Dispersion relation of water waves in finite depth

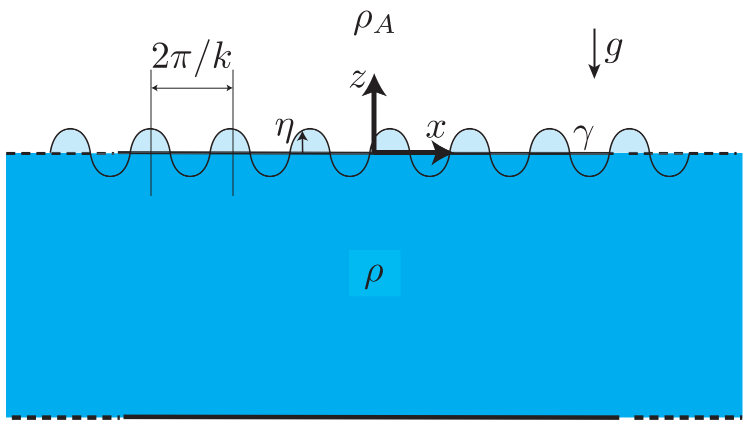

FIG. 1.

I. GOVERNING EQUATIONS AND BOUNDARY CONDITIONS

Let consider a column of water of height h and density ρ that is infinitely extended in the horizontal x-direction and that is surmounted by a semi-infinite layer of air with density ρ_A. The inviscid motion in the two media is governed by the continuity and Bernoulli equations

$$\begin{aligned} \Delta\Phi = 0, \qquad & \frac{\partial \Phi}{\partial t} + \frac{U^2+V^2}{2} + \frac{P}{\rho} + gz = 0, \\ \Delta\Phi_A = 0, \qquad & \frac{\partial \Phi_A}{\partial t} + \frac{U_A^2+V_A^2}{2} + \frac{P_A}{\rho_A} + gz = 0, \end{aligned} \tag{1}$$

where z is the vertical direction aligned with gravity g and P and P_A indicate the pressure in the liquid and air phase respectively. In addition, Φ and Φ_A are the velocity potential and their gradient corresponds to the velocity components

$$\nabla\Phi = \begin{pmatrix} U \\ V \end{pmatrix}, \qquad \nabla\Phi_A = \begin{pmatrix} U_A \\ V_A \end{pmatrix}. \tag{2}$$

The governing equations (1) have to be completed by boundary conditions. No penetration at the solid surface and null velocity in the far vertical field yield

$$\left.\frac{\partial \Phi}{\partial z}\right|_{-h} = 0, \qquad \Phi_A|_{z\to\infty} = 0. \tag{3}$$

In addition, kinematic and dynamic boundary conditions have to be imposed at the liquid-air interface $\eta(x)$

$$\begin{aligned} & \frac{\partial \eta}{\partial t} = V - U\frac{\partial \eta}{\partial x}, \qquad \frac{\partial \eta}{\partial t} = V_A - U_A\frac{\partial \eta}{\partial x} \\ & P - P_A = -\gamma\chi(\eta), \quad \text{with the curvature } \ \chi(\eta) = \frac{\frac{\partial^2\eta}{\partial x^2}}{\left(1+(\frac{\partial \eta}{\partial x})^2\right)^{3/2}}, \end{aligned} \tag{4}$$

where γ is the surface tension at the water-air interface.

We note that the governing equations of water and air are coupled through the pressure term in the dynamic condition. However, if we consider that the air remains at rest at atmospheric pressure we can decouple the equations and consider only the water phase.

II. BASE STATE

By imposing the stationarity of the motion we get the following base-state solution for the water phase

$$\Phi = 0, \quad U = 0, \quad V = 0, \quad \eta = 0, \quad P = -\rho g z. \tag{5}$$

III. LINEARIZED EQUATIONS AND BOUNDARY CONDITIONS

Let perturb the base-state solution with an unsteady small perturbation of size $\epsilon \ll 1$:

$$\Phi = 0 + \epsilon\phi, \quad U = 0 + \epsilon u, \quad V = 0 + \epsilon v, \quad \eta = 0 + \epsilon\sigma, \quad P = -\rho g z + \epsilon p \tag{6}$$

By injecting this expansion in the governing equations we get the following linearized equations and b.c.:

$$\begin{aligned}
&\text{Continuity :} && \Delta\phi = 0 \\
&\text{No penetration :} && \left.\frac{\partial\phi}{\partial z}\right|_{-h} = 0 \\
&\text{Kinematic condition :} && \left.\frac{\partial\phi}{\partial z}\right|_{z=0} = \frac{\partial\sigma}{\partial t} \\
&\text{Dynamic condition :} && \begin{cases} -\rho g\sigma + p|_{z=0} = -\gamma\frac{\partial^2\sigma}{\partial x^2}z \\ \left.\frac{\partial\phi}{\partial t}\right|_{z=0} + \frac{p_{z=0}}{\rho} = 0 \end{cases} \Rightarrow -\rho g\sigma - \rho\left.\frac{\partial\phi}{\partial t}\right|_{z=0} = -\gamma\frac{\partial^2\sigma}{\partial x^2}
\end{aligned} \tag{7}$$

IV. NORMAL MODE EXPANSION

Due to the linearity of the equations (7) and the horizontal invariance of the base-flow (6) we can seek for a solution of the type

$$\phi = f(z)e^{i(kx-\omega t)}, \quad \sigma = Be^{i(kx-\omega t)} \tag{8}$$

where k is the wavenumber of the wave and ω is its frequency. The function $f(z)$ and the constant B are unknown. Since $f(z)$ needs to satisfy the Laplace equation it is an harmonic function of the kind $f(z) = \alpha e^{kz} + \beta e^{kz}$. By imposing null derivative at the bottom boundary (homogeneous Neumann condition) in order to satisfy the condition of no-penetration we get:

$$f(z) = A\cosh(k(h+z)), \tag{9}$$

where A is an arbitrary constant. Therefore, the following ansatz

$$\phi = A\cosh(k(h+z))e^{i(kx-\omega t)}, \quad \sigma = Be^{i(kx-\omega t)}, \tag{10}$$

already satisfies continuity equation and no-penetration condition at $z = -h$.

V. DISPERSION RELATION

We need now to impose kinematic and dynamic condition to be satisfied. By substituting the ansatz (10) in the last two equations of (7) we obtain:

$$\begin{aligned} kA\sinh(kh) &= i\omega B \\ -\rho g B - i\omega\rho\cosh(kh)A &= \gamma k^2 B \end{aligned} \tag{11}$$

which is an homogeneous system of two equations in two unknowns, A and B. Therefore, in order to avoid the trivial solution $A = 0,\ B = 0$, we require the determinant of

$$\begin{bmatrix} k\sinh(kh) & -i\omega \\ -i\omega\rho\cosh(kh) & -\rho g - k^2\gamma \end{bmatrix} \tag{12}$$

to be null, yielding the famous dispersion relation:

$$\omega^2 = kg\left(1 + k^2\frac{\gamma}{\rho g}\right)\tanh(kh). \tag{13}$$

Note as the frequency is given by the product of a gravity term, a capillary term and a term accounting for the solid confinement at the bottom.