Exam ME444 Instability
20 janvier 2020

- All paper documents allowed. No electronic device tolerated.
- Duration 3h.
- Write comprehensively.
- Don't hesitate to accept the result of a question to move forward.
- Number each (say the i-st) all your n sheets as (i/n) and tag it with your name.
- There will be no answer to any question during the exam. If you find typos and errors in the exam questions, please write it as part of your responses.

Initial growth in non-normal systems [12pts]

We consider the following model linear system

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\epsilon & 0 \\ 1 & -2\epsilon \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \tag{1}$$

with initial condition $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ at $t = 0$ and $0 < \epsilon \ll 1$.

1. [1pt] What are the eigenvalues of this system ? Is this dynamical system linearly stable or unstable ?
2. [2pts] What are the eigenvectors ? Are they orthogonal for the Cartesian scalar product ?
3. [1pt] The solution (which you don't have to demonstrate) for $y(t)$ writes

$$y(t) = (y_0 - x_0/\epsilon)\exp(-2\epsilon t) + x_0/\epsilon \exp(-\epsilon t). \tag{2}$$

   What is the corresponding (and simpler) expression for $x(t)$?
4. [1pt] What are the limit values of $x(t \to \infty)$ and $y(t \to \infty)$?
5. [3pts] Considering now the short time limit $t \ll 1/\epsilon$, it is easy to show that in this limit

$$x(t) = x_0(1 - \epsilon t + \mathcal{O}(\epsilon^2)) \tag{3}$$

   What is the Taylor expansion of $y(t)$ at short times ?
6. [2pts] Deduce that for $t < 1/\epsilon$, $E = (x^2 + y^2)^{1/2}$ can grow to reach approximately x_0/ϵ. What is the scaling of the gain in terms of ϵ ? What is the nature of the initial growth ? Exponential or algebraic ?
7. [1pt] Plot the typical evolution of the curve $E(t)$ in lin-lin AND in lin-log for a few values of ϵ.
8. [1pt] What is the necessary condition on the linear operator for such transient growth phenomena to happen ? What implications can such growth have on the stability of flows ? You may want to give a few examples.

Stratified shear flow : from Rayleigh's theorem to Richardson's criterion [28pts]

We consider a stratified shear flow of velocity profile $U(z)\mathbf{e}_x$ and temperature profile $T(z)$ between two plates in $z = z_a$ and $z = z_b$, with $z_a < z_b$. We neglect the dependence along y, the transverse direction. We neglect viscous effects. The thermal expansion coefficient $\beta > 0$ connects the density variation with the temperature profile

$$\rho = \rho_0 - \beta(T - T_0) \tag{4}$$

We consider normal modes of the form $\exp(i(kx - \omega t))$. We denote the velocity components along x by $U + \epsilon u$, along z by $0 + \epsilon v$, the temperature $T + \epsilon\theta$ and pressure $P = 0 + \epsilon p$.

1. [1pt] In the case of constant temperature, what theorem justifies ignoring the transverse direction to determine the stability conditions of this flow ?
2. [1pt] Show that the linearized equations write

$$(-i\omega + ikU)u + \frac{dU}{dz}v = ik\frac{p}{\rho_0}, \tag{5}$$

$$(-i\omega + ikU)v = -\frac{1}{\rho_0}\frac{dp}{dz} + \frac{\beta g}{\rho_0}\theta, \tag{6}$$

$$(-i\omega + ikU)\theta + \frac{dT}{dz}v = 0, \tag{7}$$

$$\frac{dv}{dz} + iku = 0. \tag{8}$$

3. [1pt] What is the name of this last equation ? What does it physically express ?
4. [1pt] What is the name of the hypothesis which justifies its use while the density of the flow is not constant ?
5. [1pt] We next assume that the base temperature profile a growing function of z i.e. $dT/dz = T' > 0$. Is the flow stably stratified ? Do you expect any Rayleigh-Bénard instability to happen ?
6. [3pts] We note $U'' = d^2U/dz^2$ and denote the derivation operator with respect to z by D. Define and interpret a classical auxiliary variable $\psi(z)\exp(i(kx - \omega t))$ to obtain the so-called Taylor-Goldstein equation

$$(-i\omega + ikU)(D^2 - k^2)\psi - ikU''\psi = \frac{k^2N^2}{-i\omega + ikU}\psi \tag{9}$$

   where N is the so-called Brunt-Väisälä frequency $N^2 = \frac{\beta g U'}{\rho_0}$.
7. [1pt] Check that, from a dimensional analysis point of view, N is indeed a frequency. To what limit equation discussed in class does the Taylor-Goldstein equation simply in absence of stratification when $N = 0$?
8. [1pt] Is this a classical or a polynomial eigenvalue problem ?
9. [1pt] Separating the complex frequency in its real and imaginary part ($\omega = \omega_r + i\omega_i$), remind the condition for linear instability.

10. [3pts] Using a similar technique as that used in class, show that a necessary condition for the flow to be unstable is

$$U'' = \frac{2(-\omega_r + kU)kN^2}{\omega_i^2 + (-\omega_r + kU)^2}. \tag{10}$$

N.B. As an intermediate step, you might first divide the Taylor-Goldstein equation by $(-i\omega + ikU)$, remember that $|\psi|^2 = \psi^*\psi$ (where * denotes complex conjugation) and that for any complex number $a = a_r + ia_i$, $1/a = \frac{a_r - ia_i}{a_r^2 + a_i^2}$.

Why is that NOT a predictive criterion ?

11. [2pts] BEWARE, this is a calculus-intensive question. Consider now the new variable

$$\chi = \frac{\psi}{\sqrt{-i\omega + ikU}} \tag{11}$$

Show that the Taylor-Goldstein equation becomes

$$D[(-i\omega + ikU)D\chi] + \left[k^2 \frac{U'^2/4 - N^2}{-i\omega + ikU} - \frac{ikU''}{2} - k^2(-i\omega + ikU)\right]\chi = 0 \tag{12}$$

12. [3pts] By multiplying by $\chi*$ and integrating, show that a necessary conditions for the instability is given in terms of the Richardson number Ri is that $\exists y \in]z_a; z_b[$ such that

$$Ri = \frac{N(z)^2}{U'(z)^2} < \frac{1}{4}. \tag{13}$$

13. [1pt] What is the physical meaning behind this criterion ? Is the temperature gradient stabilizing or destabilizing ?

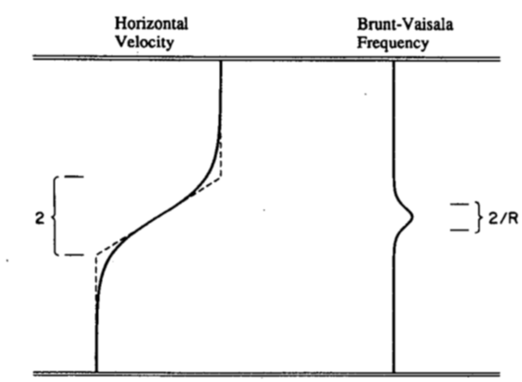

FIGURE 1 – Solid curves represent the profile of velocity and background stratification for $R = 3$.

14. [1pt] The dispersion relation of a velocity and temperature distribution represented in figure 1 (by its associated Brunt-Väisälä frequency N^2) have been computed by Smyth and Peltier in 1989. They have considered a localized temperature gradient of extension R, yielding

$$N^2(z) = J(1 - \tanh^2(Rz)), \tag{14}$$

where $J > 0$, while the velocity was chosen as

$$U(z) = \tanh(z). \tag{15}$$

Looking at the profile of N^2, comment on the stably or unstably stratified nature of the flow. Sketch a representative temperature distribution.

15. [1pt] The authors have used a numerical discretization method to determine the eigenvalues of the Taylor-Goldstein equation. With a central second order differential finite difference scheme and $N+1$ points regularly spanning the interval $[z_a; z_b]$, how many interior points are there and how many eigenvalues are expected ?
16. [1pt] We first consider $R = 1$, $Ri(z)$ is depicted in figure 2a for several values of J. Remembering the definition of the Richardson number $Ri(z)$, what is the value of its minimum Ri_{min} as a function of J ? Note that $\tanh'(z) = 1 - \tanh^2(z)$.

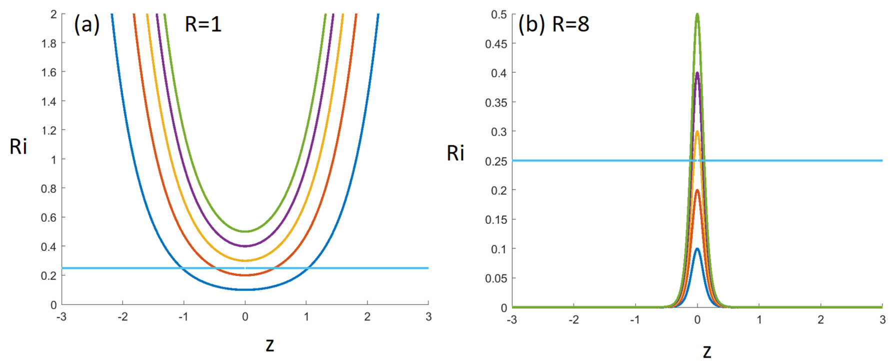

FIGURE 2 – Richardson number $Ri(z)$ as a function of z for $J = 0.1$(blue), ..., 0.5(green) for $R = 1$ (a) and $R = 8$ (b). The 1/4-limit is also depicted.

17. [1pt] The isocontours of the growth-rate of the most unstable eigenvalues are depicted in figure 3a in the k-J plane. Is the previously obtained necessary condition for instability violated ?
18. [1pt] For $R = 8$, $Ri(z)$ is depicted in figure 2b for several values of J. For which values of J do you expect the flow to be unstable ?
    The dominant eigenvalues have been computed by Smyth and Peltier (1989) in figure 4 for $R = 8$ and $J = 0.2, ..., 0.8$. Do these results confirm your previous answer ?
19. [1pt] Figure 3a refers to the Kelvin-Helmholtz instability. What is the consequence of increasing the strength of the stratification on its growing rates ? Propose a physical interpretation.
    On figure 4b, the two unstable branches associated to the Kelvin-Helmholtz instability switch to the so-called Holmboe instability for $k \geq\approx 0.25$. What is the consequence of increasing the strength of the stratification on Holmboe's growing rates ? And on its most unstable wavenumber ? Comparing figure 3a and 3b, what seems to be a necessary ingredient for an Holmboe instability to occur ?
20. [2pts] Consider the frequency curves on figure 4 (no dashed line means $\omega_r = 0$). From visual inspection only, can you predict if the Kelvin-Helmotz instability will be convective or absolute ? Why ? Same questions for Holmboe instabilities.

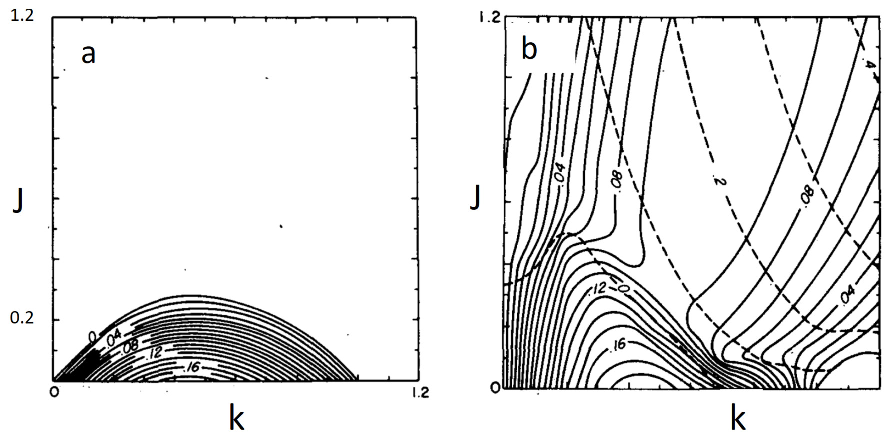

FIGURE 3 – Isocontours of the growth-rate (full line) and frequency (dashed line) of the most unstable eigenvalues for $R = 1$ (a) and $R = 8$ (b).

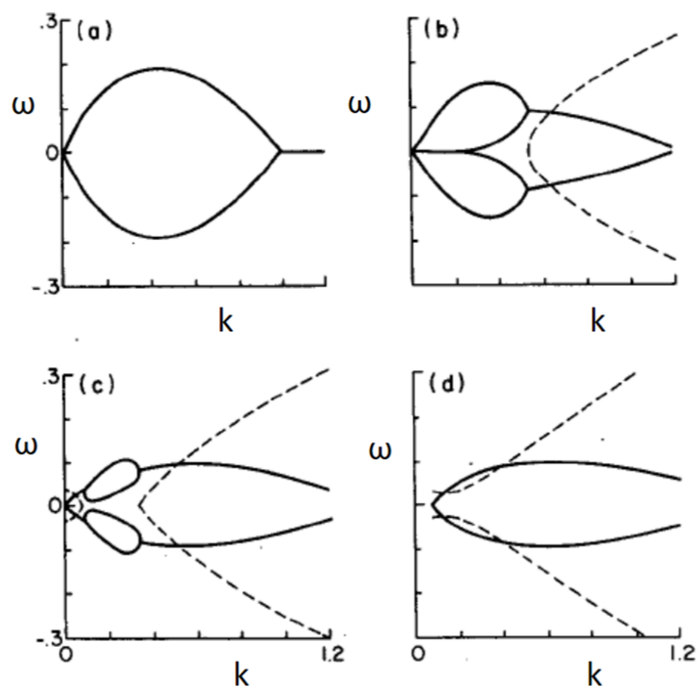

FIGURE 4 – Growth-rate (full line) and frequency (dashed line) of the dominant unstable modes for $R = 8$ and different values of $J = 0.2, 0.4, 0.6$ and 0.8.