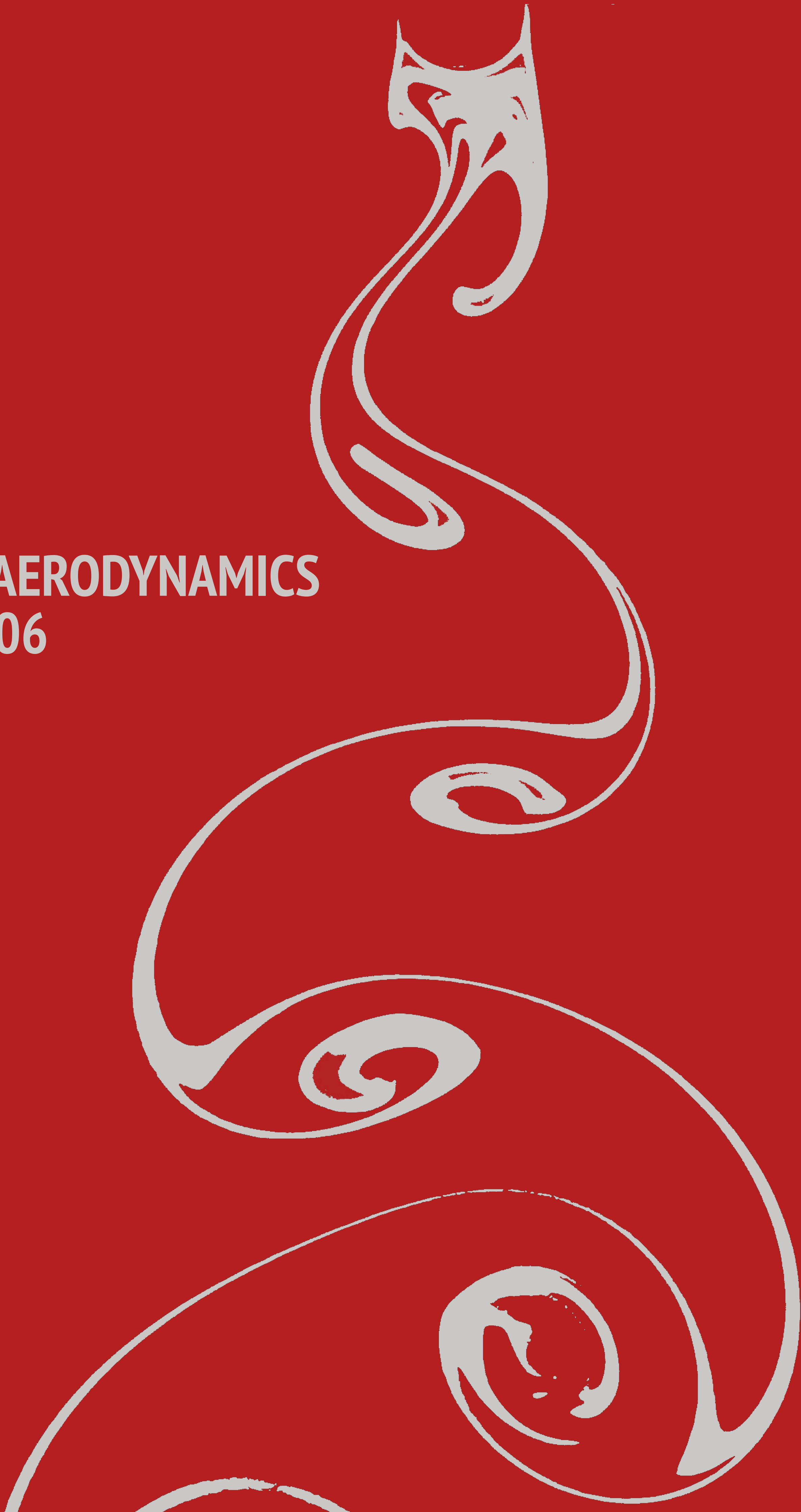
EPFL

ME-445 AERODYNAMICS
Exercise 06
Week 5



Formula sheet

Cylindrical coordinates

$$\nabla\vec{u} = \left(\frac{\partial v_r}{\partial r}, \frac{1}{r}\frac{\partial v_\theta}{\partial \theta}, 0\right)$$

$$\nabla\cdot\vec{u} = \frac{1}{r}\frac{\partial (r v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta}$$

$$\nabla\times\vec{u} = \left(0, 0, \frac{1}{r}\left[\frac{\partial (r v_\theta)}{\partial r} - \frac{\partial v_r}{\partial \theta}\right]\right)$$

Potential flow

$$v_r = \frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \quad v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r}$$

Uniform parallel flow $w = \phi + i\psi = U_\infty e^{-i\alpha} z$

Potential vortex in z_0 $w = -\dfrac{i\gamma}{2\pi}\ln(z - z_0)$

Point source or sink in z_0 $w = \dfrac{Q}{2\pi}\ln(z - z_0)$

Source-sink doublet in z_0 $w = \dfrac{\mu}{2\pi(z - z_0)}$

$$\frac{dw}{dz} = u - iv$$

Milne-Thomson circle theorem:

$$g(z) = w(z) + \overline{w\left(\frac{a^2}{\bar{z}}\right)}$$

Thin airfoil theory

For a camber line with:

$$\frac{dy_c}{dx} = A_0 + \sum_{n=1}^{\infty} A_n \cos n\theta$$

$$\frac{x}{c} = \frac{(1-\cos\theta)}{2}$$

we know:

$$k = 2U_\infty\left[(\alpha - A_0)\frac{\cos\theta + 1}{\sin\theta} + \sum_{n=1}^{\infty} A_n \sin n\theta\right]$$

$$A_0 = \frac{1}{\pi}\int_0^\pi \frac{dy_c}{dx} d\theta$$

$$A_n = \frac{2}{\pi}\int_0^\pi \frac{dy_c}{dx}\cos n\theta d\theta$$

$$C_l = 2\pi\alpha + \pi(A_1 - 2A_0)$$

$$C_{m,1/4} = -\frac{\pi}{4}(A_1 - A_2)$$

$$x_{cp} = \frac{1}{4} + \frac{\pi}{4C_l}(A_1 - A_2)$$

Finite wings with AR=b^2/S

Sign convention:

if induced velocity points downward: $w(y) > 0$, $\alpha_i(y) > 0$

if induced velocity points upward: $w < 0$, $\alpha_i < 0$

Prandtl's lifting-line theory

$$U_\infty \alpha_i(y_0) = w(y_0) = -\frac{1}{4\pi}\int_{-b/2}^{b/2}\frac{(d\Gamma/dy)}{y - y_0}dy$$

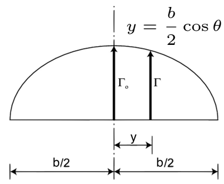
$$\alpha(y_0) = \alpha_{eff}(y_0) + \alpha_i(y_0)$$

Elliptical loading $\Gamma(y) = \Gamma_0\sqrt{1 - \left(\frac{2y}{b}\right)^2}$

$$w = \frac{\Gamma_0}{2b}$$

$$\alpha_i = \frac{C_L}{\pi AR}$$

$$C_{D,i} = \frac{C_L^2}{\pi AR}$$



General loading $\Gamma(\theta) = 2bU_\infty \sum_{n=1}^{\infty} A_n \sin n\theta$

$$w(\theta) = U_\infty \sum_{n=1}^{\infty} nA_n \frac{\sin n\theta}{\sin\theta}$$

$$C_L = \pi A_1 AR$$

$$C_{D,i} = \frac{C_L^2}{\pi AR}(1+\delta) \text{ with } \delta = \sum_{n=2}^{\infty} n\left(A_n/A_1\right)^2$$

Boundary Layer

Flat plate **laminar** boundary layer

$\dfrac{\delta}{x} = \dfrac{5}{\sqrt{Re_x}}$ boundary layer growth

$C_f = \dfrac{1.328}{\sqrt{Re_x}}$ skin friction drag coefficient

Flat plate **turbulent** boundary layer

$\dfrac{\delta}{x} = \dfrac{0.37}{Re_x^{1/5}}$ boundary layer growth

$C_f = \dfrac{0.074}{Re_x^{1/5}}$ skin friction drag coefficient

Miscellanous

θ	0°	30°	45°	60°	90°
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

water

kinematic viscosity $\nu = 1 \times 10^{-6}\,\mathrm{m^2\,s^{-1}}$

density $\rho = 1000\,\mathrm{kg\,m^{-3}}$

air

kinematic viscosity $\nu = 1.5 \times 10^{-5}\,\mathrm{m^2\,s^{-1}}$

density $\rho = 1.2\,\mathrm{kg\,m^{-3}}$

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$
$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$\cos 2\theta = 2\cos^2\theta - 1$$
$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$$
$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

$$\int_0^\pi \cos\theta \mathrm{d}\theta = 0$$

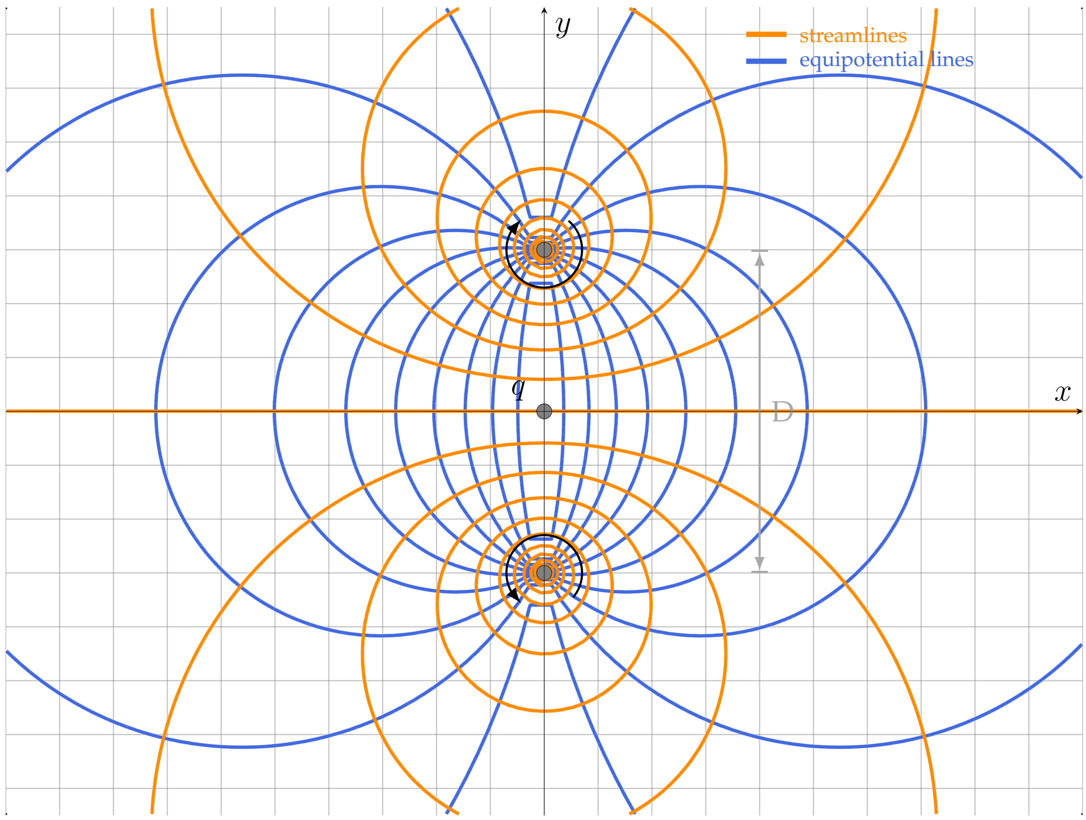
$$\int_0^\pi \sin\theta \mathrm{d}\theta = 2$$

$$\int_0^\pi \cos^2\theta \mathrm{d}\theta = \int_0^\pi \sin^2\theta \mathrm{d}\theta = \frac{\pi}{2}$$

$$\int_0^\pi \frac{\cos n\theta}{\cos\theta - \cos\theta_1}\mathrm{d}\theta = \pi\frac{\sin n\theta_1}{\sin\theta_1} \qquad n = 0, 1, 2, \ldots$$

$$\int_0^\pi \frac{\sin n\theta \sin\theta}{\cos\theta - \cos\theta_1}\mathrm{d}\theta = -\pi\cos n\theta_1 \qquad n = 1, 2, 3, \ldots$$

1. Consider a clockwise and a counterclockwise potential vortex of strength G along y separated by a distance D.



(a) Find the resultant complex potential $w(z)$.

Solution:

With $(0,0)$ in the middle between the vortices:
$w(z) = \frac{iG}{2\pi}\ln\left(z - i\frac{D}{2}\right) - \frac{iG}{2\pi}\ln\left(z + i\frac{D}{2}\right)$
If the considered the strength positive if clockwise the following result is also correct.

$$w(z) = -\frac{iG}{2\pi}\ln\left(z - i\frac{D}{2}\right) + \frac{iG}{2\pi}\ln\left(z + i\frac{D}{2}\right)$$

The sign of G needs to change in the different terms and the sign in front of $i\frac{D}{2}$

(b) Sketch the streamlines and equipotential lines in the above graph. Use a marker or colours (not red) to clearly distinguish between both and do not forget to add a legend.

(c) Find the magnitude and direction of the velocity $\vec{u}_q$ at point q in between the two vortices in function of the parameters G and D.

Solution:

Considering G positive for the counterclockwise rotating vortex and the origin in between the vortices:

$$\begin{aligned} u - iv &= \frac{\mathrm{d}w}{\mathrm{d}z}(z) \\ &= \frac{iG}{2\pi}\frac{1}{\left(z - i\frac{D}{2}\right)} - \frac{iG}{2\pi}\frac{1}{\left(z + i\frac{D}{2}\right)} \end{aligned}$$

In this coordinate system $z(q) = 0$.

$$\begin{aligned} u_\mathrm{p} - iv_\mathrm{p} &= \frac{\mathrm{d}w}{\mathrm{d}z}(p) \\ &= -\frac{iG}{2\pi}\frac{1}{i\frac{D}{2}} - \frac{iG}{2\pi}\frac{1}{i\frac{D}{2}} \\ &= -\frac{2G}{\pi D} \end{aligned}$$

$$\begin{aligned} u_q &= -\frac{2G}{\pi D} \\ v_q &= 0 \end{aligned}$$

Alternatively, from the symmetry of the set-up, we know that the induced velocity is in the negative x-direction and has a magnitude of $2\,\dfrac{G}{2\pi D/2} = \dfrac{2G}{\pi D}$

(d) Now, add a horizontal uniform free stream flow with velocity $u_\infty = \frac{1}{2}|\vec{u}_\mathrm{q}|$ to the vortex pair, with $\vec{u}_\mathrm{q}$ the velocity found in the previous part. Find the new resultant complex potential $w(z)$.

Solution: ! Solutions depend on the choice of the origin of the coordinate system !
The complex potential for the flow is $w(z) = \dfrac{iG}{2\pi}\ln\left(z - i\dfrac{D}{2}\right) - \dfrac{iG}{2\pi}\ln\left(z + i\dfrac{D}{2}\right) + u_\infty z$

(e) Find the location of all stagnation points. Write your solution(s) in function of the parameters G and D.

Solution: Considering G positive for the counterclockwise rotating vortex and the origin in between the vortices:

$$u - iv = \frac{\mathrm{d}w}{\mathrm{d}z}(z) = \frac{iG}{2\pi}\frac{1}{\left(z - i\frac{D}{2}\right)} - \frac{iG}{2\pi}\frac{1}{\left(z + i\frac{D}{2}\right)} + u_\infty$$

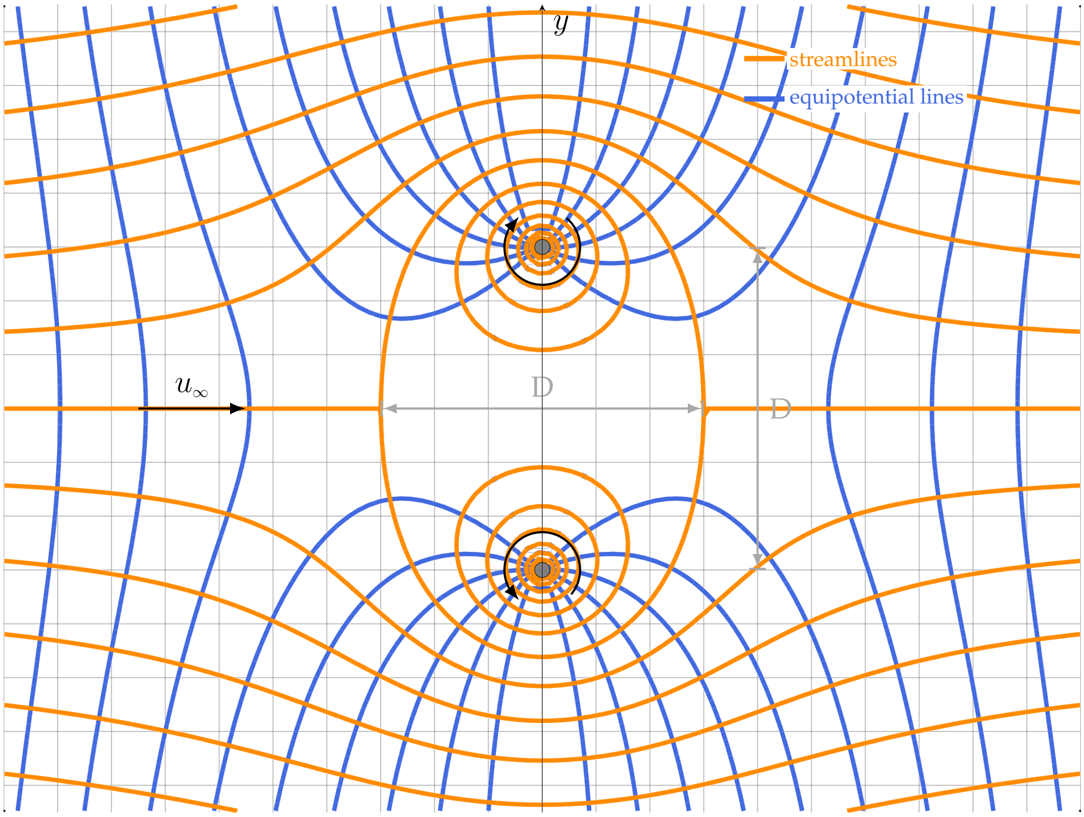
The vertical velocity component along the x-axis is zero everywhere. There will be 2 stagnation points. The stagnation points will be on the x-axis. We need to find the x-location for which the u velocity component is zero.

$$\begin{aligned} 0 &= u(x, y = 0) \\ &= \frac{iG}{2\pi}\left(\frac{1}{x - i\frac{D}{2}} - \frac{1}{x + i\frac{D}{2}}\right) + u_\infty \\ &= \frac{iG}{2\pi}\left(\frac{iD}{x^2 + D^2/4}\right) + u_\infty \end{aligned}$$

$$\begin{aligned} &\Rightarrow \frac{2GD}{\pi}\left(\frac{1}{4x^2 + D^2}\right) = u_\infty \\ &\Rightarrow x = \pm\sqrt{\frac{GD}{2\pi u_\infty} - \frac{D^2}{4}} \quad \text{with } u_\infty = 0.5u_\mathrm{p} = \frac{G}{\pi D} \\ &\Rightarrow x = \pm\sqrt{\frac{D^2}{4}} \end{aligned}$$

stagnation points in $(x, y) = (\pm\frac{D}{2}, 0)$

(f) Sketch the new streamline pattern in the graph below.



(g) Find the pressure coefficient in point q

Solution:

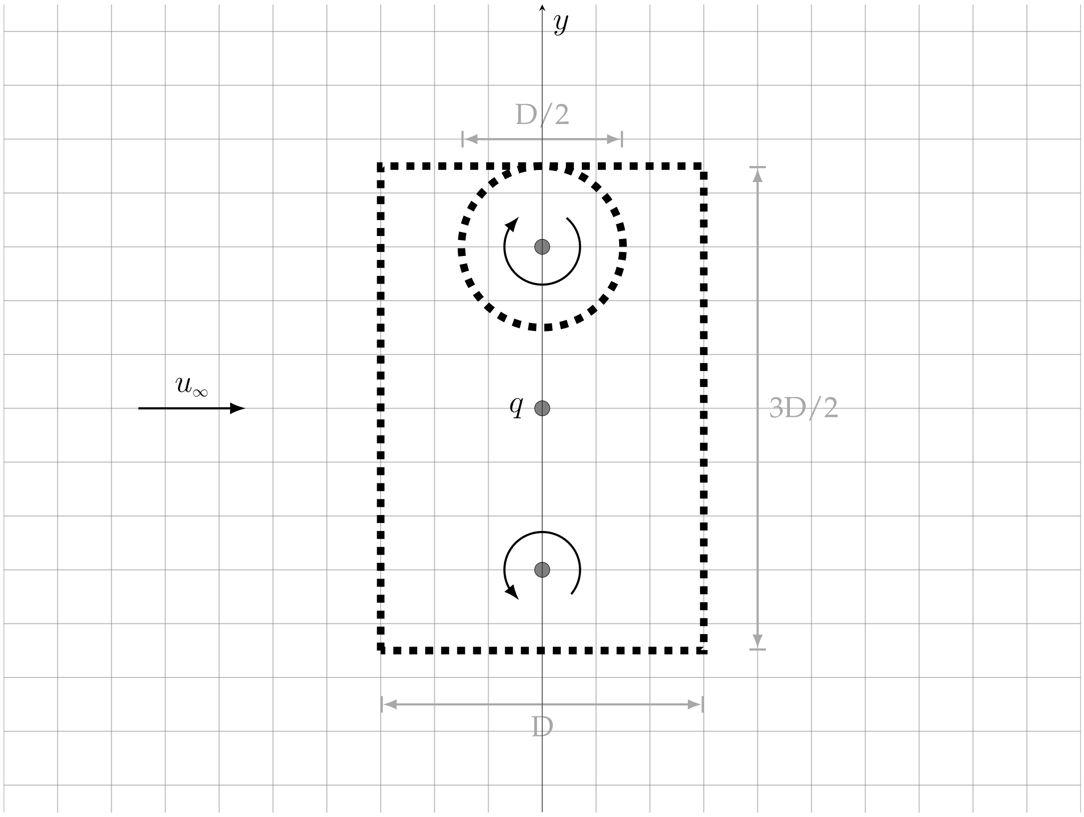
The velocity in point q was $(u_q, v_q) = (-\frac{2G}{\pi D}, 0)$ before including the horizontal uniform free stream $u_\infty = \frac{G}{\pi D}$. The new velocity components in point q are thus $(u_q, v_q) = (-\frac{G}{\pi D}, 0) = (-u_\infty, 0)$ and the pressure coefficient $c_p(q) = 1 - (u_q^2/u_\infty^2)$ hence $c_p = 0$.

(h) Find the total circulation Γ in the area enclosed by the large rectangle in the figure below.

Solution: $\Gamma = 0$

(i) Find the total circulation Γ in the area enclosed by the circle in the figure below.

Solution: $\Gamma = -G$


y
D/2
u_∞
q
3D/2
D