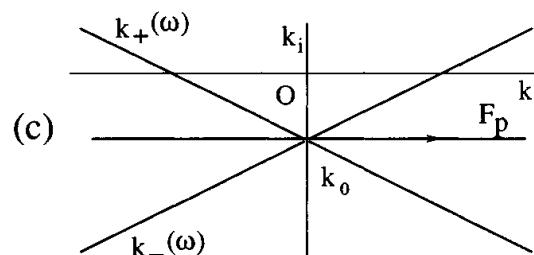
Saddle point



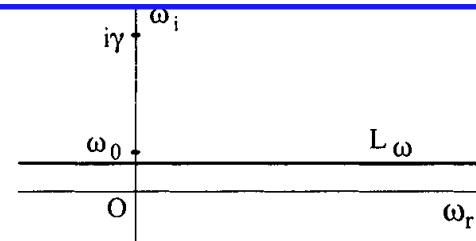
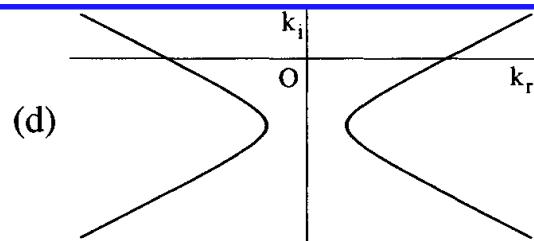
Add ω_i offset and vary ω_r
 Saddle point condition



Saddle point

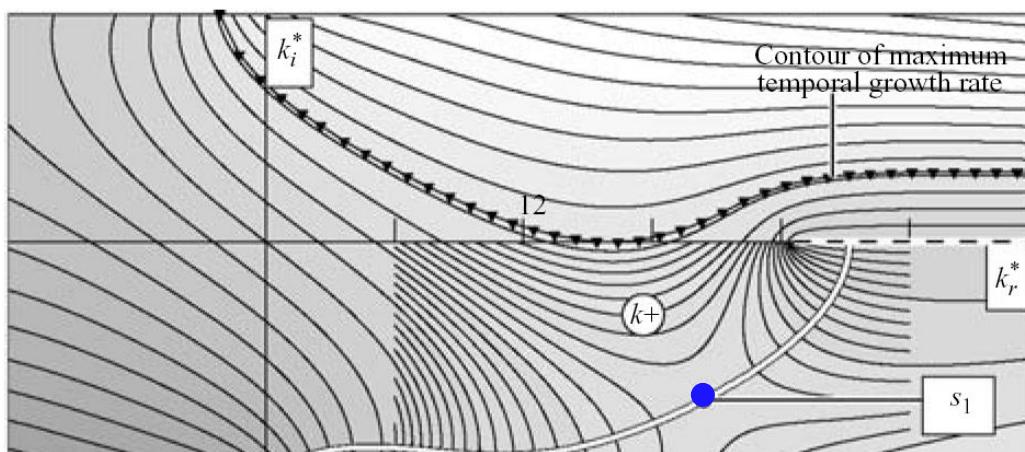


Absolute instability



Spatio-temporal analysis (vary both kr and ki)

Isocontours of ω_i in the $kr-ki$ plane



$$\boxed{\partial\omega/\partial k = 0}$$

Saddle point condition

Absolute frequency $\omega_0 = \omega(k_0)$

Absolute wavenumber k_0

ANALYSIS IN COMPLEX FOURIER SPACE: AU/CU CRITERION

Instability criteria

$\omega_{i,max} < 0$ linearly stable

$\omega_{i,max} > 0$ linearly unstable

$\omega_{0,i} < 0$ convectively unstable

$\omega_{0,i} > 0$ absolutely unstable

Hyperbolic tangent mixing layer

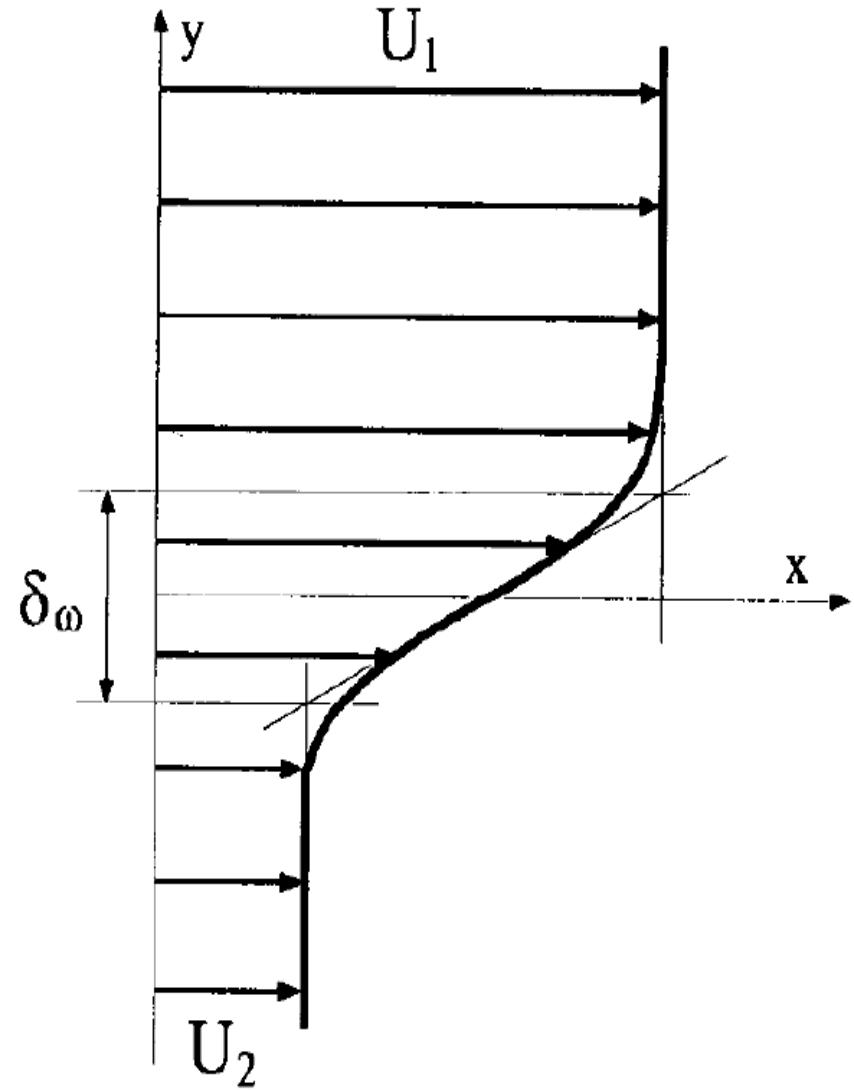
$$U(y) = \bar{U} + \frac{\Delta U}{2} \tanh\left(\frac{2y}{\delta_\omega}\right)$$

$$\delta_\omega(x) \equiv \frac{(U_1 - U_2)}{(dU/dy)_{\max}}$$

Velocity ratio

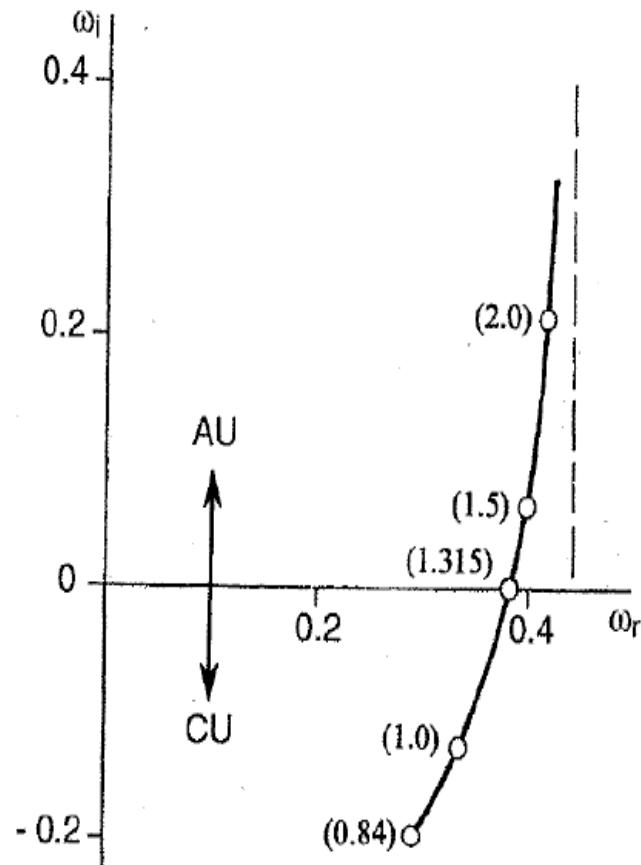
$$R \equiv \frac{U_1 - U_2}{U_1 + U_2} = \frac{\Delta U}{2\bar{U}}$$

$$U(y; R) = 1 + R \tanh y$$

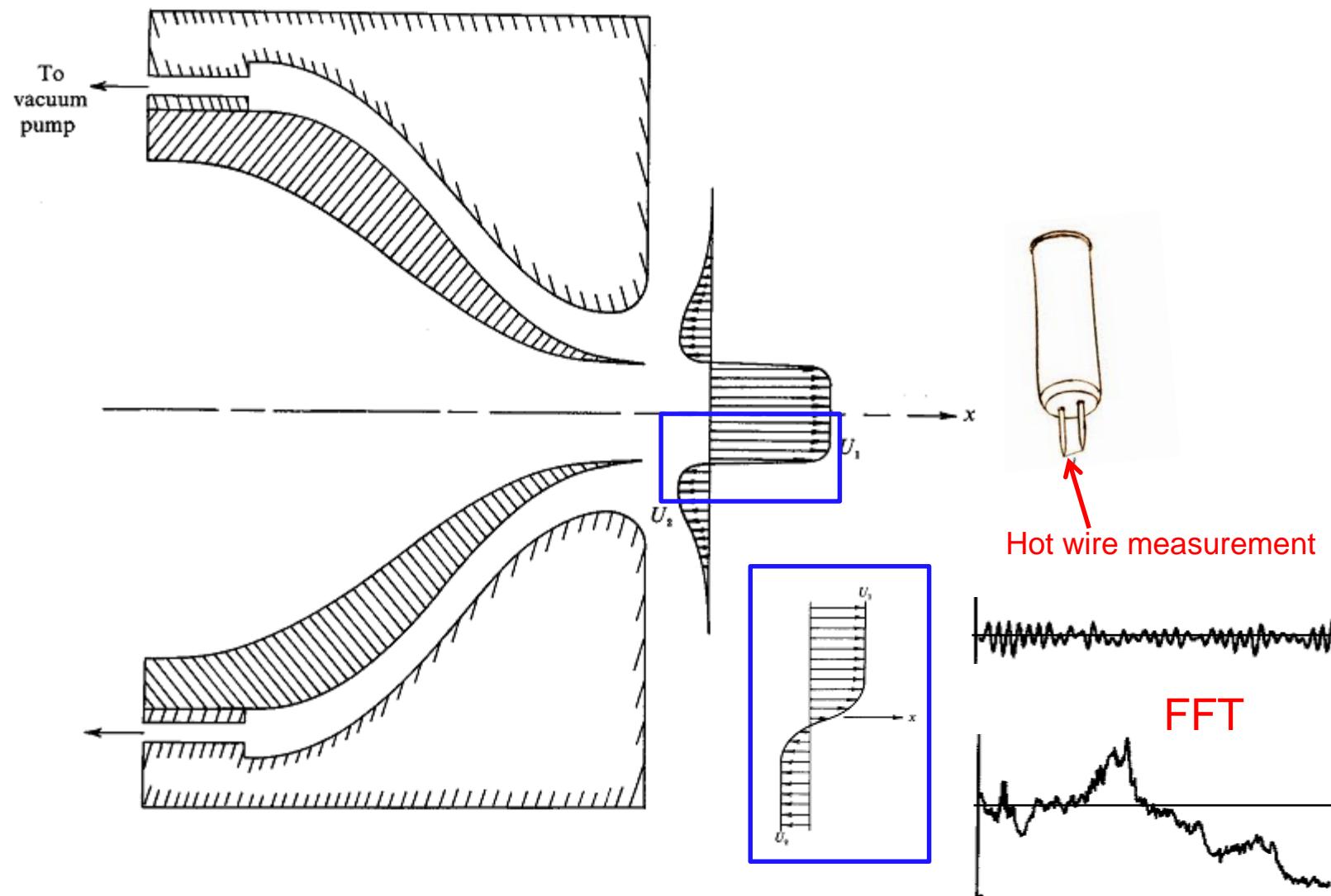


APPLICATION TO MIXING LAYERS

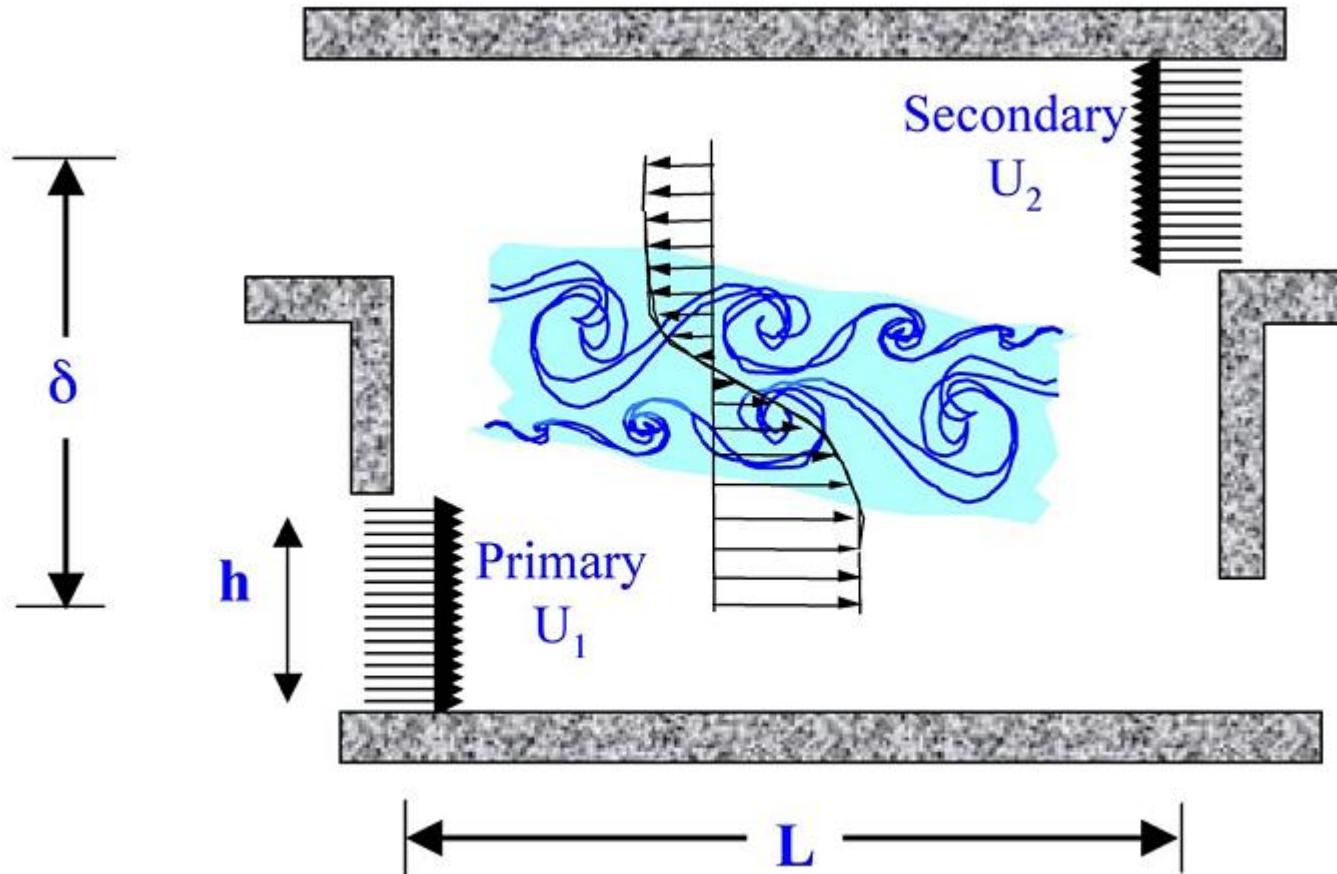
Locus of complex absolute frequency



H.&Monkewitz (1985)

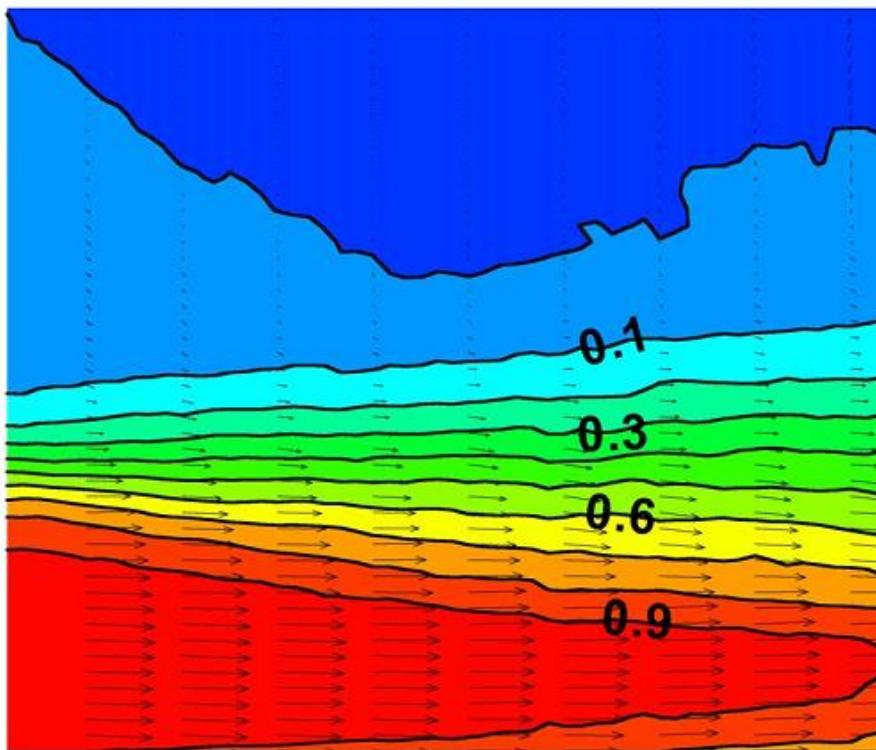


Influence of countercurrent shear on turbulence level

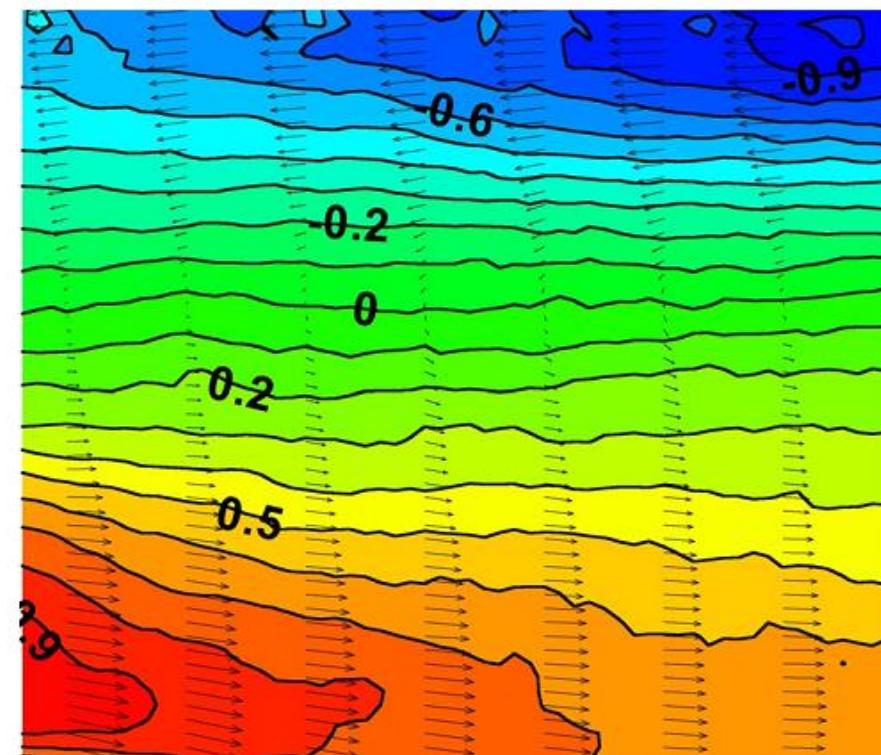


Influence of countercurrent shear on turbulence level

Base flow



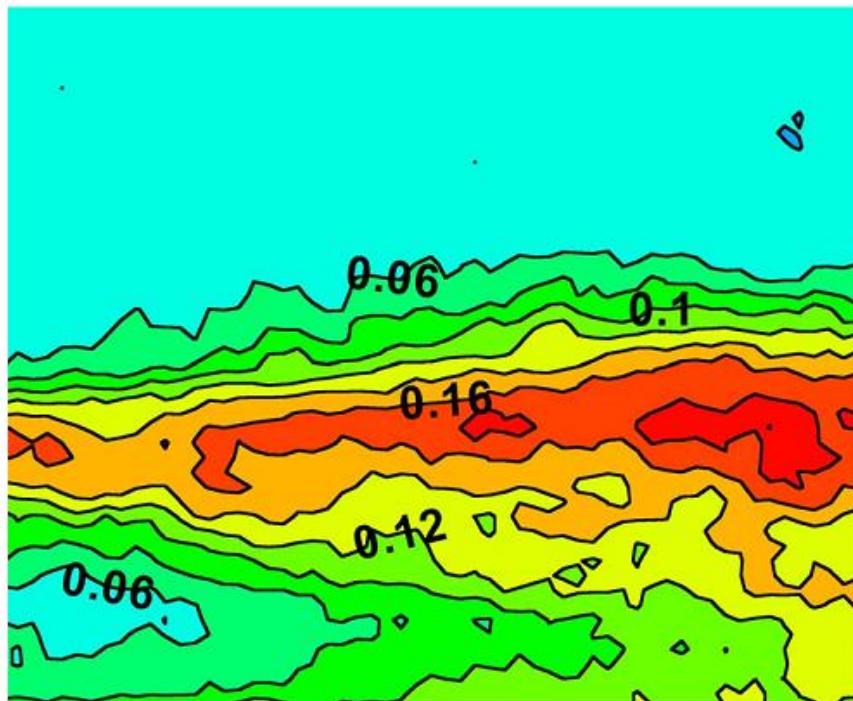
a) Single stream shear layer



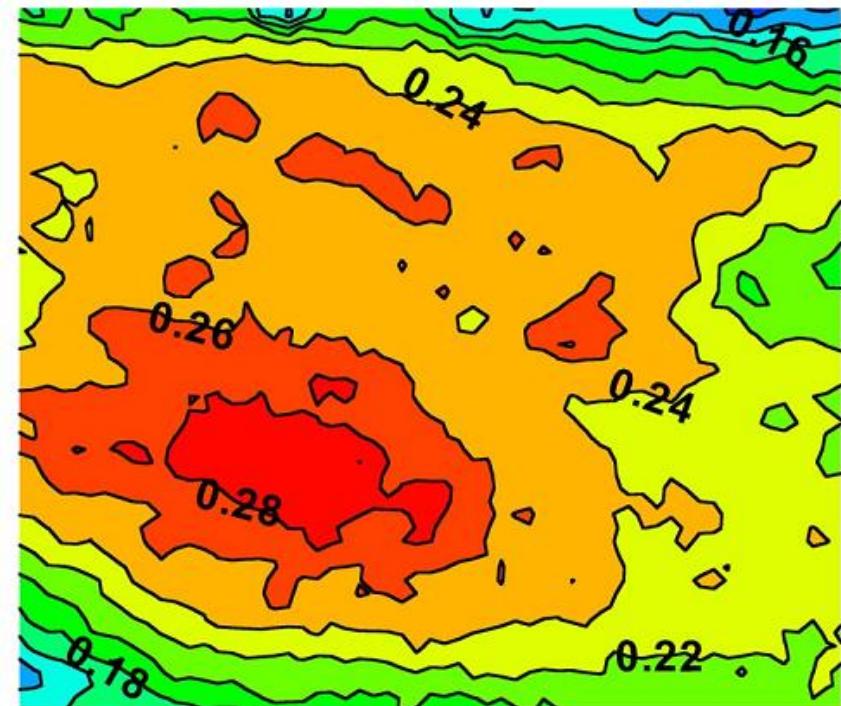
b) Countercurrent shear layer

Influence of countercurrent shear on turbulence level

Turbulence intensity

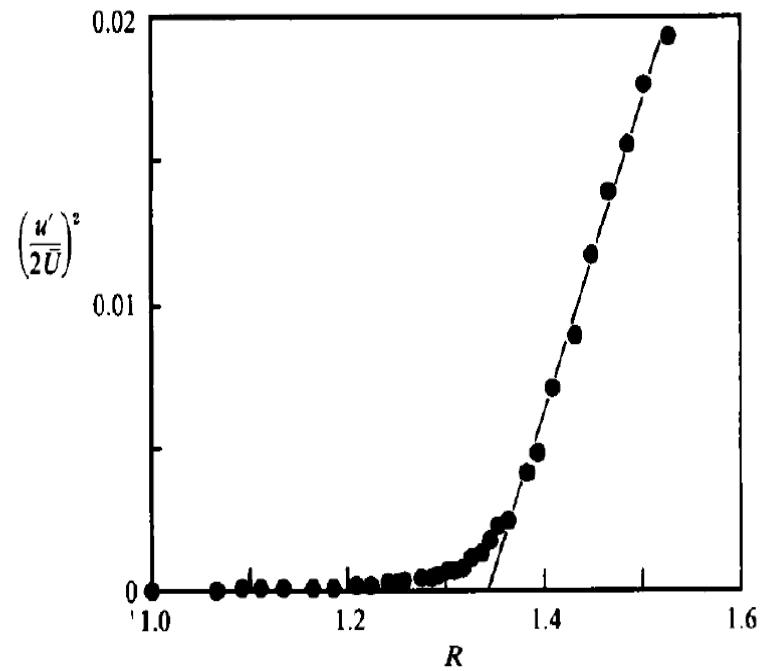
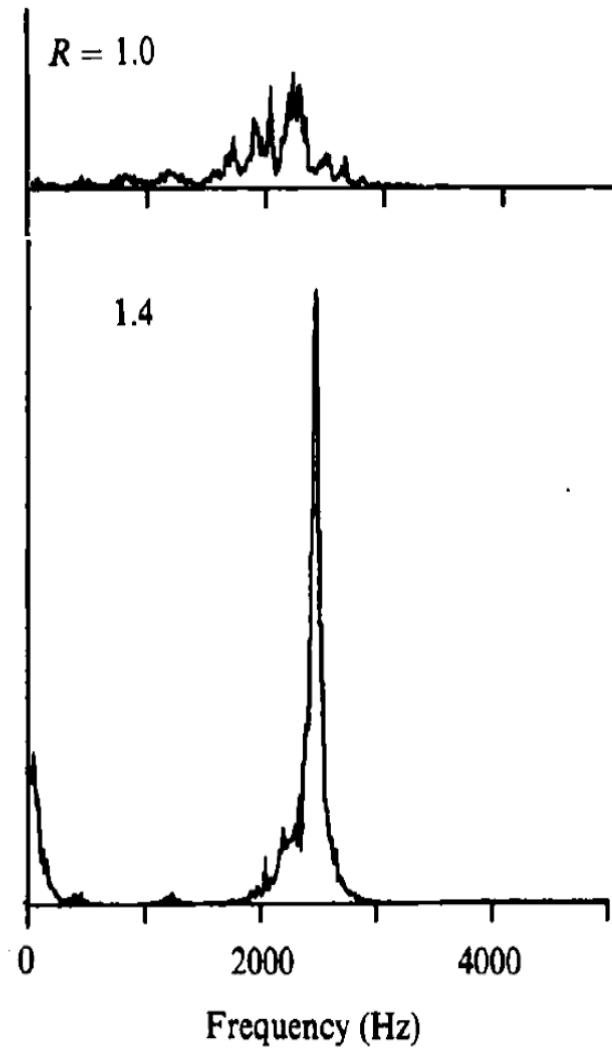


a) Single-stream shear layer



b) Countercurrent shear layer

THE MIXING LAYER: SHIFT TO OSCILLATOR !

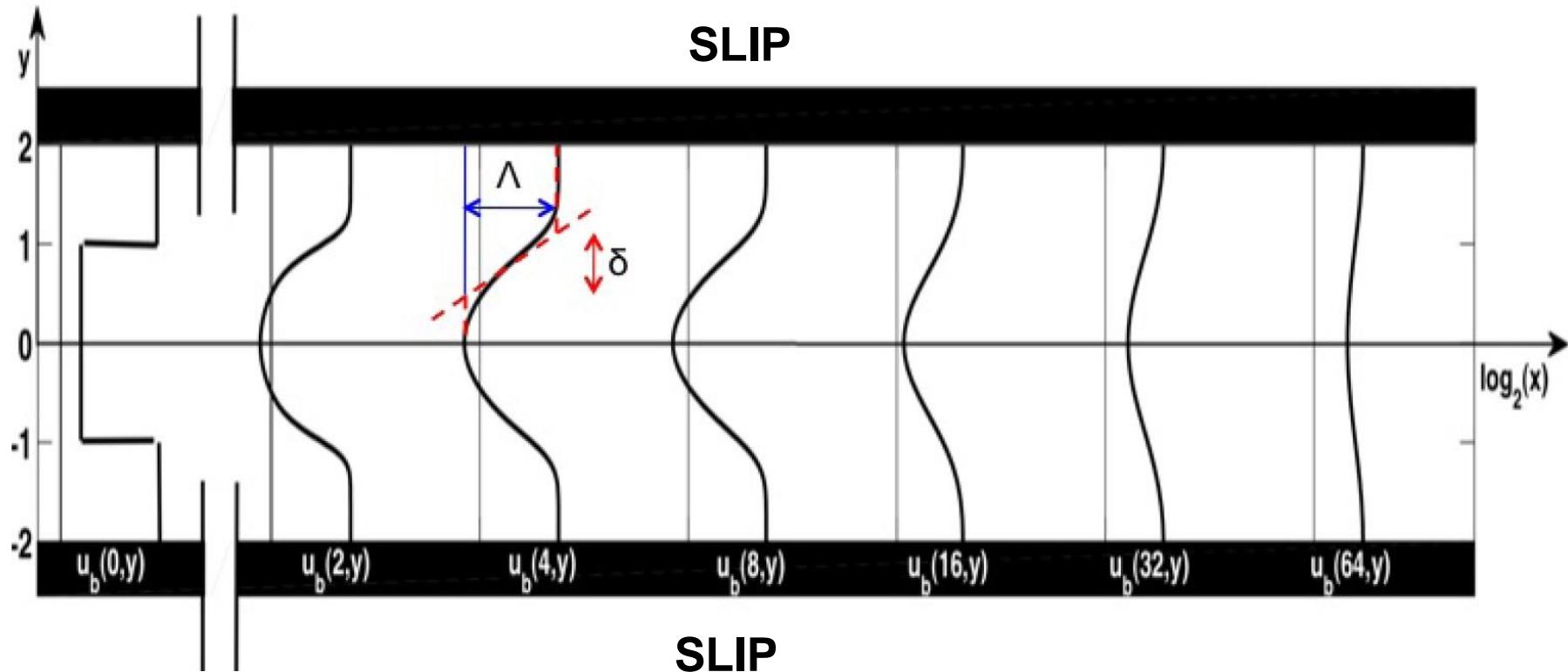


Strykowski & Niccum (1991)

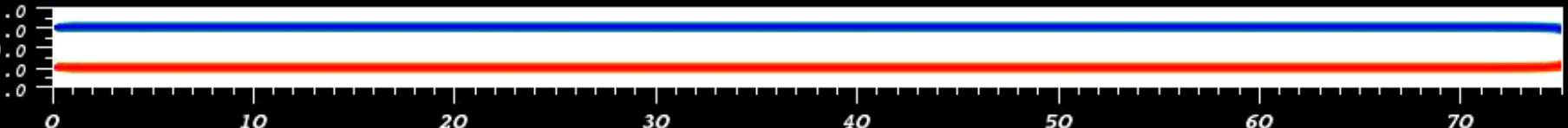
Direct Numerical Simulations with top-hat profile at inlet

Viscous diffusion \longrightarrow Non-parallel flow

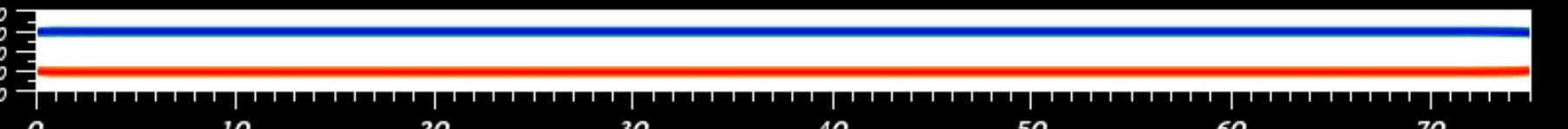
- $\Lambda_{loc} = (U_{max} - U_{min})/(U_{max} + U_{min})$
- $\delta = (U_{max} - U_{min})/(|dU/dy|_{max})$



Vorticity field: $Re = 100$, $h = 1$



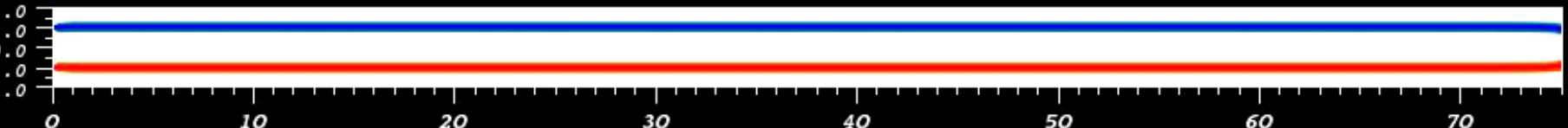
$$\Lambda = -0.739$$



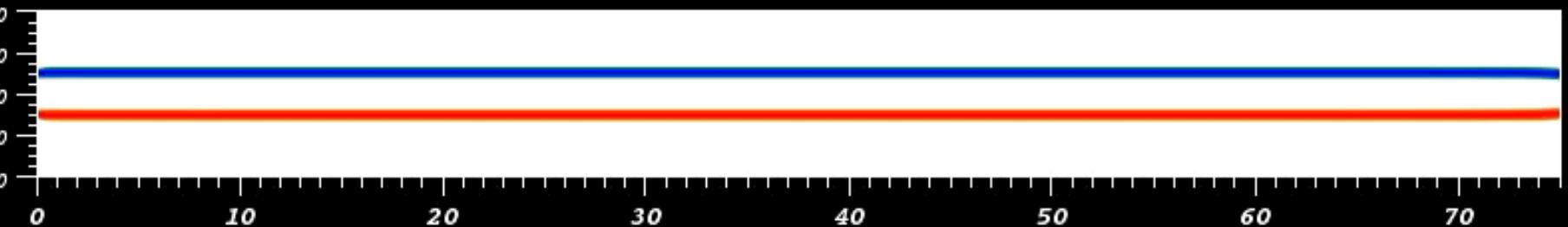
$$\Delta = -0.667$$

An increase in Λ (more coflow) advects the perturbation

Vorticity field: $Re = 100$, $\Lambda = -0.739$



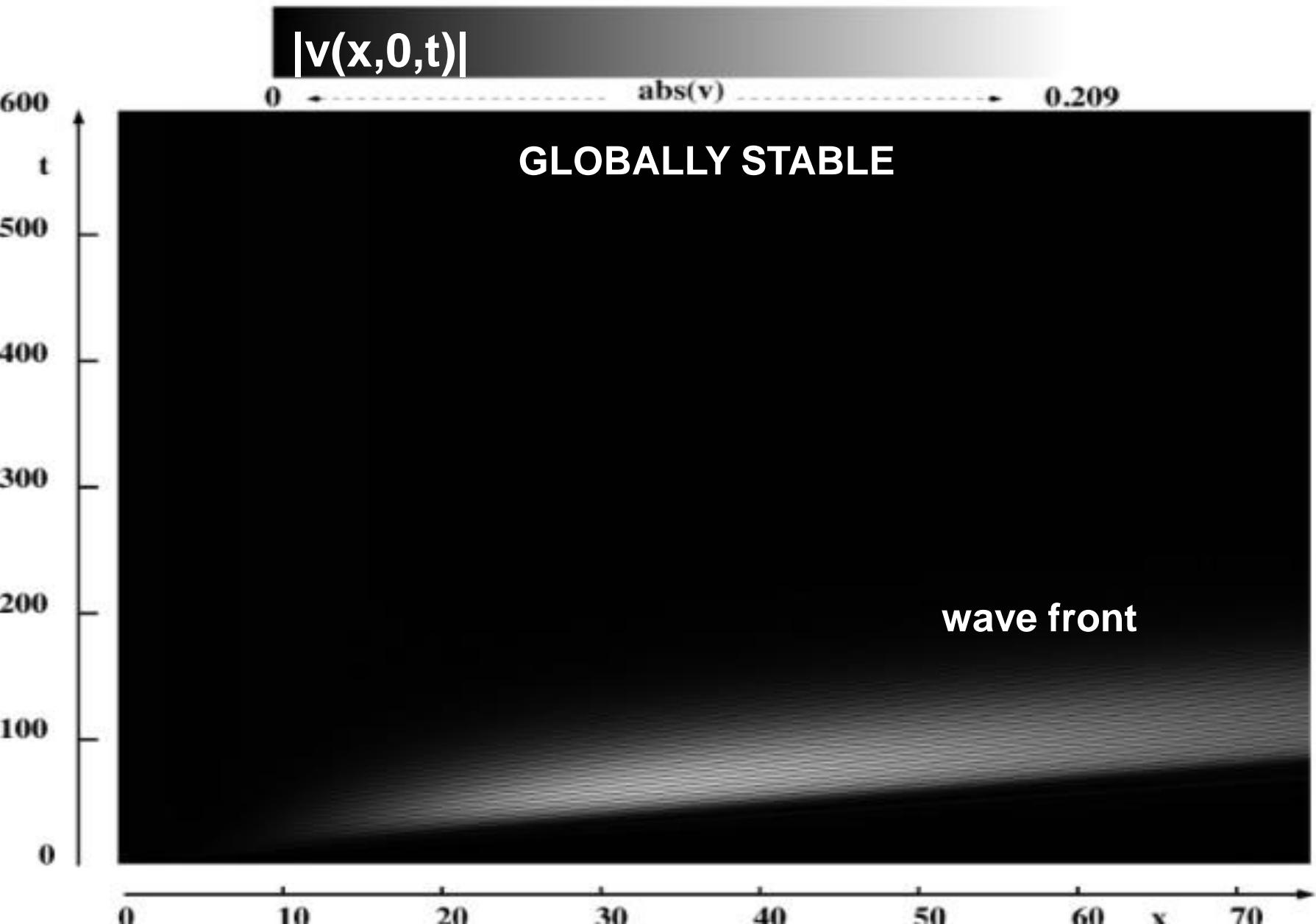
$h=1$



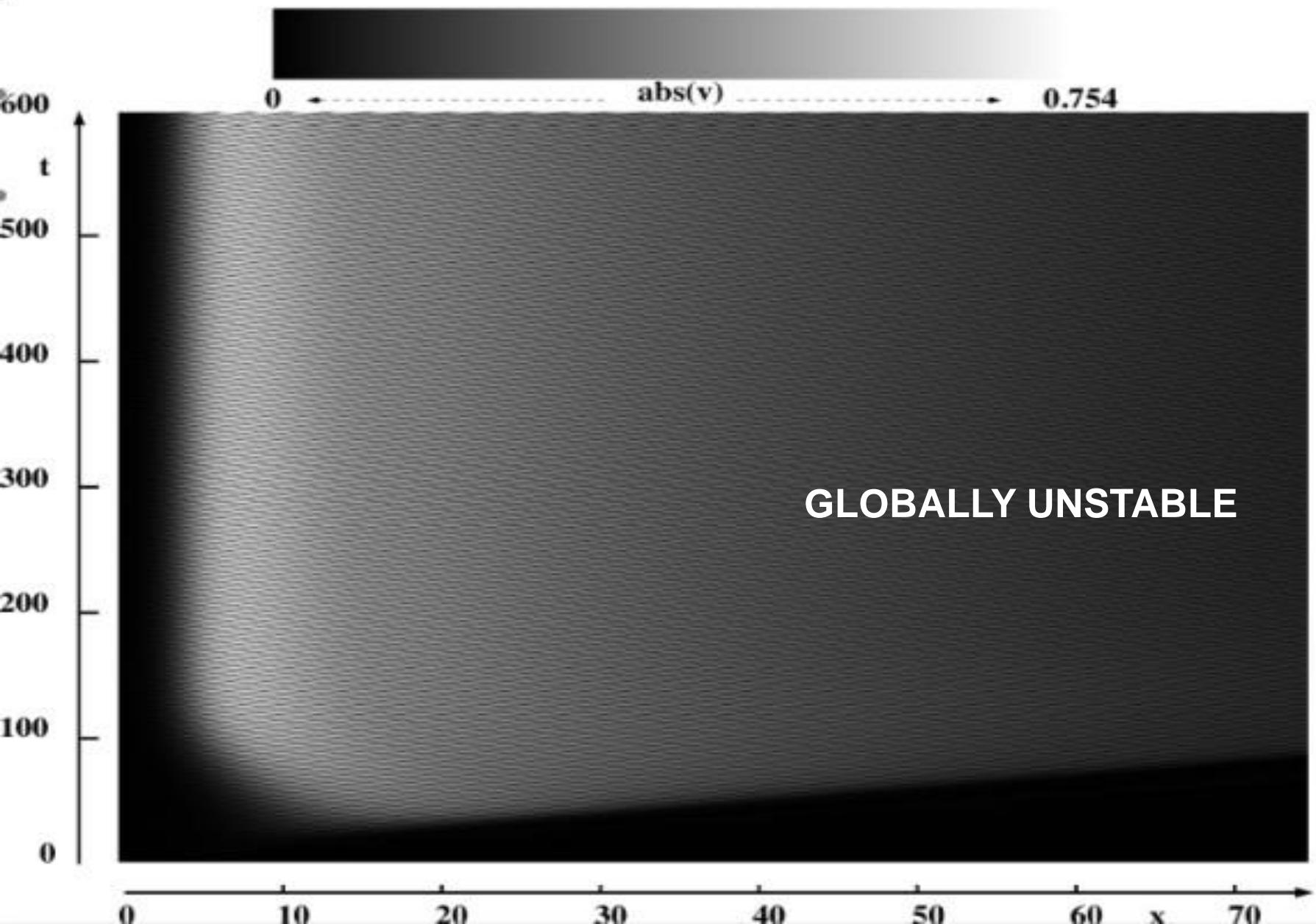
$h=3$

Destabilizing influence of confinement!

Spatio-temporal diagram, $h=1$ and $\Lambda = -0.667$



Spatio-temporal diagram, $h=1$ and $\Lambda = -0.739$



THE BLUFF BODY WAKE: A TYPICAL FLOW OSCILLATOR



$Re = 140$
Periodic
flow

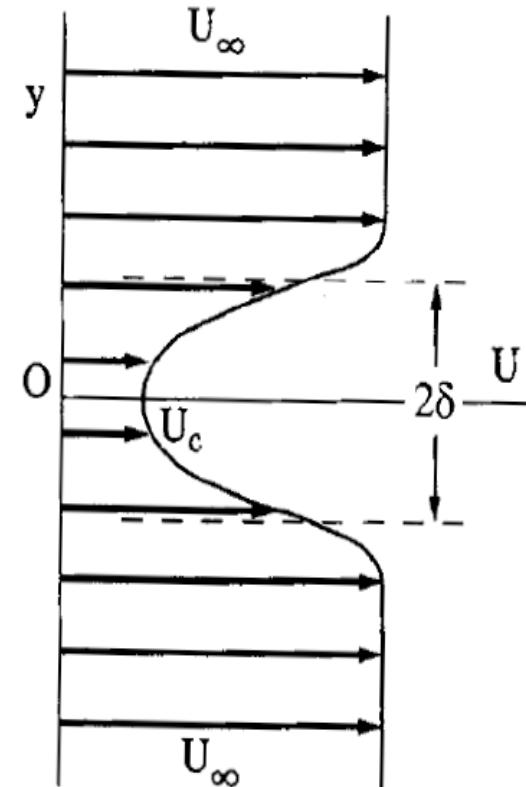
Taneda (1982)

ABSOLUTE/CONVECTIVE INSTABILITY IN PARALLEL WAKES

Family of wake profiles

$$U(y) = U_\infty + (U_c - U_\infty) U_1 \left(\frac{y}{\delta}; N \right)$$

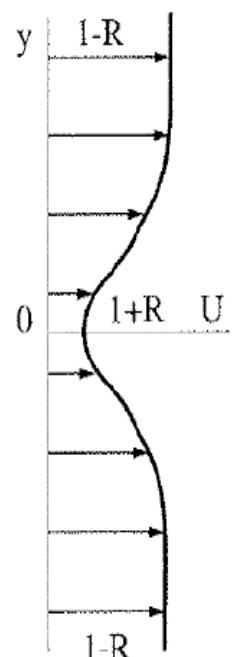
$$U_1(\xi; N) = [1 + \sinh^{2N} \{ \xi \sinh^{-1}(1) \}]^{-1}$$



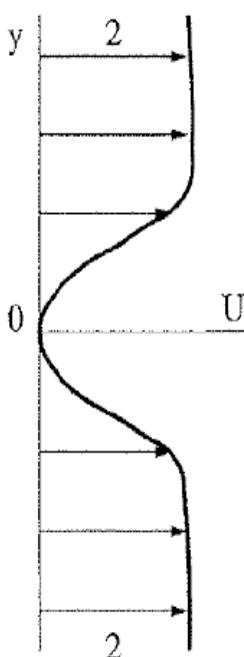
Monkewitz (1988)

ABSOLUTE/CONVECTIVE INSTABILITY IN PARALLEL WAKES

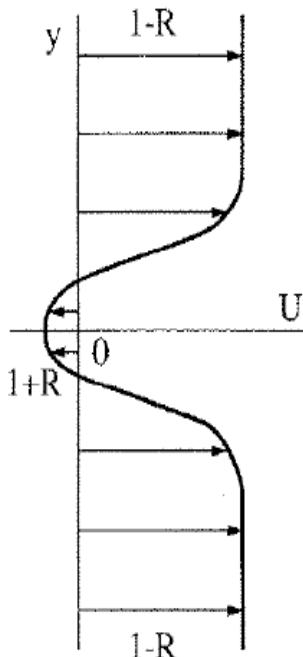
Family of wake profiles



$$(a) \quad -1 < R < 0$$

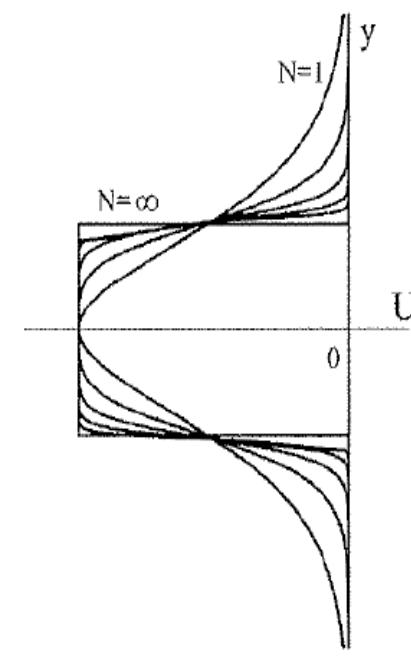


$$(b) \quad R = -1$$



$$(c) \quad R < -1$$

Effect of velocity ratio R



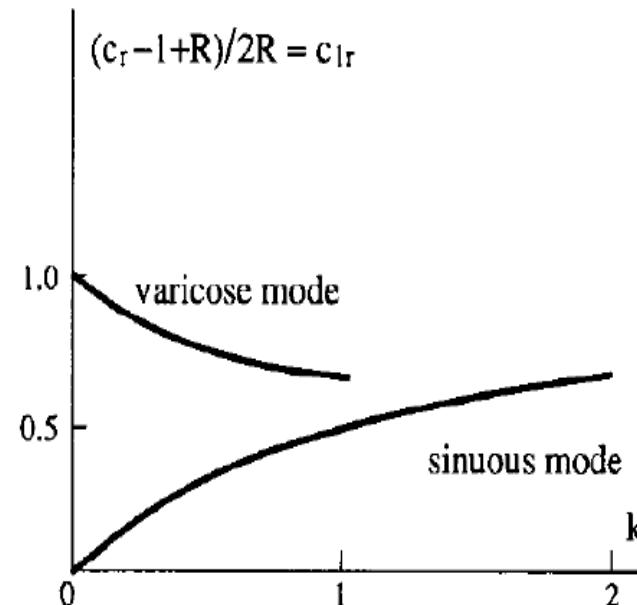
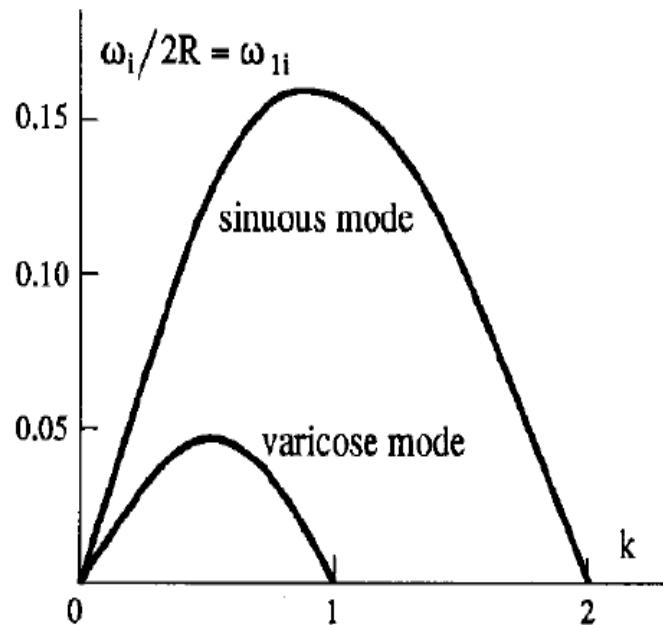
$$(d) \quad N$$

Effect of N

ABSOLUTE/CONVECTIVE INSTABILITY IN PARALLEL 2D PARALLEL FLOW CONCEPTS

$\text{sech}^2 y$ wake

Temporal approach

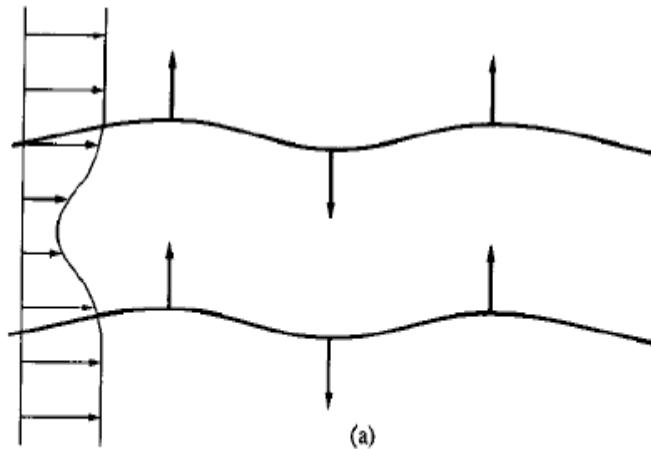


Betchov & Criminale (1966)

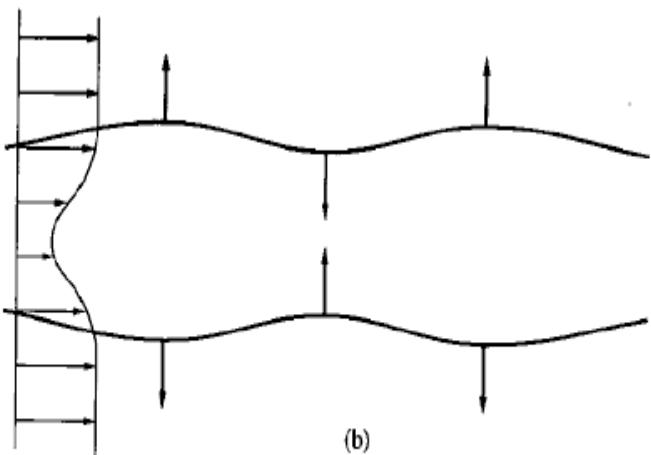
2D PARALLEL FLOW CONCEPTS

$\text{sech}^2 y$ wake

Sinuous and varicose modes



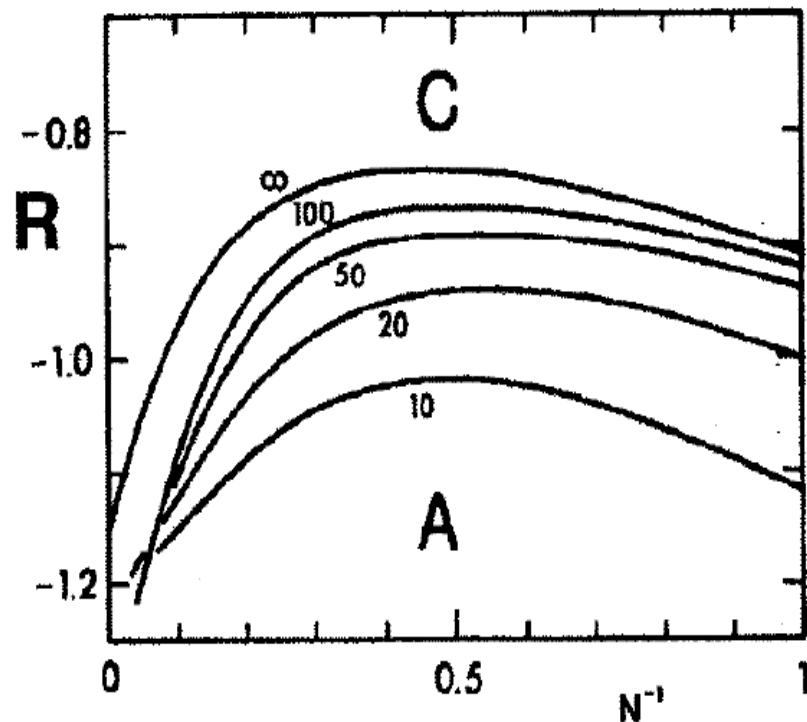
sinuous



varicose

ABSOLUTE/CONVECTIVE INSTABILITY IN PARALLEL WAKES

Effect of steepness, velocity ratio and Reynolds number



Monkewitz (1988)

LOCAL INSTABILITY BEHAVIOR OF CYLINDER WAKE

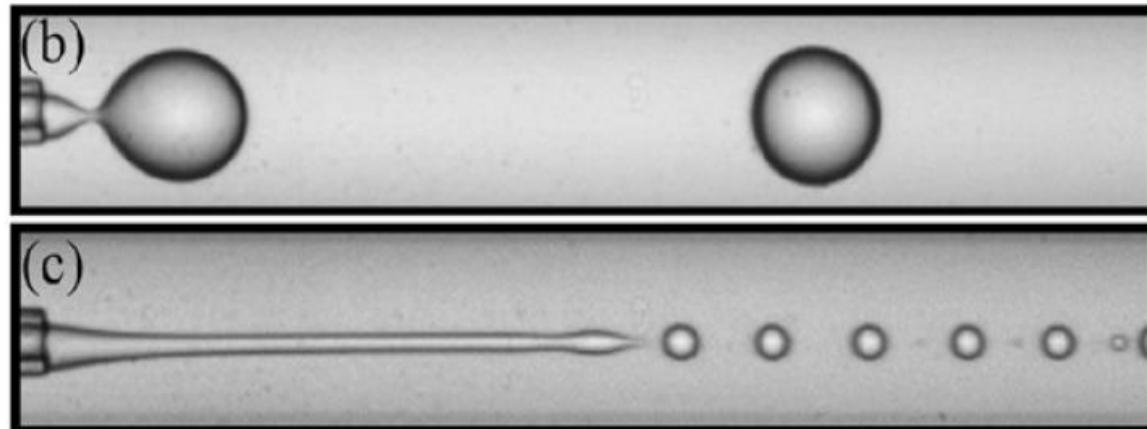
$5 < Re < 25$

Convective instability

$25 < Re < 48.5$

Absolute instability

Dripping/Jetting transition linked to absolute/convective transition?



Absolutely unstable

Convectively unstable

Guillot et al. (2008), Utada et al. (2008)

5. Dispersion relation

$$\omega = U k \pm \sqrt{\frac{\gamma k^2}{\rho} \left(k^2 - \frac{1}{R_0^2} \right) \frac{I'_0(kR)}{I_0(kR)}}$$

•Unstable if there exists one ω , $\text{Im}(\omega) > 0$ at $k < 1/R_0$

•Neutral if for all ω , $\text{Im}(\omega) = 0$ at $k > 1/R_0$

•Stable (or damped) if for all ω , $\text{Im}(\omega) < 0$:

The flow considered is not damped, we have neglected dissipation by neglecting viscosity

Destabilisation d'un jet

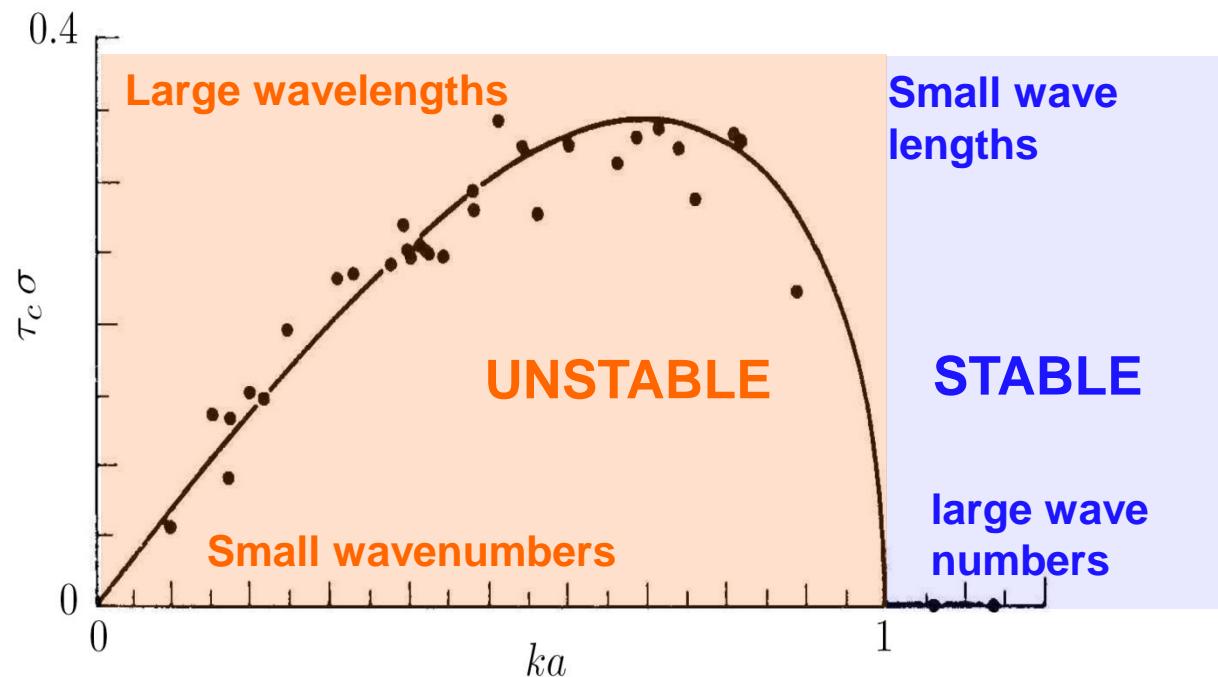
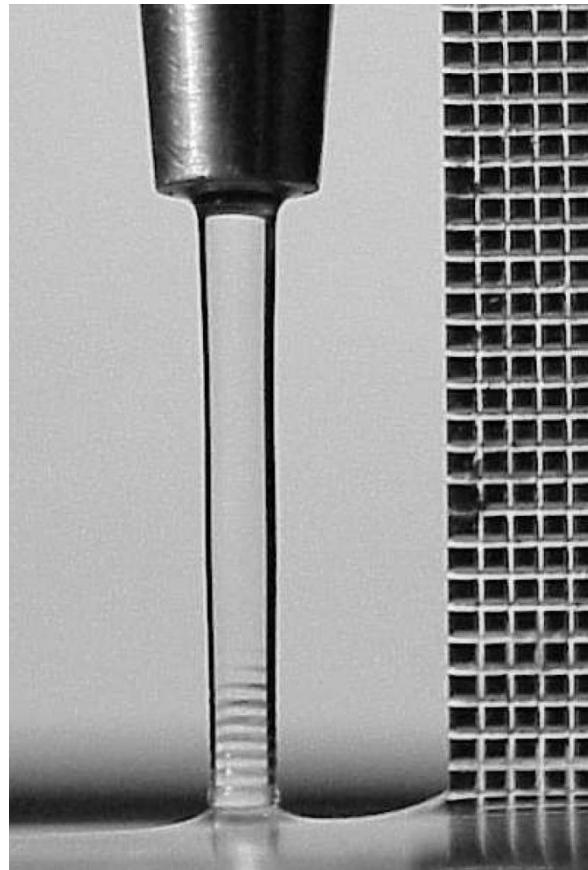


FIG. 2.10 – Taux de croissance $\tau_c \sigma$, avec $\tau_c = \sqrt{\rho a^3 / \gamma}$, de l'instabilité d'un filet fluide non visqueux, et points expérimentaux. D'après (Drazin & Reid 2004).



Surface tension is **destabilizing** as a consequence of the **radial** curvature
Surface tension is **stabilizing** as a consequence of the **axial** curvature