22.05 Reactor Physics - Part Twenty-Seven

Analytic Solution of Point Kinetics Equations[1]

1. Analytic solutions of the point kinetic equations allow one to observe the impact of major reactivity changes on the behavior of both the neutron and precursor populations. In particular, these cases are of interest:

   - Large negative reactivity (reactor scram)
   - Small positive reactivity such that $\rho << \bar{\beta}$
   - Positive reactivity such that $\rho > \bar{\beta}$

   We begin by reviewing some of the fundamentals.

2. <u>Review:</u>

Neutrons may be classified by when they appear following the fission event. Prompt neutrons are produced at the time of fission. They have a lifetime of about 1×10^{-4} s with lifetime defined as the interval for the neutrons to appear, thermalize, and be absorbed. Delayed neutrons are produced from precursors which are fission products that undergo a beta decay to a daughter nuclide that emits a neutron. The average lifetime is 12.2 seconds with most of this time being that required for the beta decay.

The fraction of neutrons that are delayed at birth is denoted as "beta" or β. For reactors with D_2O or Be reflectors, the fraction of delayed neutrons is larger because of photoneutrons produced by a (γ, n) reaction on deuterium. Fission products are the source of the photons. Delayed neutrons are born at lower energies than fast neutrons. Hence, the percent of the delayed neutrons lost in the slowing down process is smaller than that of the prompt neutrons. The fraction of delayed neutrons at thermal energies is denoted as $\bar{\beta}$. It is 0.0065 for light water reactors and 0.0075 for D_2O reflected reactors.

Reactivity is defined as the fractional change in a neutron population per generation. If reactivity is negative, the reactor is subcritical. If it is zero, the reactor is critical, and if it is positive, the reactor is super critical. In order to raise a reactor's power level, one increases the reactivity by removal of an absorber, allows the power to rise, and then reinserts the absorber so as to return the

[1] Material in Sections 4-8 is paraphrased from Henry pp 307-312. Portions that are verbatim are indicated by quotations.

reactivity to zero. It is absolutely essential that the excess reactivity present in a reactor during such transients be significantly less than the delayed neutron fraction. That is, we desire $\rho << \overline{\beta}$. For that case, the delayed neutrons, with their relatively long lifetime, are the rate determining step.

If the reactivity approaches or exceeds the delayed neutron fraction, the reactor is referred to as being "prompt-critical." This means that prompt neutrons alone are sufficient to sustain the neutron chain reaction. The reactor power rises uncontrollably and the reactor is destroyed. (Note: A misconception about prompt criticality with some arguing that safety itself is step-like. That is, if the reactivity is less than $\overline{\beta}$, the reactor is safe. If $\rho > \overline{\beta}$, all is lost. This is myopic reasoning. There is no abrupt transition between safe and unsafe. As reactivity increases, the period shortens, power rises more rapidly. If the rate of rise exceeds the response time of the safety system, the reactor will be damaged. This can occur for reactivities that are less than the delayed neutron fractions.)

Reactor period is defined as the power level divided by the rate of change of power. So, a period of infinity is steady-state and one equal to a small positive number represents a rapid power rise. Period is readily obtained from nuclear instruments because a neutron detector can yield signals that correspond to both the power level and its rate of change. Reactor operators control a reactor by adjusting reactivity to obtain a certain reactor period. Hence, it is desirable to have equations that relate period to reactivity. One very effective way of doing this is the dynamic period equation which in its full form (three terms in the denominator) is a relatively new relation. The previous approach, which remains invaluable as well, is an analytic solution of the kinetics equations and a relation known as the "in-hour" equation. The latter can be shown to be a special case of the dynamic period equation.

The definition of reactor period is:

$$\tau(t) = \frac{n(t)}{dn(t)/dt}$$

If the period is constant, then

$$n(t) = n_o e^{t/\tau}$$

Where n is reactor power.

3. Amplitude Function:

In Part 26 of these notes, the kinetics equations were written in terms of the reactor power, n(t). This is actually incorrect, although a good approximation. A

formal derivation of the kinetics equations would incorporate a key assumption – the shape of the neutron scalar flux, $\phi(r,E)$ does not change during the transient. We allow its amplitude to change but not its shape. The value or importance of a neutron to the chain reaction depends on its location in the core with ones in the center being more valuable than ones in the periphery because the center ones have a higher likelihood of causing further fissions. So, in order to avoid changes in importance, one imposes the requirement that the flux shape remain a constant. Thus, we define an amplitude function, T(t), which is a weighted integral (weighted by importance) of the neutrons present.

$$T(t) \equiv \int_{\text{reactor}} dv \int_0^\infty dE \; W(r,E) \frac{1}{\nu(E)} \phi(r,E,t)$$

Where $W(r,E)$ is the importance. For $W(r,E) = 1$, $T(t) = n(t)$. We will write the kinetics equations here using T(t) instead of n(t). (Note: To a first approximation, the importance function has the same shape as the flux.)

4. Analytic Solution of One-Group Point Kinetics:

There are many fission products that act as precursors. They are normally grouped by half-life into six groups. To treat all six groups in an analytic solution is not practical. So, we assume one group. Thus, we will see the impact of the prompt and the delayed neutrons but not the transition from one group of delayeds to another. The point kinetics equations, with one group that has a decay constant λ, are:

$$\dot{T}(t) = \frac{(\rho - \bar{\beta})}{\ell^*} T(t) + \lambda C(t)$$

$$\dot{C}(t) = \frac{\bar{\beta}}{\ell^*} T(t) - \lambda \, C(t)$$

Where we take ρ as fixed (i.e., a step function).

In matrix form, these become:

$$\frac{d}{dt}\begin{bmatrix} T(t) \\ C(t) \end{bmatrix} = \begin{bmatrix} \frac{\rho - \beta}{\Lambda} & \lambda \\ \frac{\beta}{\Lambda} & -\lambda \end{bmatrix} \begin{bmatrix} T(t) \\ C(t) \end{bmatrix}$$

Where for ease of notation we have replaced $\bar{\beta}$ by β and ℓ^* by Λ. Those can be solved by LaPlace transforms or by assuming a particular response. We take the latter here. "Assume particular solutions such that $T(t) \approx C(t) \approx \exp(\omega t)$ and

then take linear combinations of these particular solutions to find the general solution obeying the initial conditions of the problem." (Henry, p. 307) The parameter ω is the inverse of the reactor period. That is, $\omega = 1/\tau$.

"If the assumed exponential behavior is correct, it follows that dT(t)/dt = ωT(t) and dC(t)/dt = ωC(t). These relationships can be valid only if:

$$\begin{bmatrix} \frac{\rho-\beta}{\Lambda}-\omega & \lambda \\ \frac{\beta}{\Lambda} & -(\omega+\lambda) \end{bmatrix} \begin{bmatrix} T(t) \\ C(t) \end{bmatrix} = 0$$

This relationship, in turn, implies that either T = C = 0 or that the determinant is zero. If the latter, then the above relation implies that for each of the two roots ω_i, T(t) and C(t) have a fixed ratio. Thus,

$$\left.\frac{C(t)}{T(t)}\right|_i = -\frac{(\rho-\beta)/\Lambda-\omega_i}{\lambda} = \frac{\beta}{\Lambda(\omega_i+\lambda)}$$

It follows that the general solution may be written as:

$$\begin{bmatrix} T(t) \\ C(t) \end{bmatrix} = a_1 \begin{bmatrix} 1 \\ \frac{\beta}{\Lambda(\omega_1+\lambda)} \end{bmatrix} \exp(\omega_1 t) + a_2 \begin{bmatrix} 1 \\ \frac{\beta}{\Lambda(\omega_2+\lambda)} \end{bmatrix} \exp(\omega_2 t)$$

Where a_1 and a_2 are constants to be determined by the initial conditions." (Henry, pp. 307-308)

To continue with the objective of finding the roots ω_1 and ω_2, we evaluate the determinant and obtain:

$$\omega^2 - \left(\frac{\rho-\beta}{\Lambda}-\lambda\right)\omega - \frac{\lambda\rho}{\Lambda} = 0$$

or, because $(\rho-\beta)/\Lambda$ is much greater than λ

$$\omega^2 - \left(\frac{\rho-\beta}{\Lambda}\right)\omega - \frac{\lambda\rho}{\Lambda} = 0$$

For appropriate transients, reactivity will be much less than the delayed neutron fraction. In that case, it is known from prior analysis that one root will be very

large in magnitude and the other very small. Let ω_1 be the large root. Then ω_1^2 will greatly exceed $\lambda\rho/\Lambda$ and we obtain:

$$\omega_1 \cong -\frac{\beta-\rho}{\Lambda}$$

For the second root, ω_2, we have that ω_2^2 is much less than $\lambda\rho/\Lambda$ and hence ω_2^2 can be neglected. Thus,

$$\omega_2 \cong \frac{\lambda\rho}{\beta-\rho}$$

"The general solution is therefore:

$$\begin{bmatrix} T(t) \\ C(t) \end{bmatrix} \approx a_1 \begin{bmatrix} 1 \\ \dfrac{-\beta}{\beta-\rho} \end{bmatrix} \exp\left(-\frac{\beta-\rho}{\Lambda}t\right) + a_2 \begin{bmatrix} 1 \\ \dfrac{\beta-\rho}{\Lambda\lambda} \end{bmatrix} \exp\left(\frac{\lambda\rho}{\beta-\rho}t\right)$$

Where we have once more neglected λ in comparison with ω_1.

We see immediately from this result that if, ρ=0, the first term dies away quickly $\left(-\beta/\Lambda \cong .65\right)$ and steady-state values of T(t) and C(t) are given by:

$$\begin{bmatrix} T_0 \\ C_0 \end{bmatrix} \approx a_2 \begin{bmatrix} 1 \\ \dfrac{\beta}{\Lambda\lambda} \end{bmatrix} = T_0 \begin{bmatrix} 1 \\ \dfrac{\beta}{\Lambda\lambda} \end{bmatrix}$$

Given that C_0 and T_0 are approximately the number of precursors and neutrons in the reactor, it is evident that the precursor population far outweighs that of the neutrons. For $\beta = 0.0065$, $\lambda = 0.076$, $\Lambda = 1 \times 10^{-4,}$, the ratio is ~850:1.

The constants a_1 and a_2 can be found by setting t = 0 in the general solution above and requiring the result to give the steady state values for T_0 and C_0. Thus,

$$a_1 = -\rho T_0/(\beta-\rho)$$

$$a_2 = \beta T_0/(\beta-\rho)$$

And we have for the complete solution (one group)

”

$$\begin{bmatrix} T(t) \\ C(t) \end{bmatrix} \approx T_0 \left\{ \begin{bmatrix} \dfrac{-\rho}{\beta-\rho} \\ \dfrac{\rho\beta}{(\beta-\rho)^2} \end{bmatrix} \exp\left(-\frac{\beta-\rho}{\Lambda}t\right) + \begin{bmatrix} \dfrac{\beta}{\beta-\rho} \\ \dfrac{\beta}{\Lambda\lambda} \end{bmatrix} \exp\left(\frac{\lambda\rho}{\beta-\rho}t\right) \right\}$$

(Henry, pp. 309-310)

5. Reactor Shutdown

Assume β=0.0065, $\Lambda = 1 \times 10^{-4}$, and $\lambda = 0.076\ s^{-1}$. Let ρ=-0.05 $\Delta k/k$ (or -7.7 Beta). We obtain

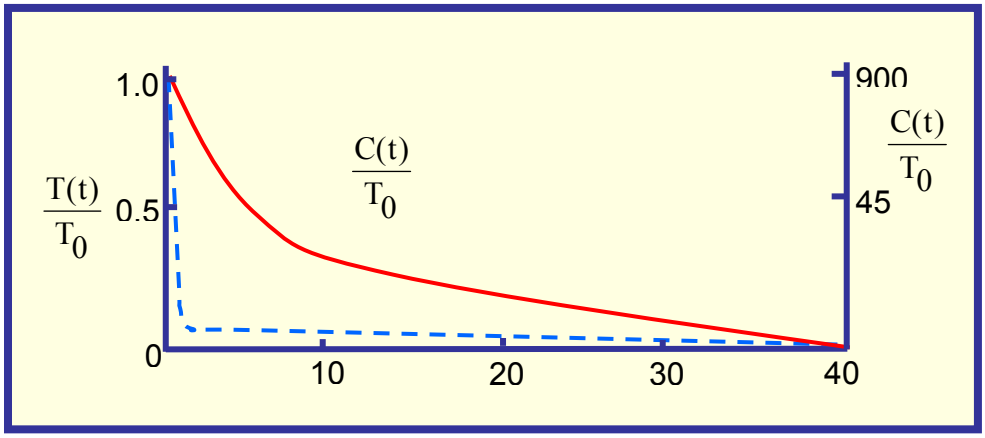
$$\begin{bmatrix} T(t) \\ C(t) \end{bmatrix} \approx T_0 \left\{ \begin{bmatrix} 0.88 \\ 0.10 \end{bmatrix} \exp(-565t) + \begin{bmatrix} 0.12 \\ 855 \end{bmatrix} \exp(-0.067t) \right\}$$

or

$$T(t) = 0.88\ T_0\ e^{-565t} + 0.12\ e^{-0.067t}$$

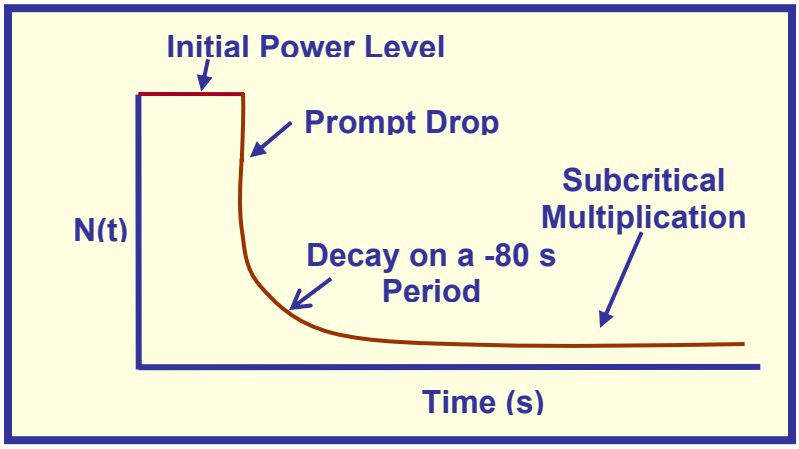
$$C(t) = 0.10\ T_0\ e^{-565t} + 855\ e^{-.067t}$$

It is apparent that 88% of the neutron population is gone almost immediately (0.01 s) while the precursor population is virtually unchanged. This is because the vast majority of neutrons have a 1×10^{-4} s lifetime while the precursors have a 12.2 s one. This initial sudden decrease in the neutron population is called the “prompt drop.” Thereafter, our equation predicts that both the remaining neutrons and the precursors will die off at a 15 s period ($\tau = \omega_2 = 1/0.067$). The following figure illustrates the behavior:



Neutron and precursor population following a change in reactivity of -0.05 from an initially critical state – one-group model.

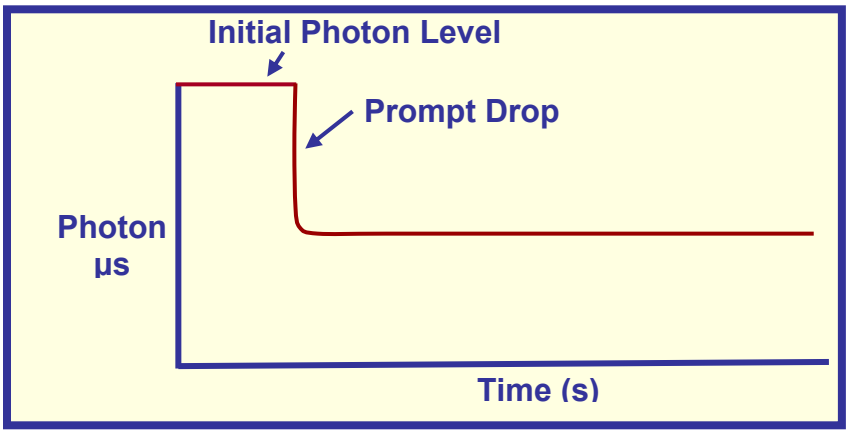
Is this correct? We assumed one group of delayed neutrons. In reality there are six. The observed transient for reactor power following a scram shows three distinct regions:



Neutron population following a scram (not to scale).

The -80 s period is the result of the longest-lived precursor group ($t_{1/2}$ = 55 s). The short-lived precursor groups die out first eventually leaving only the longest one. Once that group dies out as well, the reactor power level is given by the subcritical multiplication relation with photo neutrons as the source. The prompt drop occurs in a small fraction of a second, the decay on the -80 s period in a few minutes, the subcritical level can last weeks to months.

The photon population in a nuclear reactor would also experience a prompt drop and then level off without any decay in a -80 s period. At full power, the photon to neutron ratio is 1:1. After shutdown, the ratio is 100:1. Photons vastly outnumber neutrons. This means that nuclear instruments must be designed to screen out photons. One way to tell if a detector is properly calibrated is to verify that it indicates a – 80 s period following a scram. The following figure illustrates the behavior of the photons.



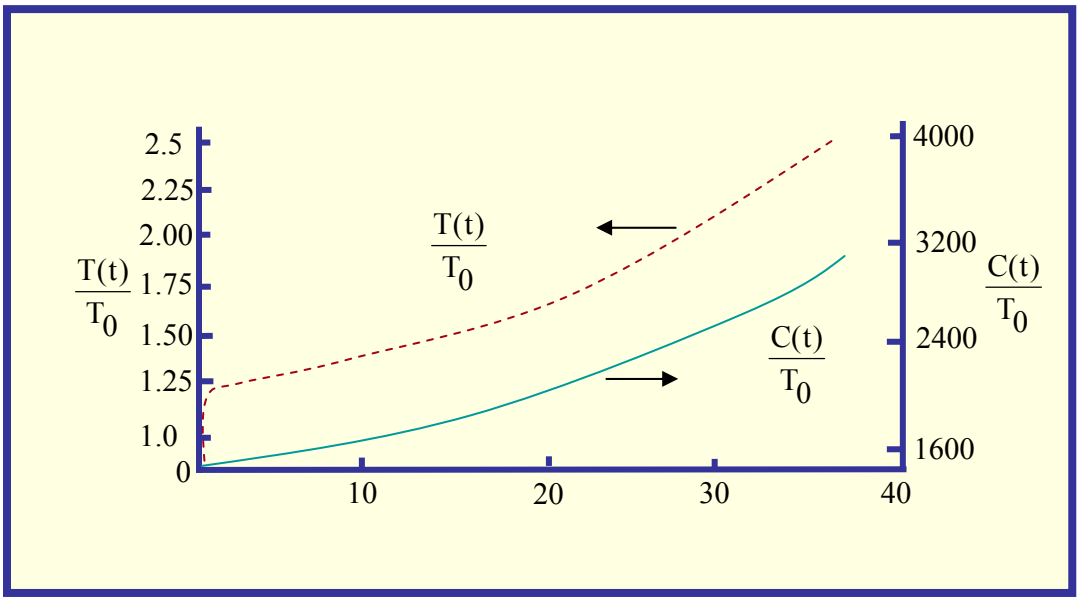
Photon population following a scram (not to scale)

6. Supercritical Transient:

Assume a step increase in reactivity of 0.0015 Δk/k. Thus,

$$\begin{bmatrix} T(t) \\ C(t) \end{bmatrix} \approx T_0 \left\{ \begin{bmatrix} -0.30 \\ 0.39 \end{bmatrix} \exp(-20t) + \begin{bmatrix} 1.30 \\ 855 \end{bmatrix} \exp(0.02t) \right\}$$

The following figure illustrates the transient.



Neutron and precursor population following a change in reactivity of +0.0015 from an initially critical state: One-group model.

This result is very similar in shape to the one given in the previous section of these notes. "The initial rapid rise of the prompt neutrons is called the "prompt jump." It is the start of a neutron runaway but it can't continue because $\rho << \beta$ and the delayed neutrons are needed in order to sustain the chain reaction. Note that mathematically the prompt jump is the result of the dying out of a negative term." (Henry, p. 311)

7. Prompt-Jump Approximation:

"The prompt jump occurs so quickly that, in many cases, it can be assumed to be instantaneous. Physically such an assumption implies that T(t) is in instant equilibrium with the $C_i(t)$. Mathematically this situation can be expressed by neglecting dT(t)/dt in comparison to $[(\rho - \beta)/\Lambda]T$. Such neglect simplifies the point kinetics equations." (Henry, p. 311) Thus, they become for t > 0,

$$0 = \frac{\rho - \beta}{\Lambda} T(t) + \lambda C(t)$$

$$\frac{dC(t)}{dt} = \frac{\beta}{\Lambda} T(t) - \lambda C(t)$$

Thus, from the first of the above equations:

$$T(t) = \Lambda\lambda C(t)/(\beta - \rho)$$

And, upon substitution of this result for T(t) into the second equation:

$$\frac{dC(t)}{dt} = \left(\frac{\beta}{\Lambda}\right)\left(\frac{\Lambda\lambda C(t)}{\beta - \rho}\right) - \lambda C(t)$$

$$= \frac{\beta\lambda C(t) - \beta\lambda C(t) + \rho\lambda C(t)}{\beta - \rho}$$

$$= \frac{\rho\lambda C(t)}{\beta - \rho}$$

Thus, the effect of making the prompt-jump approximation is to reduce the two coupled point kinetics equations to a single first-order differential equation:

$$\frac{dC(t)}{dt} = \frac{\rho\lambda}{\beta - \rho} C(t)$$

The solution for C(t) and T(t) for t > 0 is, then,

$$C(t) = C_0 \exp\left(\frac{\rho\lambda}{\beta - \rho} t\right),$$

$$T(t) = \frac{\lambda\Lambda}{\beta - \rho} C(t) = \frac{\lambda\Lambda}{\beta - \rho} C_0 \exp\left(\frac{\rho\lambda}{\beta - \rho} t\right)$$

Note that the quantity $\rho\lambda/(\beta - \rho)$ is the inverse of the dynamic period as defined in the previous section of these notes for the case where the rate of change of reactivity ($\dot{\rho}$) is zero. That is,

$$\omega = \frac{\beta - \rho}{\lambda\rho}$$

8. Super Prompt-Critical Transient:

"For this case assume that $\rho = 0.0115\ \Delta k/k$. Recall that β is $0.0065\ \Delta k/k$. Thus,

$$\begin{bmatrix} T(t) \\ C(t) \end{bmatrix} \approx T_0 \left\{ \begin{bmatrix} 2.30 \\ 2.99 \end{bmatrix} \exp(50.0\,t) + \begin{bmatrix} -1.30 \\ 855 \end{bmatrix} \exp(-0.17\,t) \right\}$$

The reactor power rises by a factor of 340 in the first tenth of a second and it will be destroyed. This is one reason, a very major reason, why reactors are designed with negative feedback effects. We discuss these later. For the present, we note their importance." (Henry, p. 312)