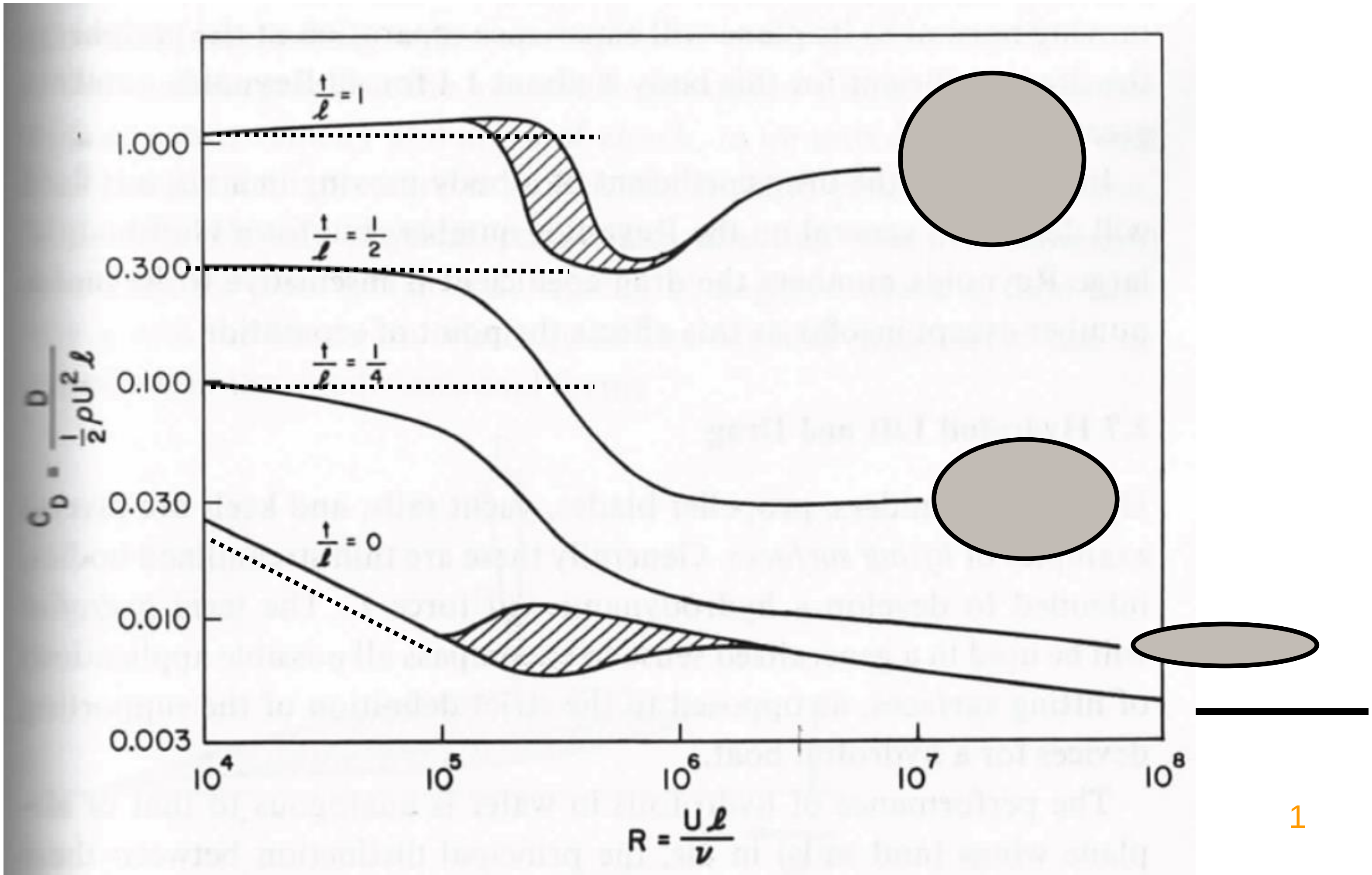
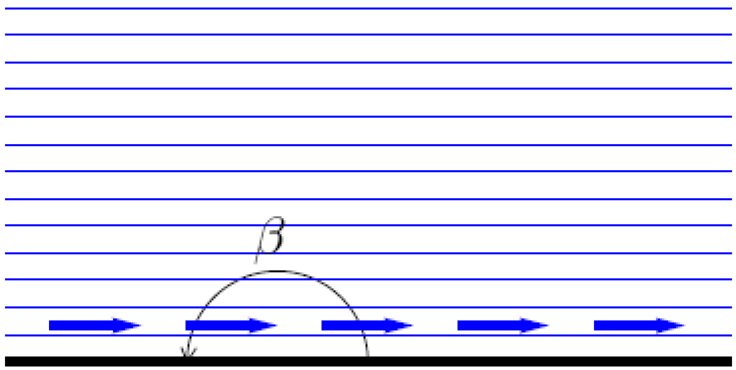
But this does still not explain the aerodynamic drag scaling: $\frac{1}{2}\rho_\infty U_\infty{}^2 S$



Pressure gradient effect

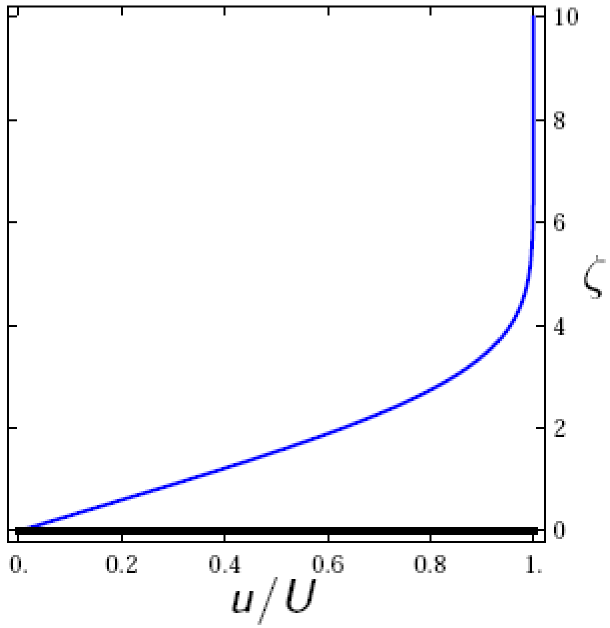
Consider the flow along an angle $\beta = \pi/(m+1) = \pi$ i.e. $m = 0$

This is the flow along a flat plate



Uniform flow- no pressure gradient

$\overline{\nabla} p$ =0

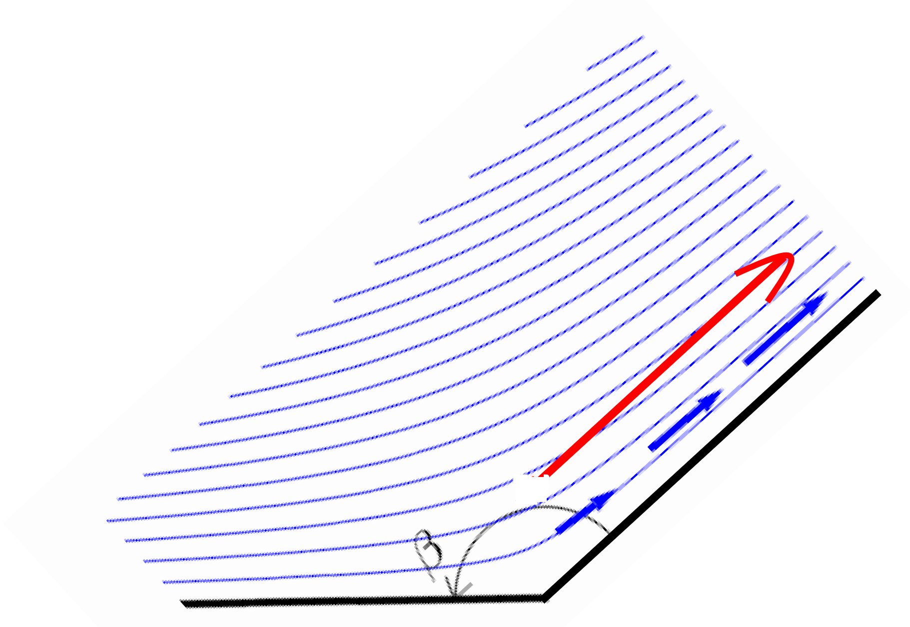


Blasius boundary layer

Pressure gradient effect

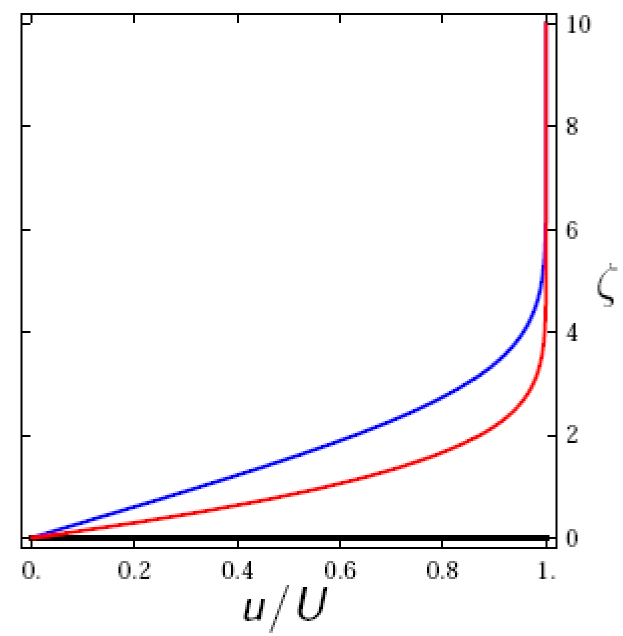
Consider the flow along an angle $\beta = \pi/(m+1) < \pi$ i.e. $m > 0$

This is the flow along a « forward wedge »



Accelerated flow
favorable pressure gradient

$\overline{\nabla} p > 0$

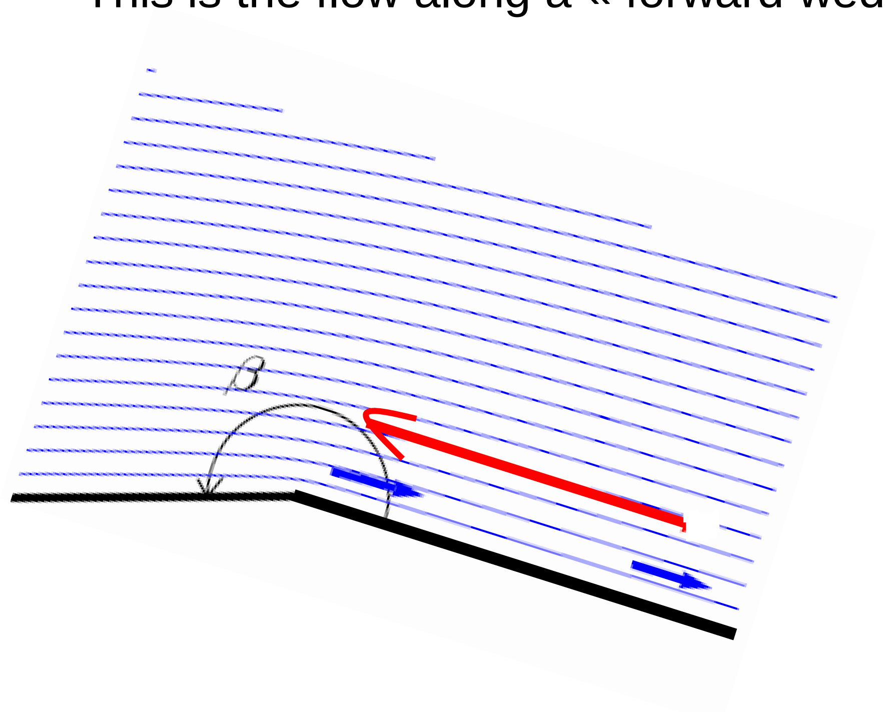


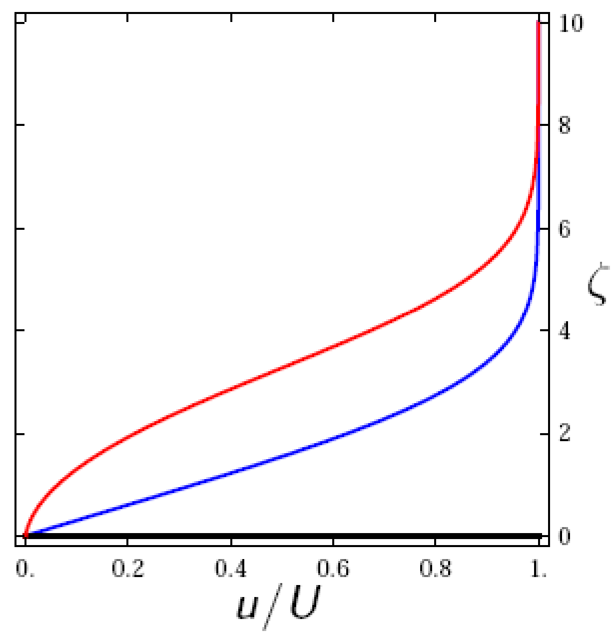
Thinner boundary layer

Pressure gradient effect

Consider the flow along an angle $\beta = \pi/(m+1) > \pi$ i.e. $m < 0$

This is the flow along a « forward wedge »





Decelerated flow
unfavorable (adverse) pressure gradient

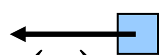
$\vec{\nabla} p < 0$

Thicker boundary layer

Pressure gradient effect

Adverse pressure gradient : $$\frac{\partial p}{\partial x} > 0$$

p_1 $\quad$ $p_2 > p_1$ $\quad$ $p_3 > p_2$

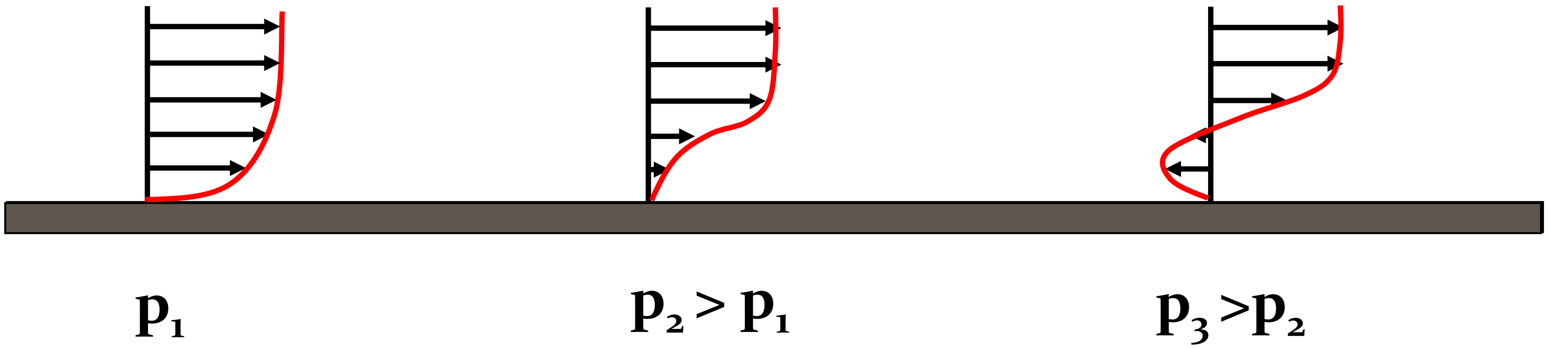


$p(x)$ $\quad$ $p(x+dx)$

Resulting pressure force

Pressure gradient effect

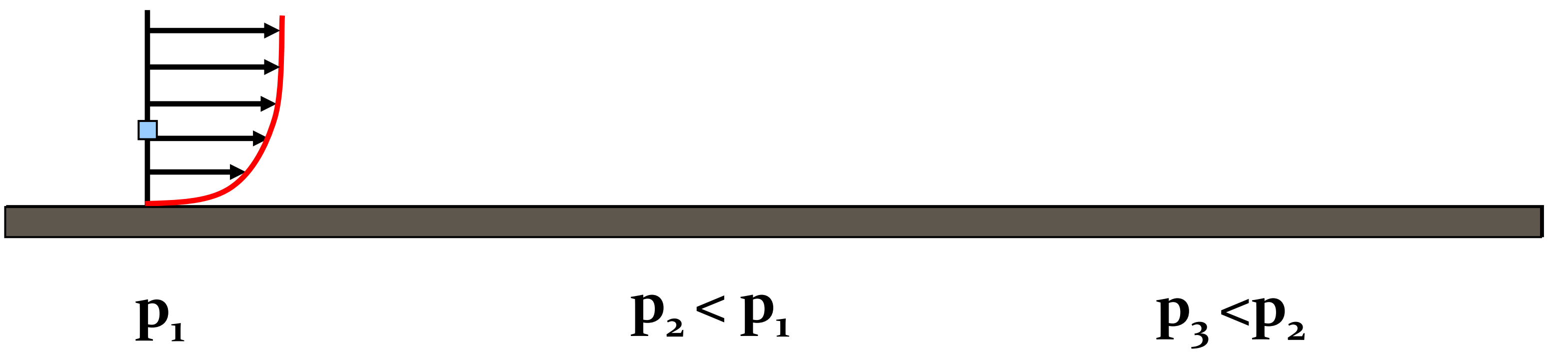
Adverse pressure gradient : $\frac{\partial p}{\partial x} > 0$

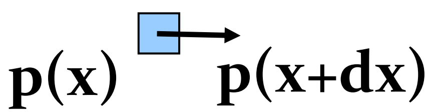


Close to the wall, the viscous effects dominate
The pressure gradient further decreases the velocity
⇨Detachement

Pressure gradient effect

Favorable pressure gradient: $\dfrac{\partial p}{\partial x} < 0$

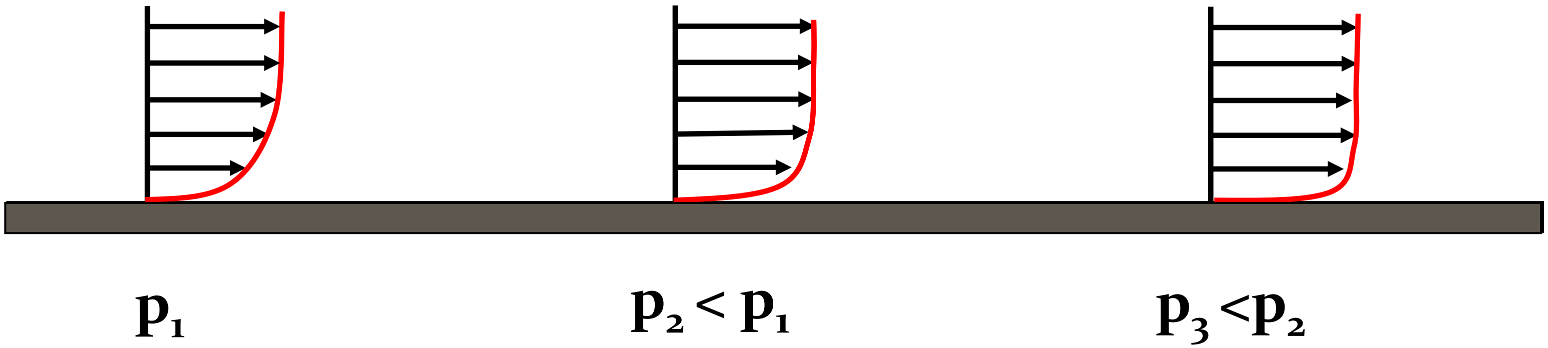




Resulting pressure force

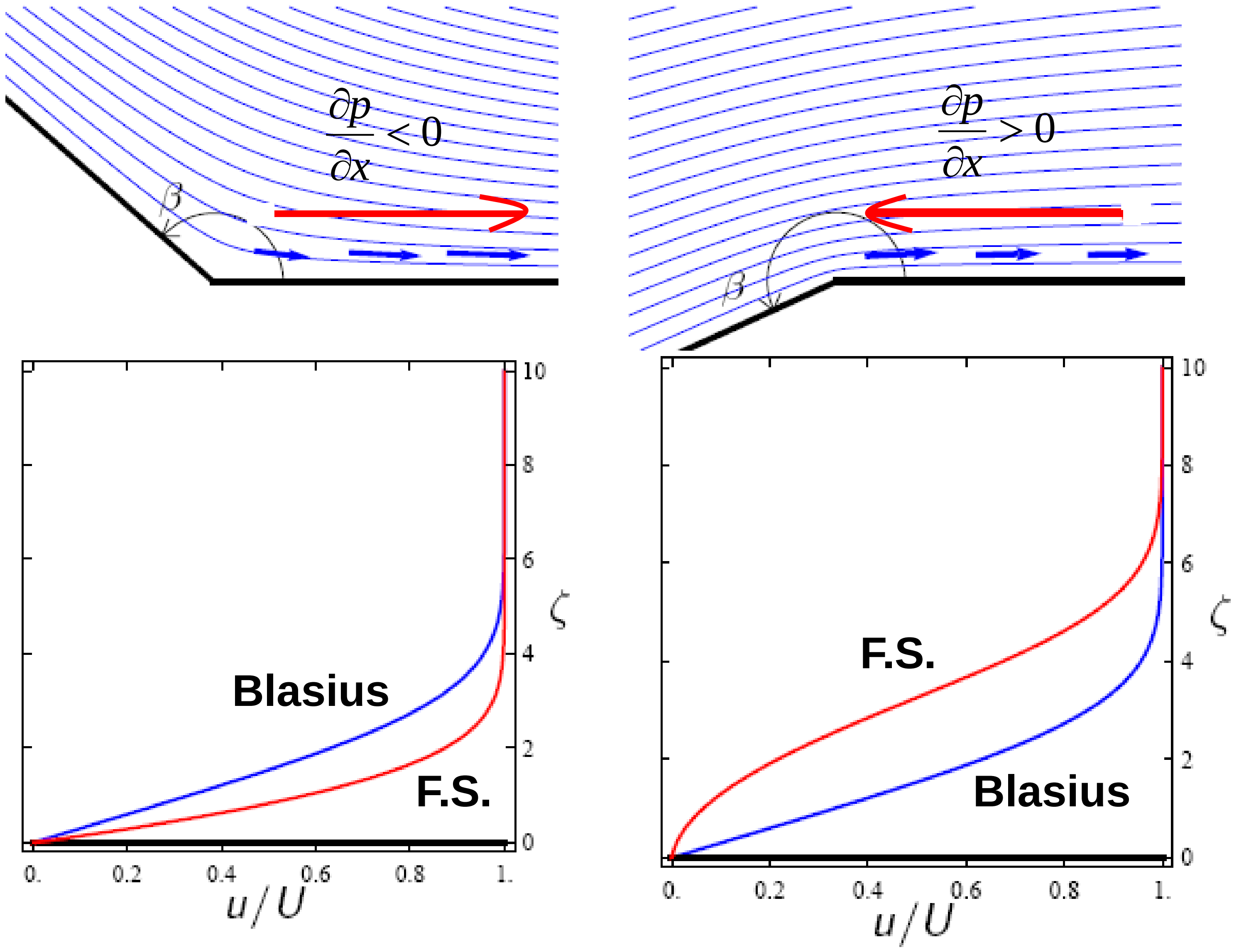
Pressure gradient effect

Favorable pressure gradient: $\frac{\partial p}{\partial x} < 0$



Close to the wall, the pressure gradient further increases the velocity of the flow ⇨ no detachement

Pressure gradient effect



Falkner-Skan solutions

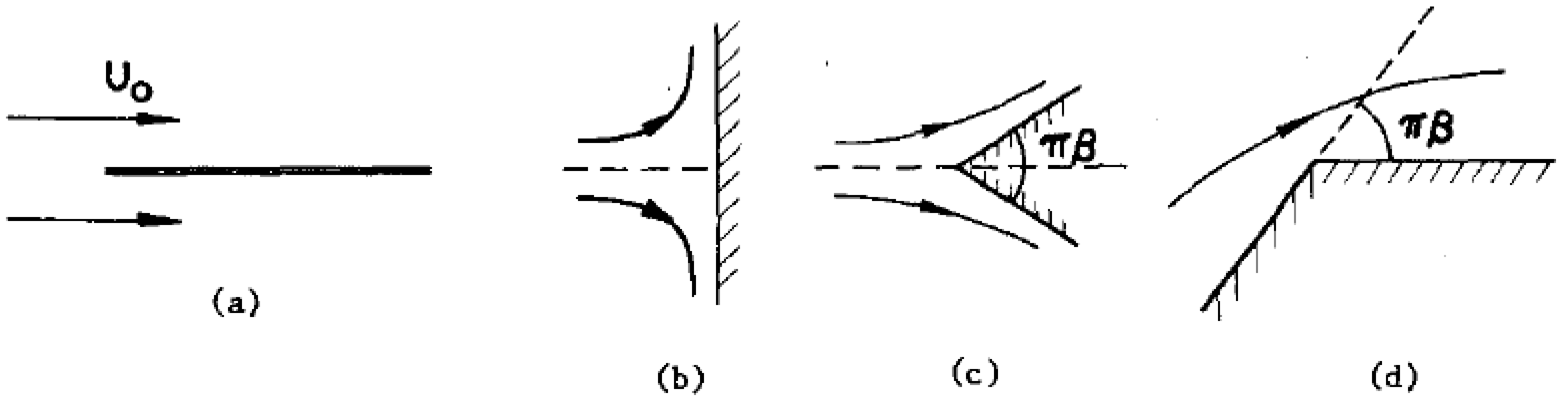
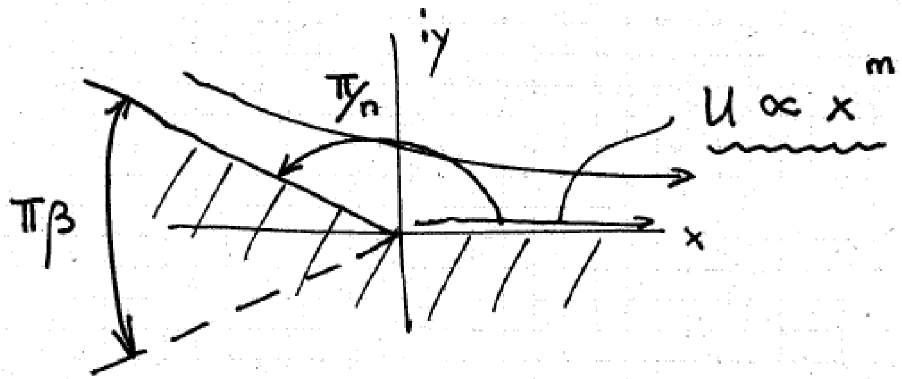


Figure 5.2 Boundary layer flows represented by solutions of the Falkner-Skan equation for different values of the parameter m: (a) $m = 0$; (b) $m = 1$; (c) $0 < m < 1$; (d) $-1/2 < m < 0$

Falkner-Skan far field solutions

pot. complexe

$$F(z) = C z^{n}$$



$$\frac{dF}{dz} = C n z^{n-1} = v_x - i v_y \longrightarrow \boxed{n = 1 + m}$$

$$\frac{\pi}{n} + \frac{\pi\beta}{2} = \pi \longrightarrow \frac{1}{1+m} + \frac{\beta}{2} = 1$$

$$\boxed{\beta = \frac{2m}{1+m}}$$

Falkner-Skan boundary layer equations

1. Prandtl equations

$$\hat{\psi}_{\hat{y}}\,\hat{\psi}_{x\hat{y}} - \hat{\psi}_{x}\,\hat{\psi}_{\hat{y}\hat{y}} = U\frac{\mathrm{d}U}{\mathrm{d}x} + \hat{\psi}_{\hat{y}\hat{y}\hat{y}},$$

$$\hat{\psi} = \hat{\psi}_{\hat{y}} = 0 \text{ on } \hat{y} = 0, \quad \hat{\psi}_{\hat{y}} \to U(x) \text{ as } \hat{y} \to \infty.$$

Falkner-Skan boundary layer equations

1. Prandtl equations

$$\hat{\psi}_{\hat{y}}\,\hat{\psi}_{x\hat{y}} - \hat{\psi}_x\,\hat{\psi}_{\hat{y}\hat{y}} = U\frac{\mathrm{d}U}{\mathrm{d}x} + \hat{\psi}_{\hat{y}\hat{y}\hat{y}},$$

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2. Self-similar solution

$$\hat{\psi}(x, \hat{y}) = (Ax^{m+1})^{1/2} f(\eta) \text{ where } \eta = \hat{y}(Ax^{m-1})^{1/2}.$$

Falkner-Skan boundary layer equations

1. Prandtl equations

$$\hat{\psi}_{\hat{y}}\,\hat{\psi}_{x\hat{y}} - \hat{\psi}_x\,\hat{\psi}_{\hat{y}\hat{y}} = U\frac{\mathrm{d}U}{\mathrm{d}x} + \hat{\psi}_{\hat{y}\hat{y}\hat{y}},$$

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$$\hat{\psi}(x,\hat{y}) = (Ax^{m+1})^{1/2} f(\eta) \text{ where } \eta = \hat{y}(Ax^{m-1})^{1/2}.$$

3. Falkner-Skan equation

$$f''' + \tfrac{1}{2}(m+1)\,ff'' + m(1 - f'^2) = 0$$

$$f(0) = f'(0) = 0, \quad f'(\infty) = 1$$

Falkner-Skan boundary layer solutions

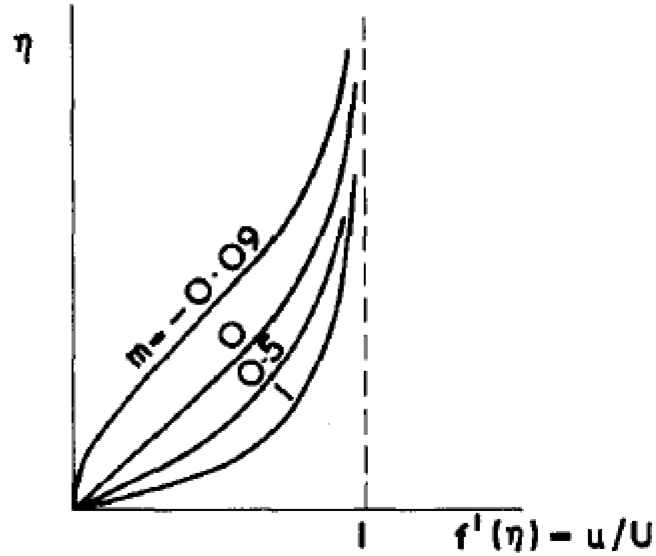
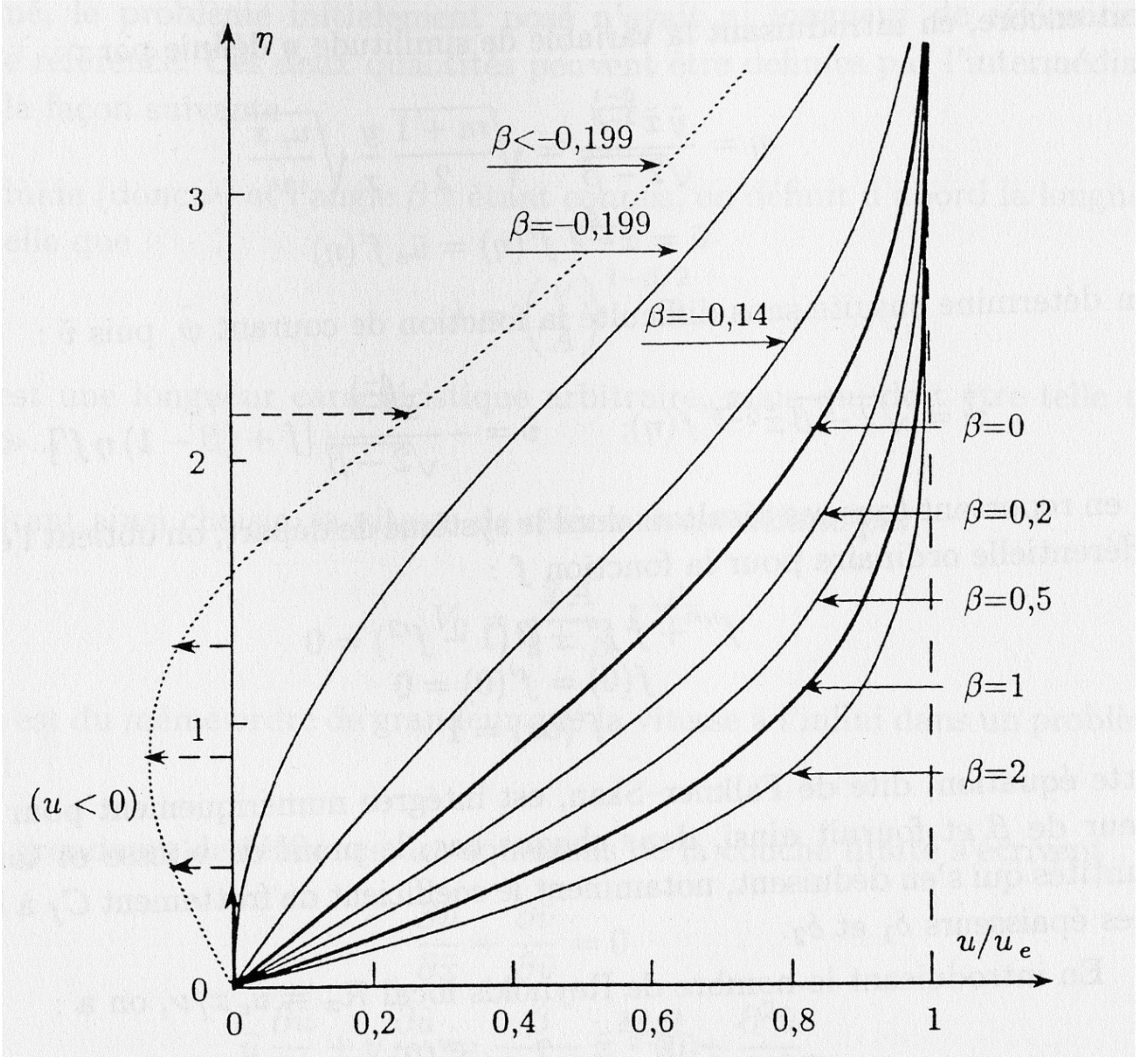
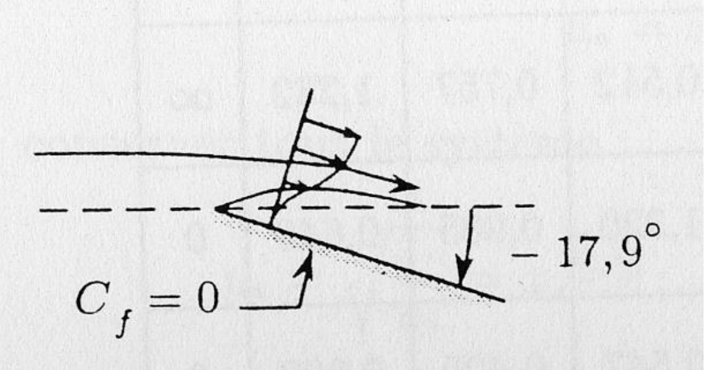
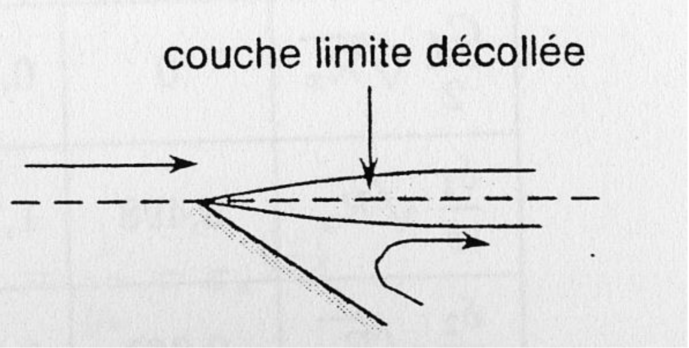


Figure 5.3 Sketch of velocity profiles given by solutions of the Falkner–Skan equation

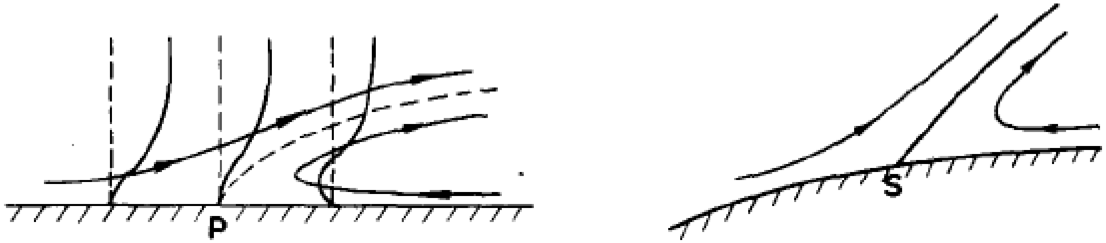
Falkner-Skan boundary layer solutions





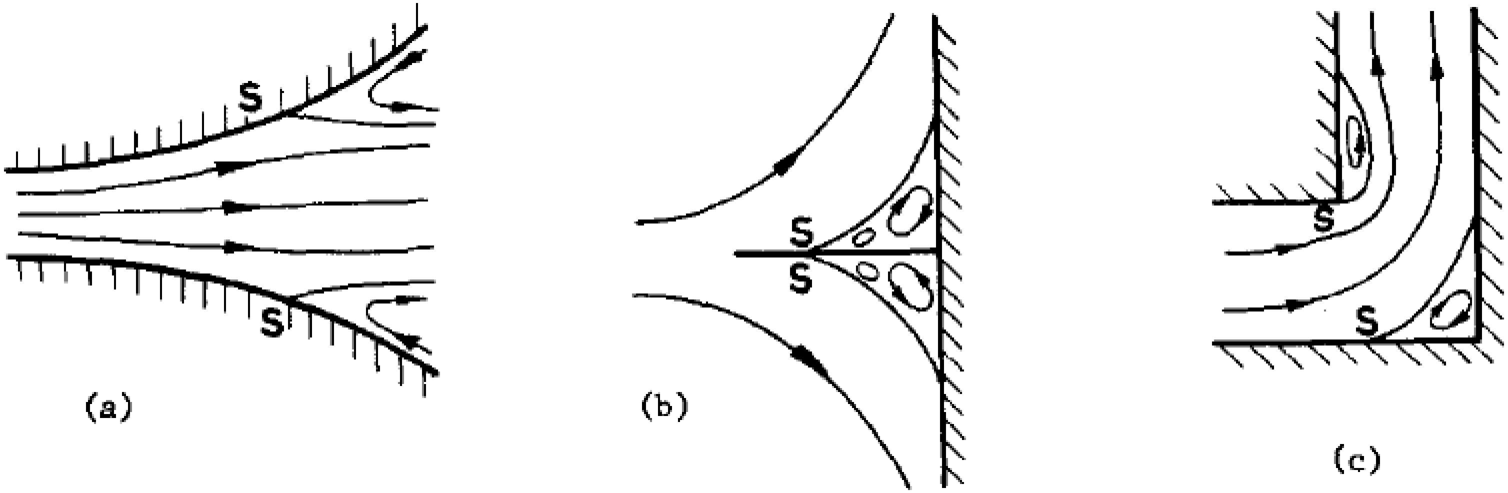


Boundary layer separation



$$\hat{\psi} \sim (x - x_S)^{1/2}, \quad \text{so that } \frac{\partial \hat{\psi}}{\partial x} \sim (x - x_S)^{-1/2} \text{ as } x \to x_S.$$

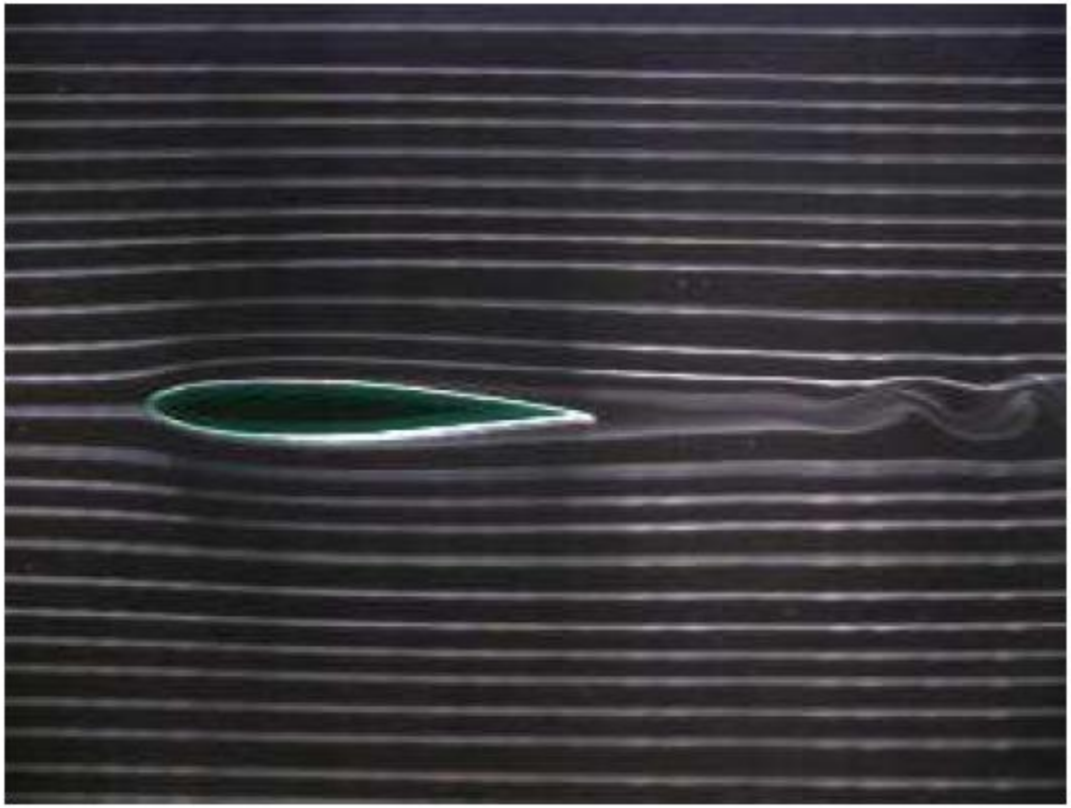
Boundary layer separation



Decollement sur un profil d'aile

Expériences en soufflerie menées à l'université de Stanford,
l'écoulement est visualisé grâce à des fumées :

angle d'incidence $\boxed{\gamma = 0^\circ}$:

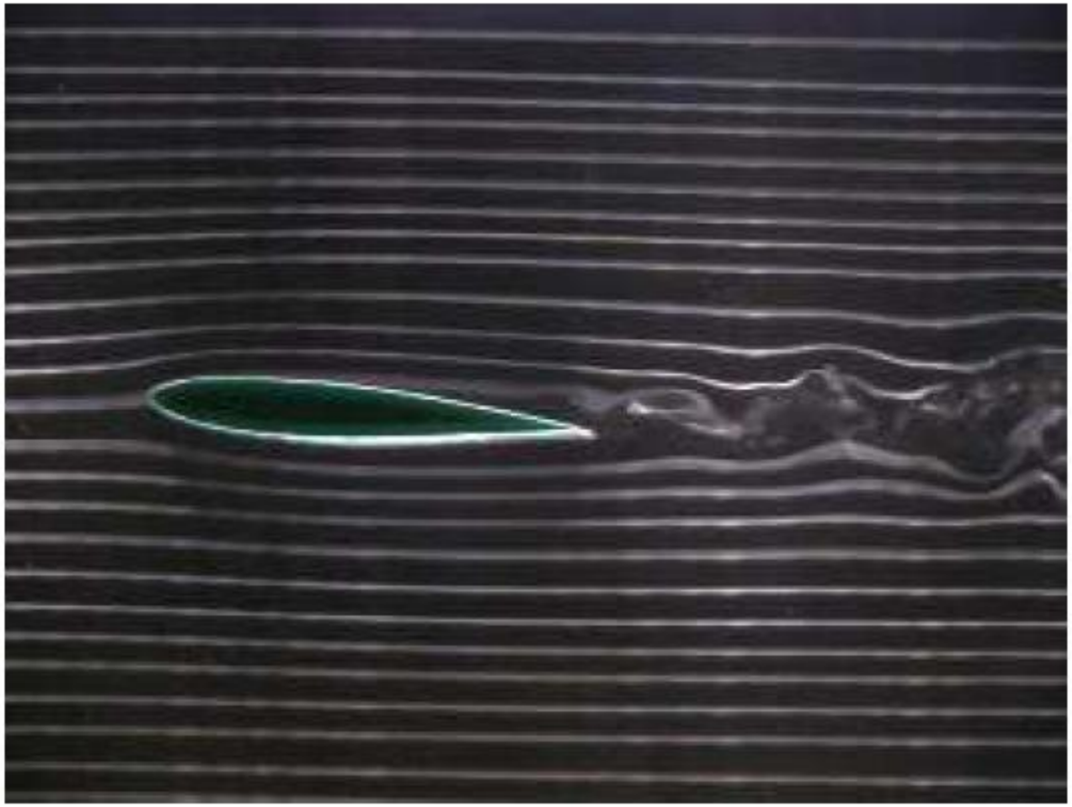


[DVD 'Multimedia Fluid Mechanics', Homsy et al. 2004, Cambridge University Press]

Décollement sur un profil d'aile

Expériences en soufflerie menées à l'université de Stanford, l'écoulement est visualisé grâce à des fumées :

angle d'incidence $\boxed{\gamma = 5^\circ}$:

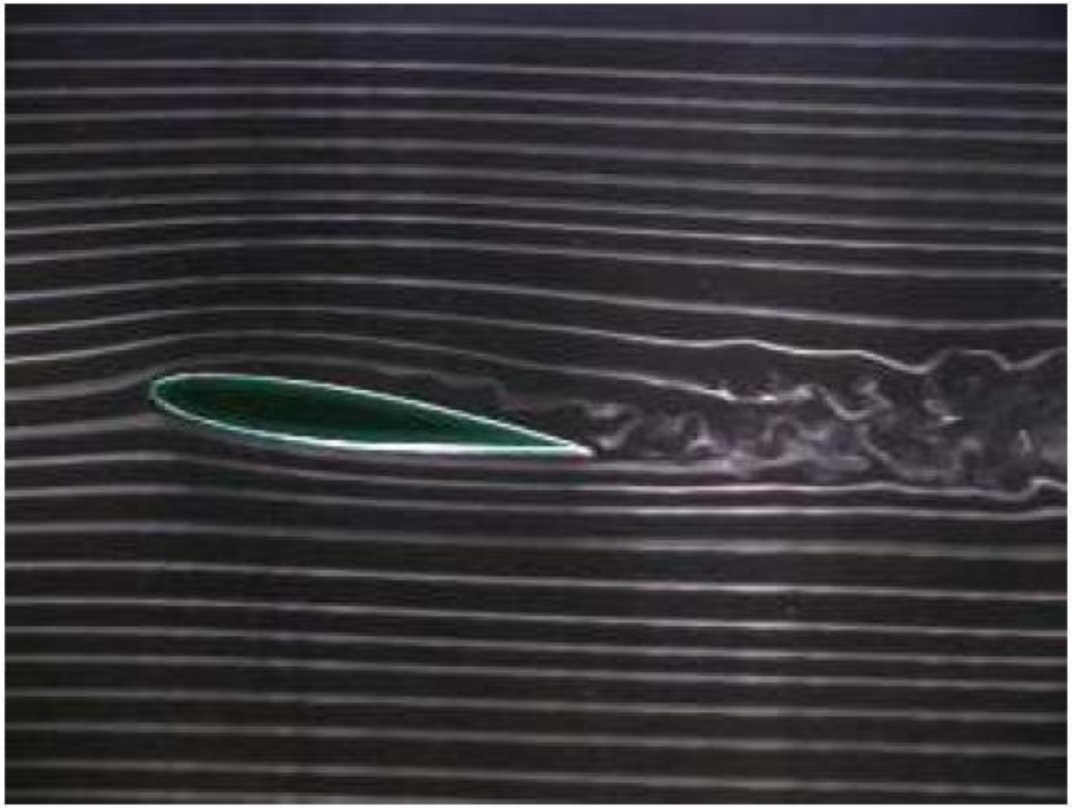


[DVD 'Multimedia Fluid Mechanics', Homsy et al. 2004, Cambridge University Press]

Décollement sur un profil d'aile

Expériences en soufflerie menées à l'université de Stanford, l'écoulement est visualisé grâce à des fumées :

angle d'incidence $\boxed{\gamma = 10^\circ}$:

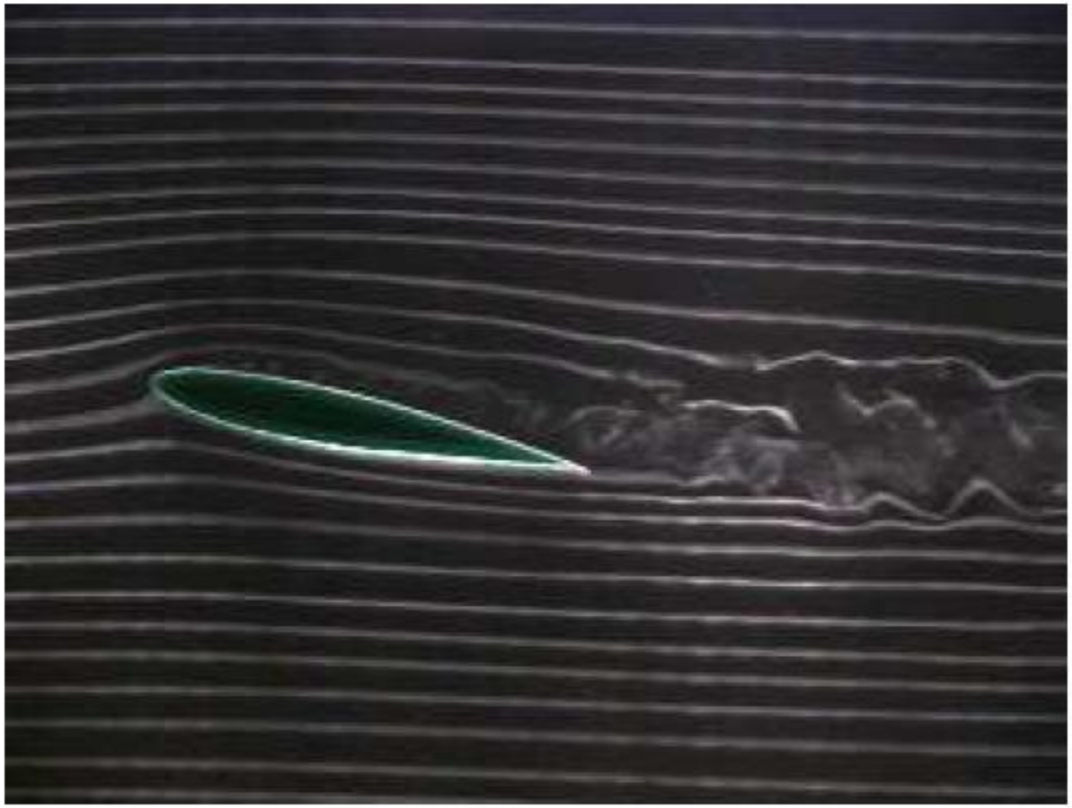


[DVD 'Multimedia Fluid Mechanics', Homsy et al. 2004, Cambridge University Press]

Décollement sur un profil d'aile

Expériences en soufflerie menées à l'université de Stanford, l'écoulement est visualisé grâce à des fumées :

angle d'incidence $\boxed{\gamma = 15^\circ}$:

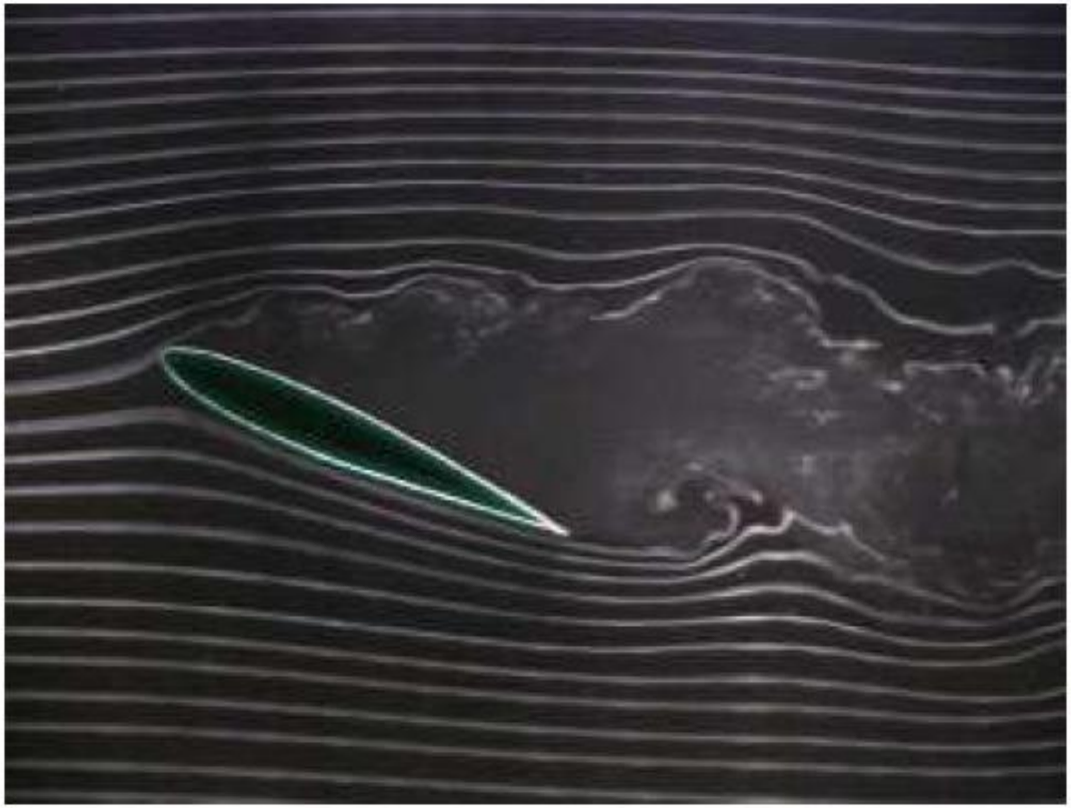


[DVD 'Multimedia Fluid Mechanics', Homsy et al. 2004, Cambridge University Press]

Décollement sur un profil d'aile

Expériences en soufflerie menées à l'université de Stanford, l'écoulement est visualisé grâce à des fumées :

angle d'incidence $\boxed{\gamma = 25^\circ}$:



[DVD 'Multimedia Fluid Mechanics', Homsy et al. 2004, Cambridge University Press]

Décollement sur un profil d'aile

Expériences en soufflerie menées à l'université de Stanford, l'écoulement est visualisé grâce à des fumées :

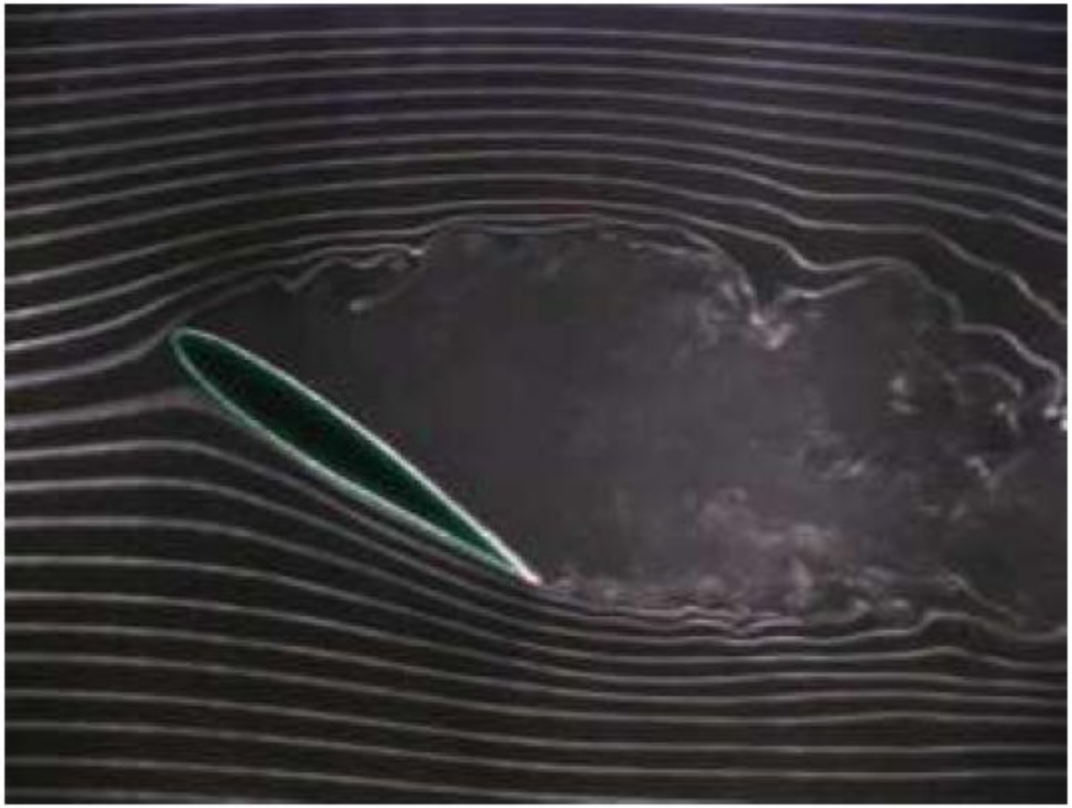
angle d'incidence $\boxed{\gamma = 30^\circ}$:

[DVD 'Multimedia Fluid Mechanics', Homsy et al. 2004, Cambridge University Press]

Décollement sur un profil d'aile

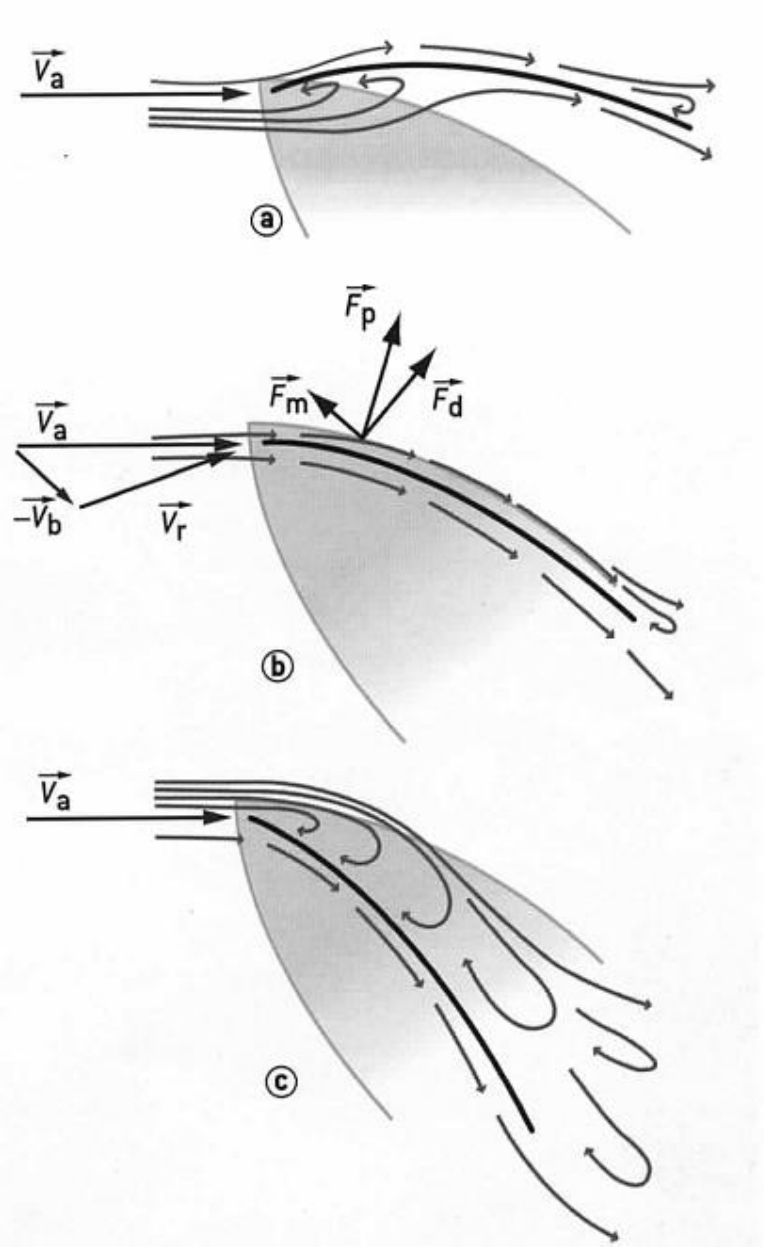
Expériences en soufflerie menées à l'université de Stanford,
l'écoulement est visualisé grâce à des fumées :

angle d'incidence $\boxed{\gamma = 35^o}$:



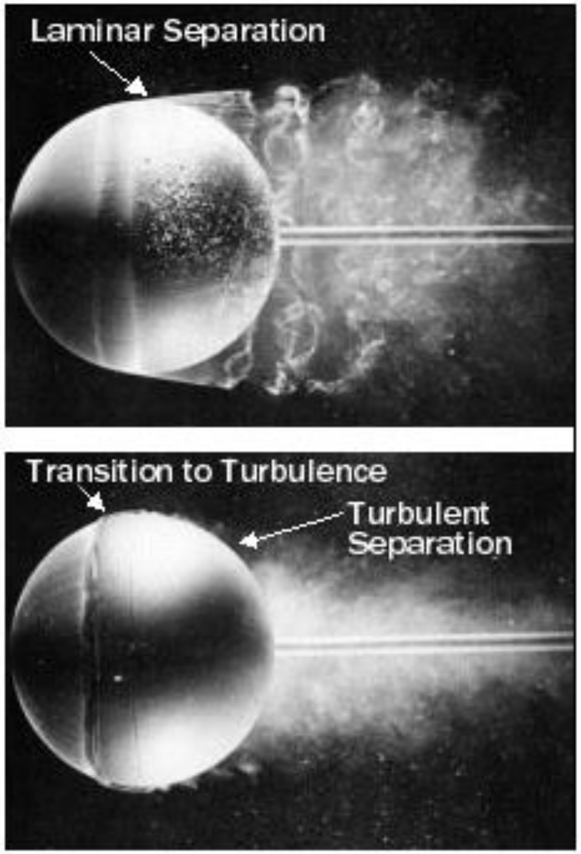
[DVD 'Multimedia Fluid Mechanics', Homsy et al. 2004, Cambridge University Press]

Application to sailing

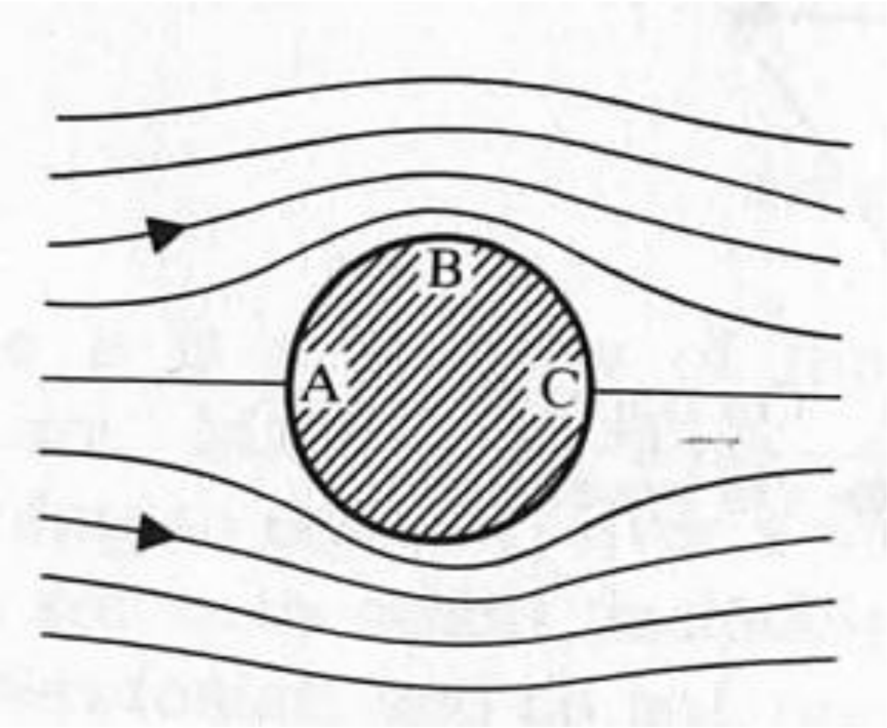


Application to sailing

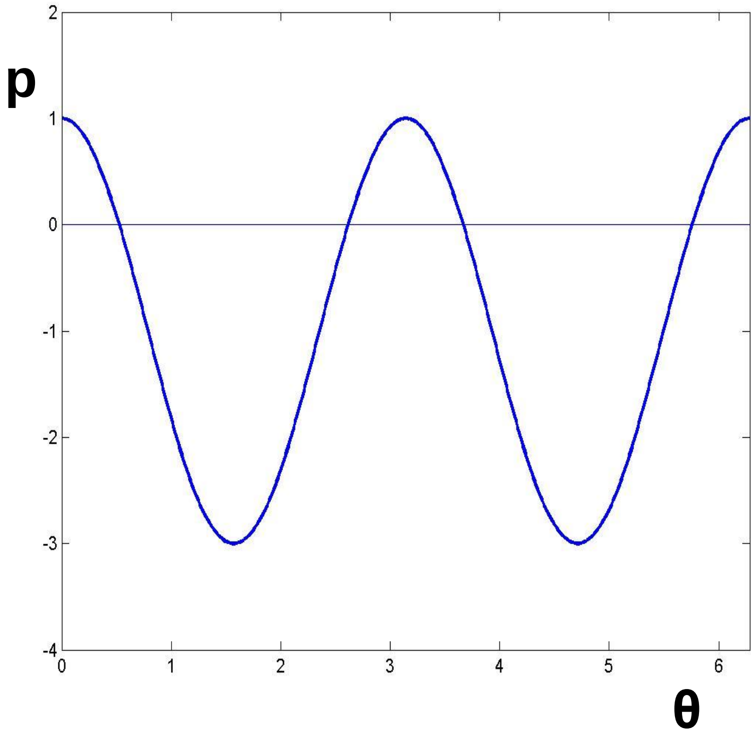
Example: Flow around a sphere



Flow around a cylinder

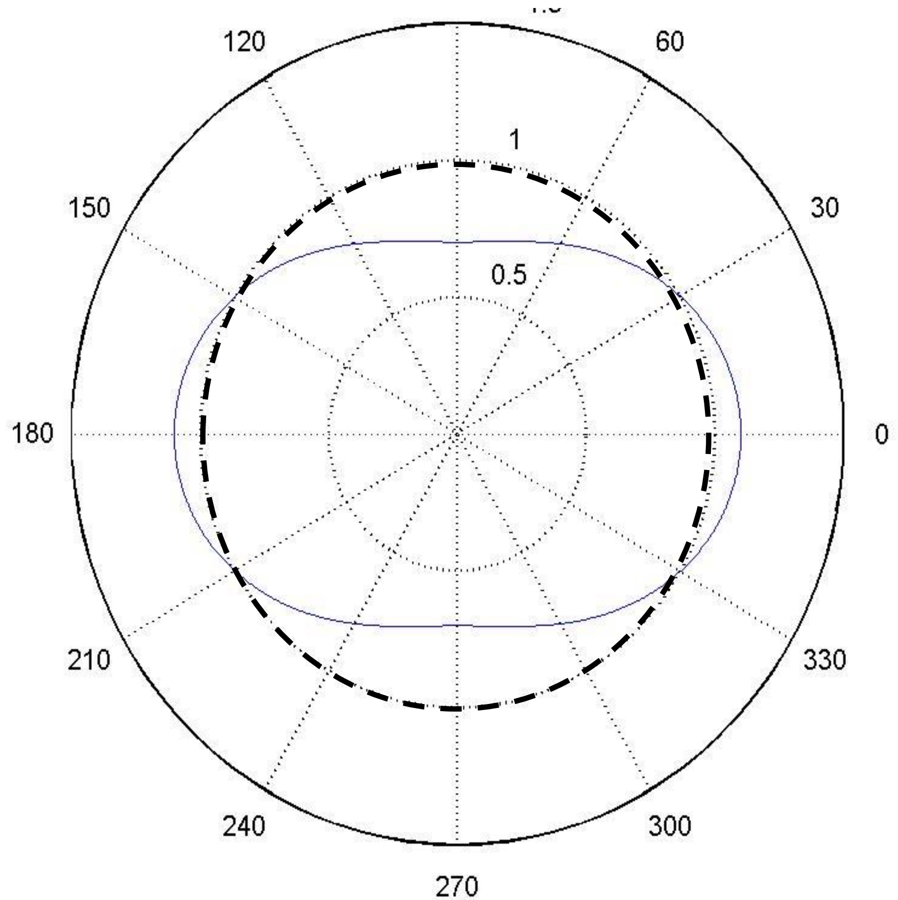


$$p(a,\theta) = \frac{1}{2}\rho U_{\infty}^{2}(1 - 4\sin\theta^{2})$$

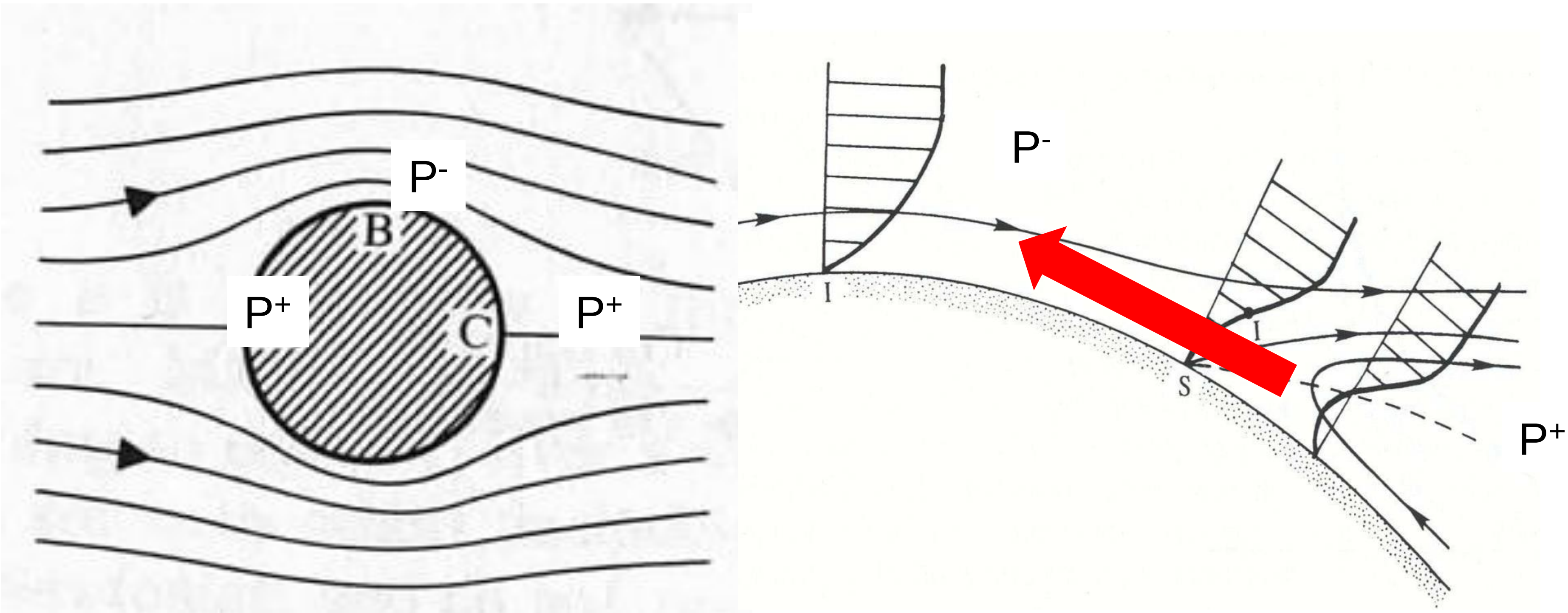


Flow around a cylinder

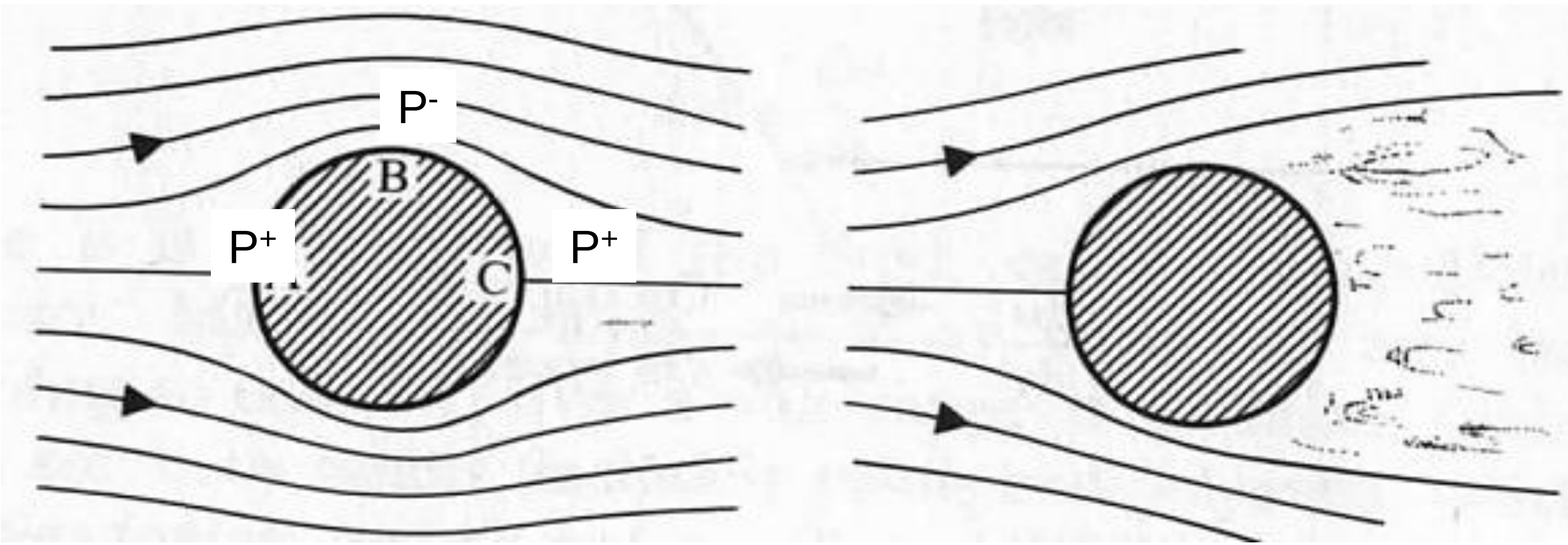
$$p(a, \theta) = \frac{1}{2}\rho U_\infty^2 (1 - 4\sin\theta^2)$$



Origin of detachment: pressure gradient

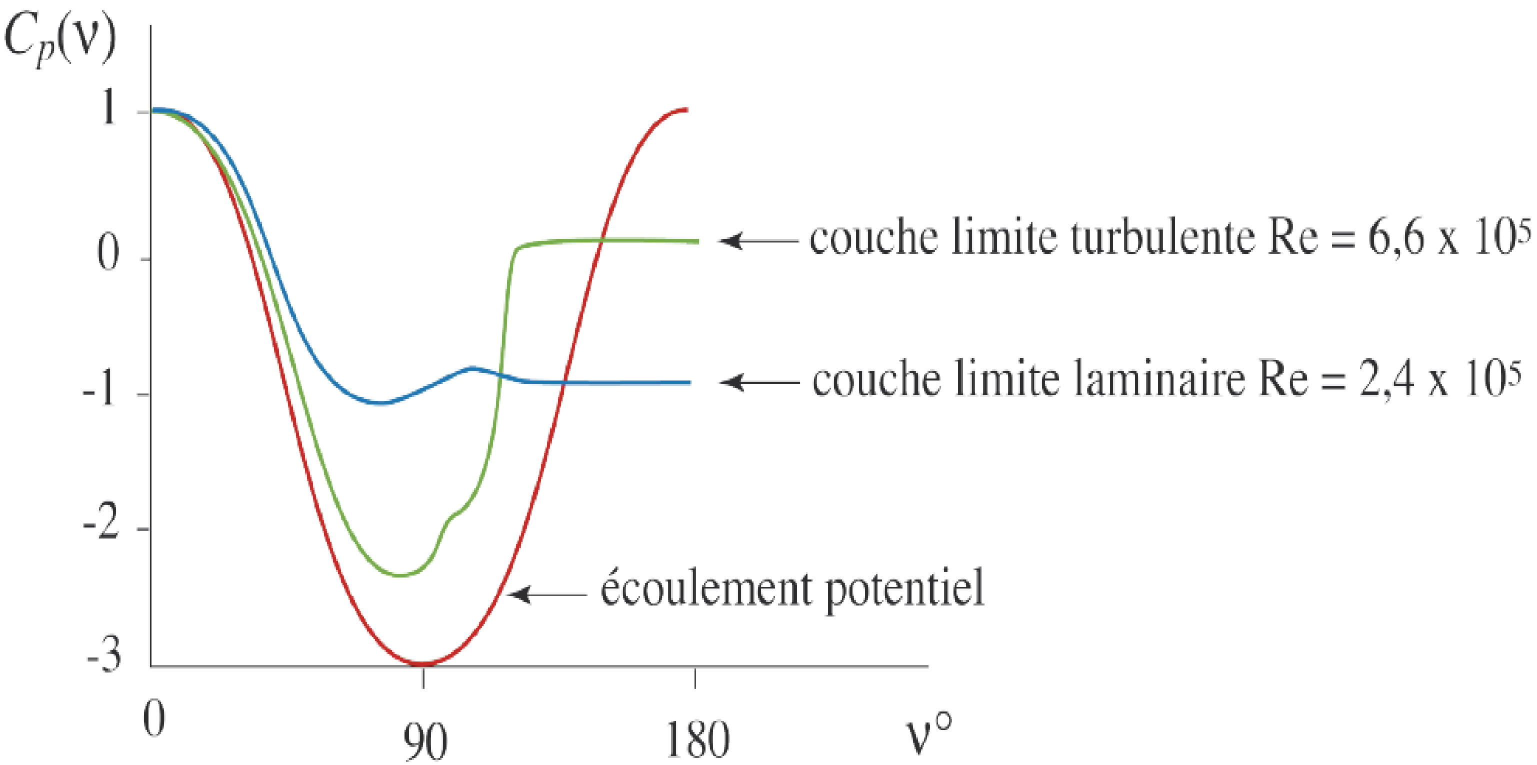


A viscous flow close to the wall opposes the free-stream

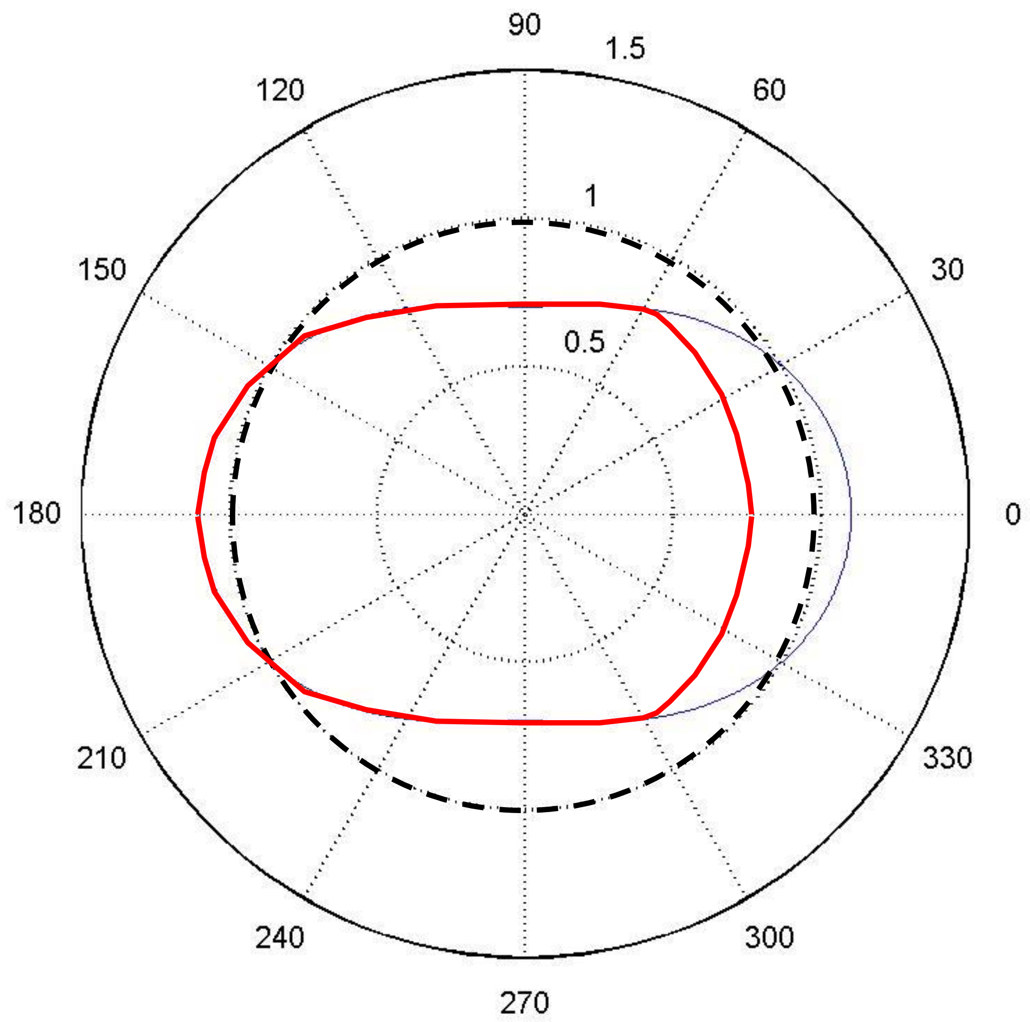


P-
B
P+
C
P+

Pressure coefficient

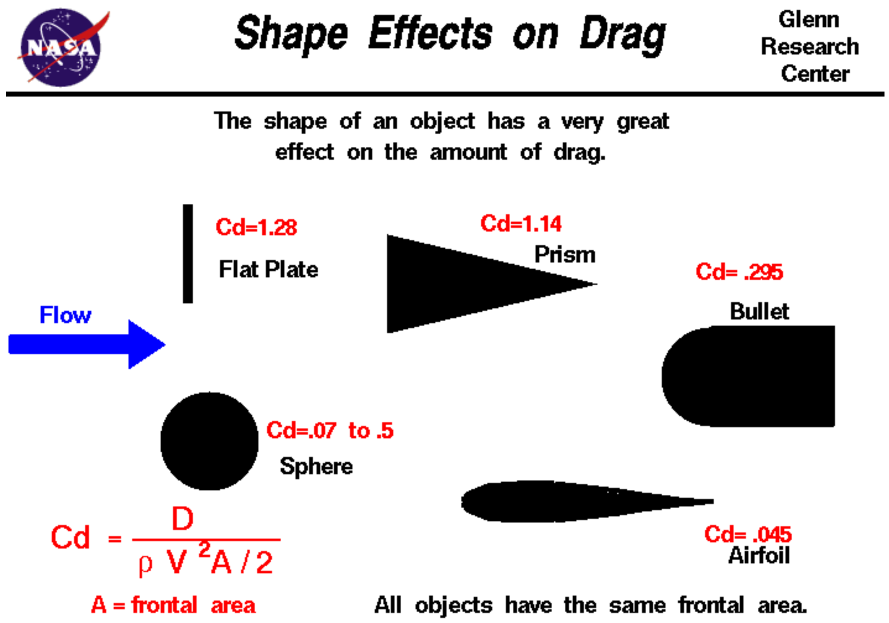


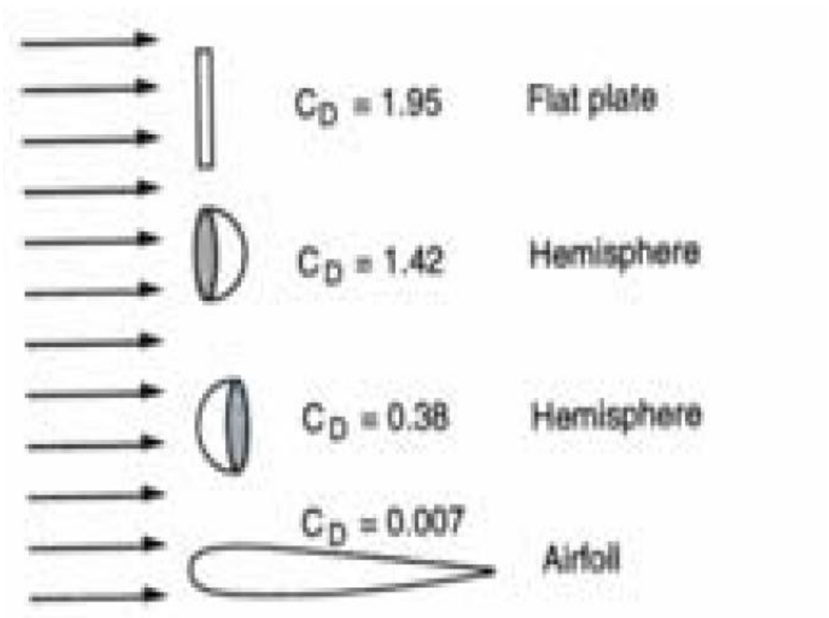
Form drag



Drag coefficient

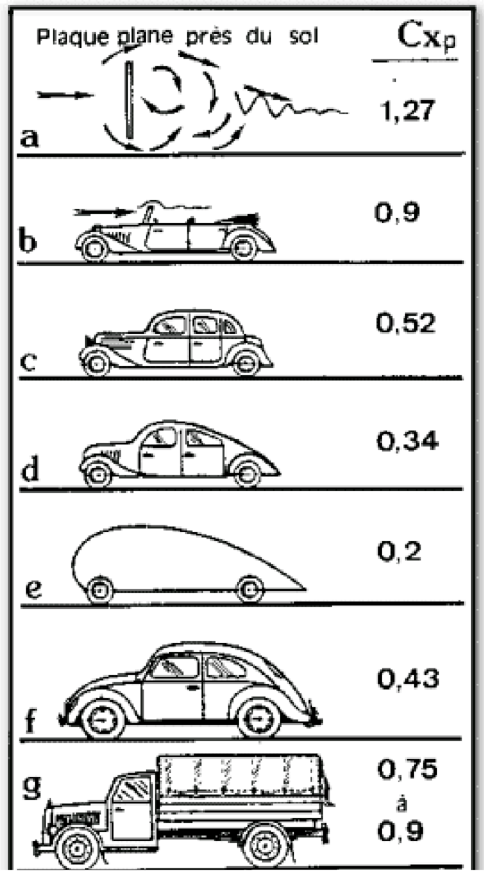
$$C_X = \frac{trainée}{\frac{1}{2}\rho U^2 A}$$



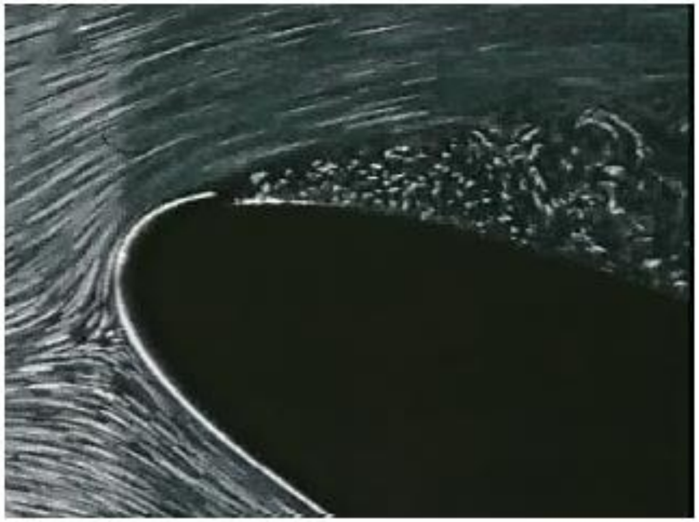


$$C_x = \frac{\textbf{drag}}{\frac{1}{2}\rho U^2 A}$$

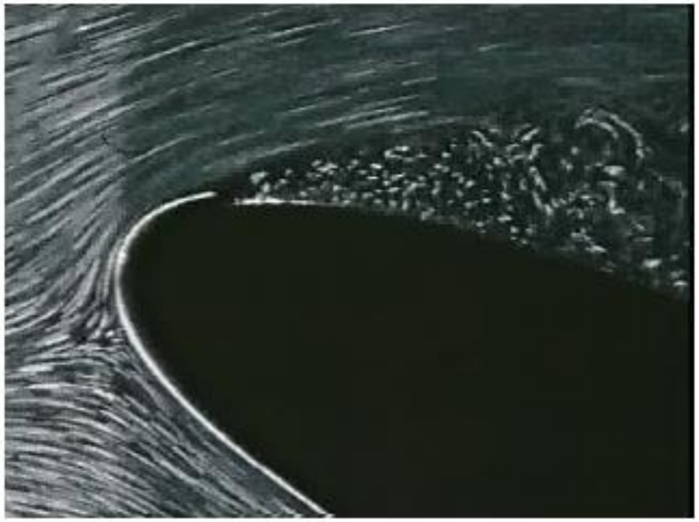
section, somewhat arbitrary…



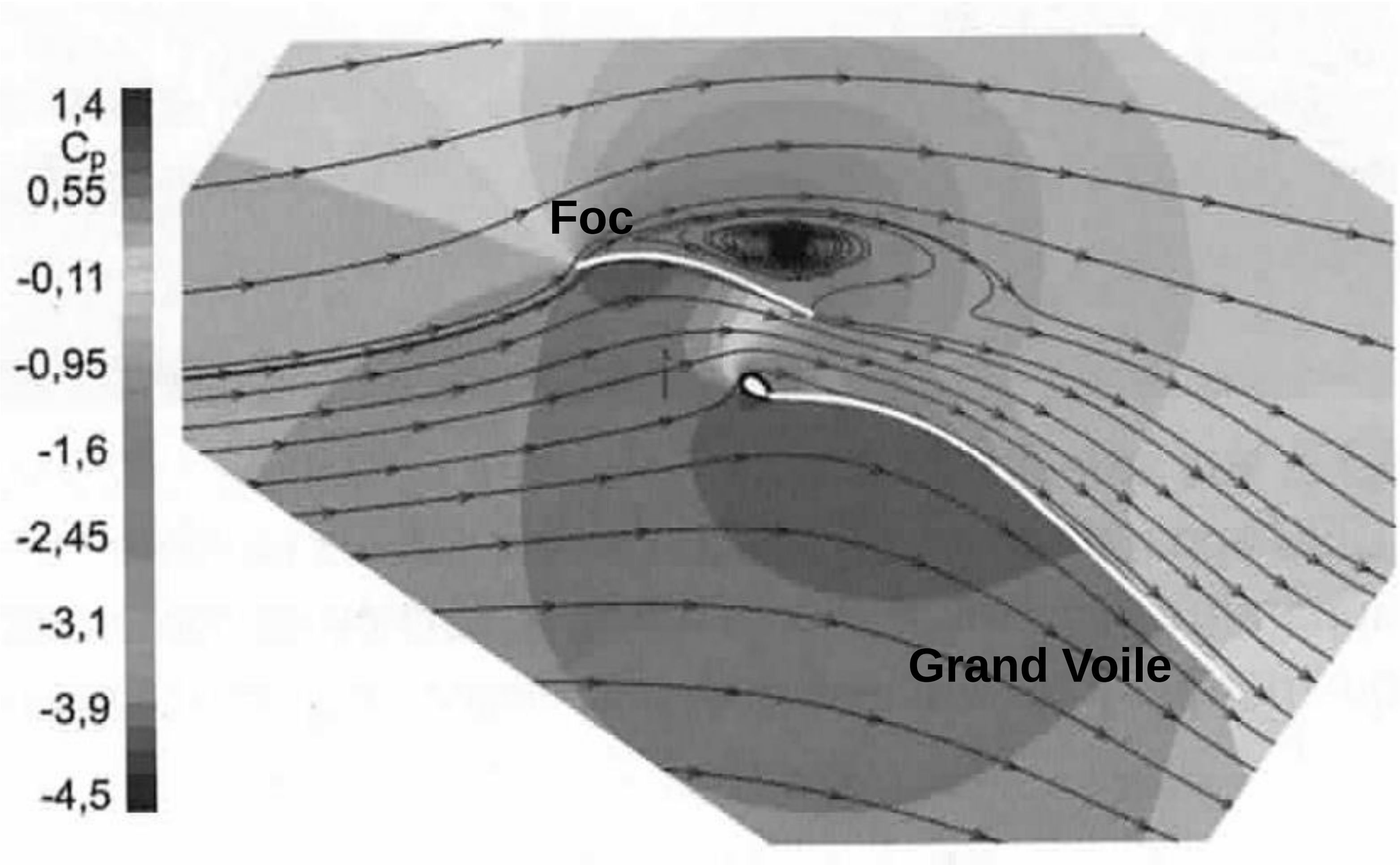
Separation control



Separation control



Application to sailing

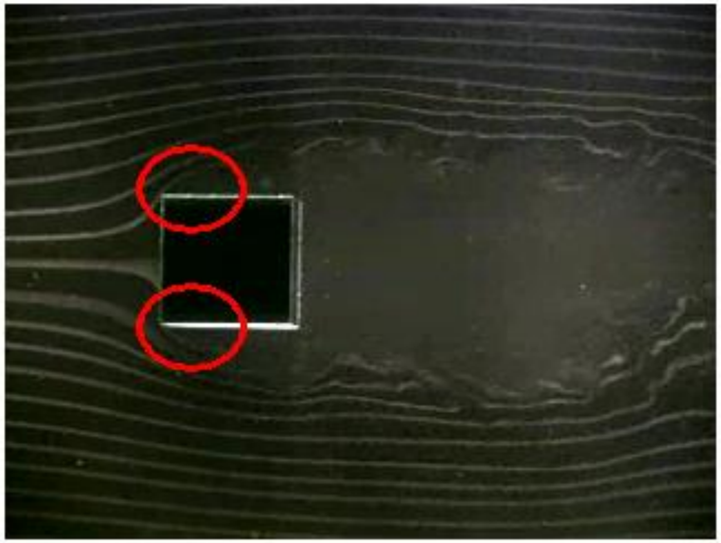


Thickness effect

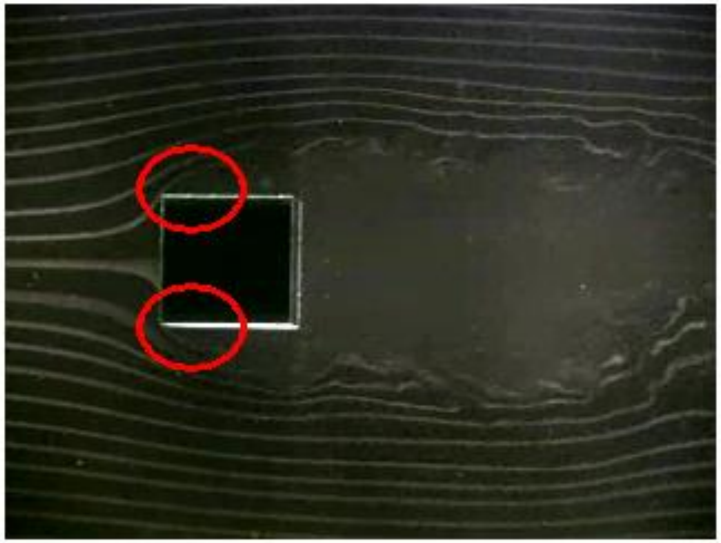
Attached

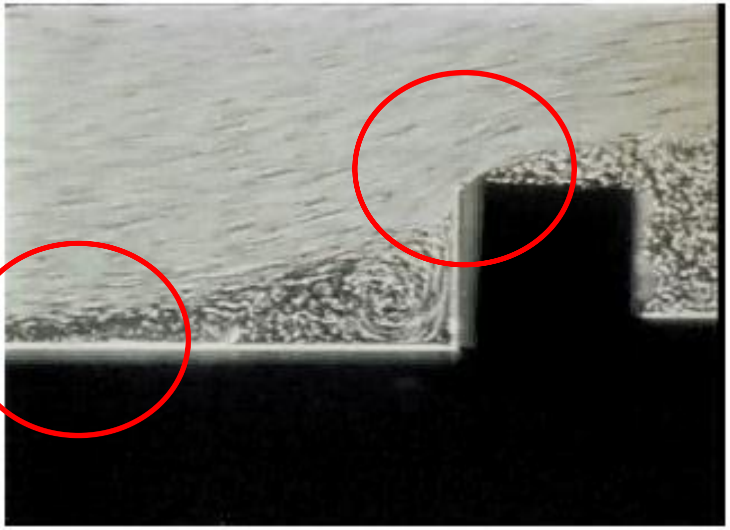
Detached

A gallery of detached flows

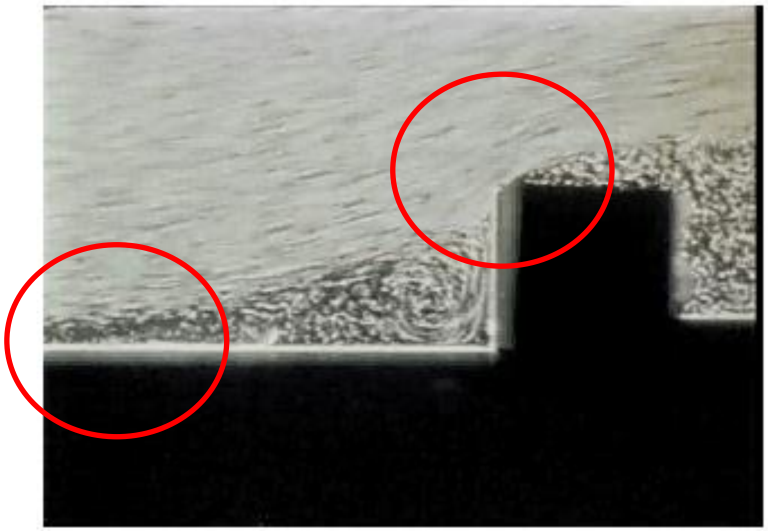


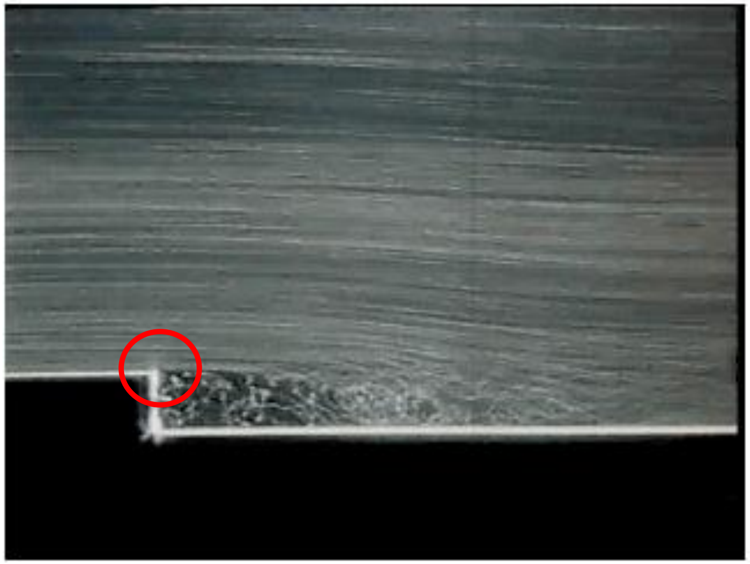
A gallery of detached flows





A gallery of detached flows





Classical Boundary layer

Outer flow dictates boundary layer which does not feedback

$$\begin{cases} \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0 \\ u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = \boxed{u_e \dfrac{d u_e}{dx}} + \dfrac{1}{Re}\dfrac{\partial^2 u}{\partial y^2} \\ 0 = -\dfrac{\partial p}{\partial y} \end{cases}$$

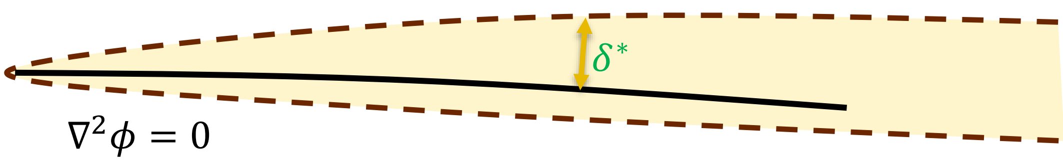
Obtained by solution of irrotational flow, assuming $Re = \infty$

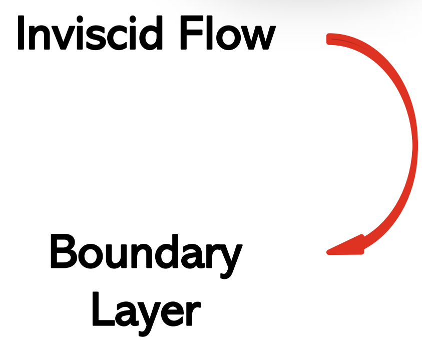
This is a unilateral coupling

Boundary layer deflects outer inviscid flow by δ^*

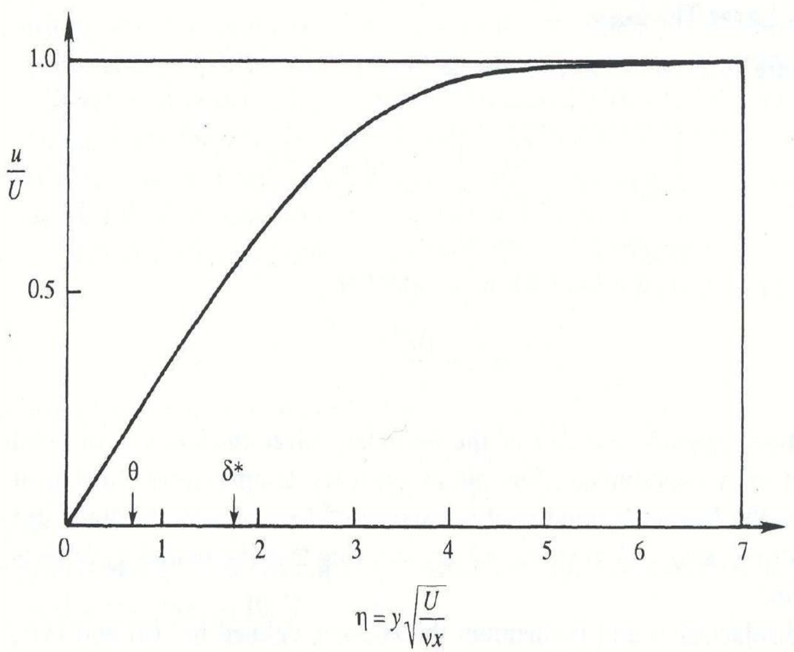
Boundary Layer Displacement Thickness

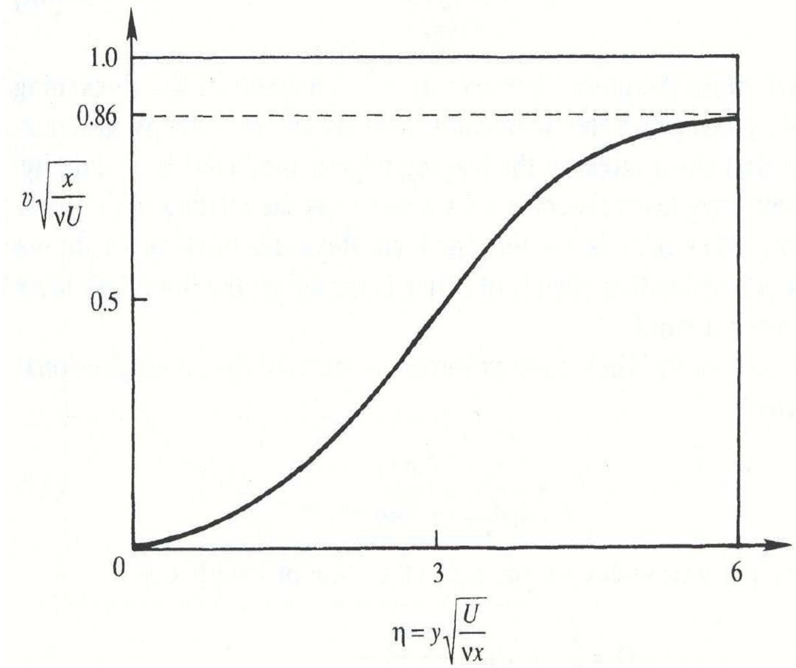
$$\delta^* = \int_0^\infty \left(1 - \frac{u}{u_e}\right) dy$$





BUT REMEMBER that the wall normal velocity is not zero in the boundary layer, it is just small!





It therefore makes sense to correct the potential flow which has to meet a small transpiration velocity at the wall

More quantitatively

Starting from the incompressibility equation and adding and substracting the same derivative of the velocity (in the spirit of Von Kármán integral equations):

$$\frac{\partial \tilde{v}}{\partial \tilde{y}} = (-\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \bar{u}_e}{\partial \bar{x}}) - \frac{\partial \bar{u}_e}{\partial \bar{x}},$$

we obtain, after integration up to an $\tilde{y}$ ($\bar{x}$ and $\tilde{y}$ are independent variables) the velocity is:

$$\tilde{v}(\tilde{y}) - \tilde{v}(0) = -\frac{\partial}{\partial \bar{x}} \int_0^{\tilde{y}} (\tilde{u} - \bar{u}_e) d\tilde{y} - \tilde{y} \frac{\partial \bar{u}_e}{\partial \bar{x}}$$

so, if $\tilde{y}$ is large enough and as $\tilde{v}(0) = 0$ we obtain the behavior for large enough $\tilde{y}$:

$$\tilde{v}(\tilde{y}) \simeq \frac{\partial}{\partial \bar{x}} (\bar{u}_e \tilde{\delta}_1) - \tilde{y} \frac{\partial \bar{u}_e}{\partial \bar{x}}$$

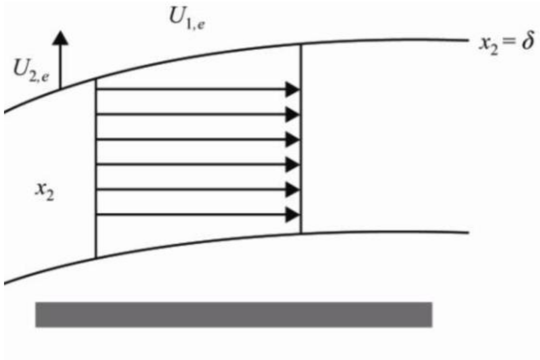
This velocity must be multiplied by $Re^{-1/2}$; and $\bar{y} = Re^{-1/2} \tilde{y}$. Now, we write the velocity in the ideal fluid as a Taylor expansion near the wall for small $\bar{y}$:

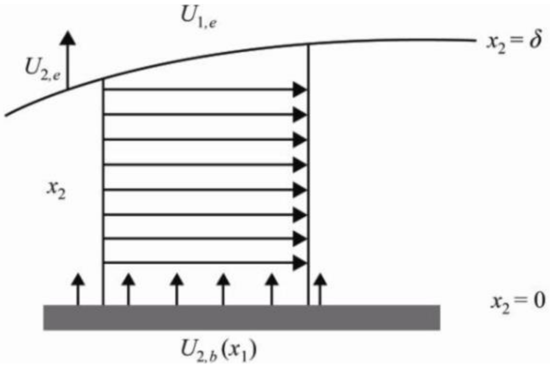
$$\bar{v} = \bar{v}(\bar{x}, 0) + \bar{y} \frac{\partial \bar{v}}{\partial \bar{y}} + ... = \bar{v}(\bar{x}, 0) - \bar{y} \frac{\partial \bar{u}_e}{\partial \bar{x}} + ...$$

matching this velocity and the boundary layer velocity show that:

$$\boxed{\bar{v}(\bar{x}, 0) = Re^{-1/2} \frac{\partial}{\partial \bar{x}} (\bar{u}_e \tilde{\delta}_1)}$$

The viscous-inviscid coupling has two interpretations





The potential flow flows on an effective wall, slighlty displaced by δ

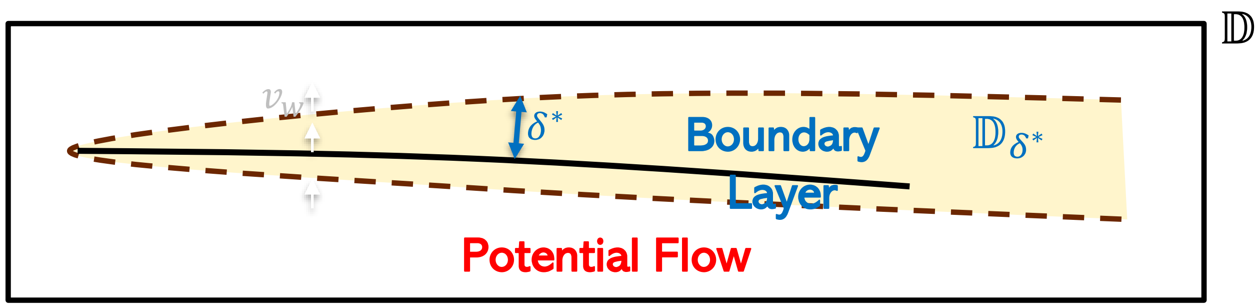
The potential flow flows on wall crossed through transpiration velocity

Interactive Boundary layer

Potential flow and boundary layer are solved in a coupled way

Boundary Layer Displacement Thickness

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{u_e}\right) dy$$



$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}.\nabla \boldsymbol{u} = -\nabla p + \frac{1}{Re}\nabla^2 \boldsymbol{u}$$

$$\nabla^2 \phi = 0 \qquad in\ \mathbb{D} - \mathbb{D}_{\delta^*}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{dp}{dx} + \frac{1}{Re}\frac{\partial^2 u}{\partial y^2}$$

Easier to Solve

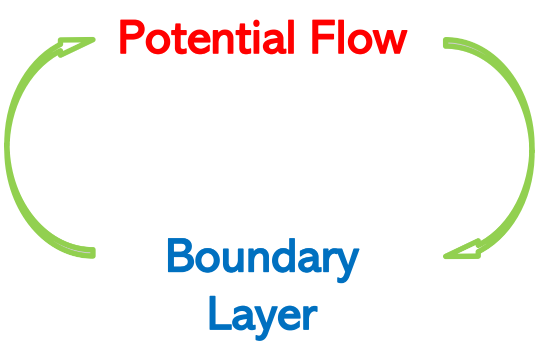
Correction to purely inviscid models

Accounts for Drag Force

Applicable to Moderate Reynolds Regimes

Predicts Separation and Stall

Predicts Instabilities and Turbulence



INTERACTIVE BOUNDARY LAYER

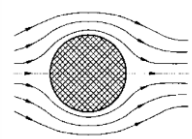
How to solve potential flow?

Thin Airfoil Theory

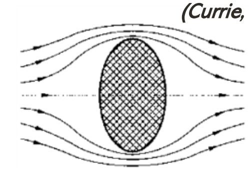
Exact Solutions

Analytical Solution

$$F = U(\zeta + \frac{a^2}{\zeta})$$

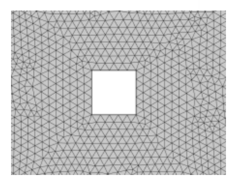


$$z = \zeta + \frac{c^2}{\zeta}$$

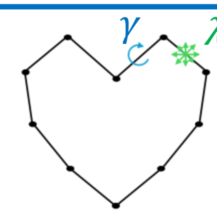


Photos adapted from (Currie, 2002)

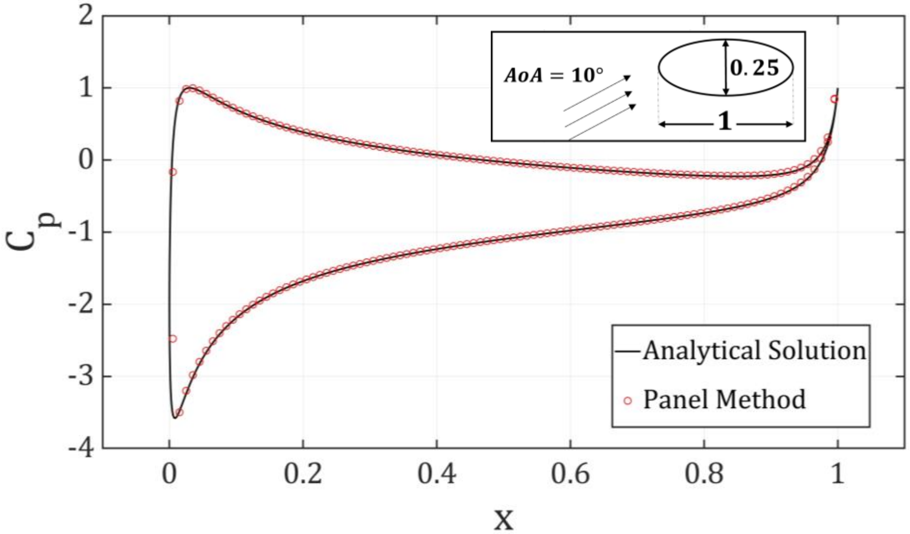
Finite Element



Panel Method



Example of potential flow solver



How to solve the boundary layer ?

Similarity Solution

$$f''' + ff'' + \beta\left(1 - f'^2\right) = 0$$

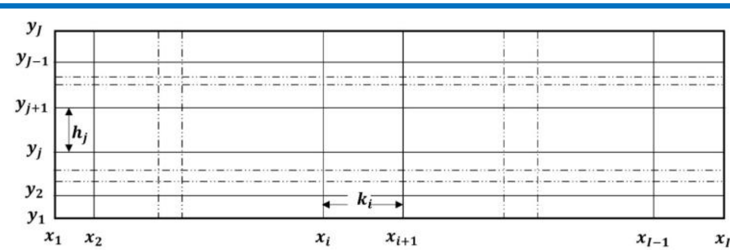
Falkner-Skan Solution

Momentum Integral Equation

$$\frac{d\theta}{dx} + (H+2)\theta\frac{1}{U}\frac{dU}{dx} = \frac{C_f}{2}$$

- Karman-Pohlhausen Method
- Thwaites Method
- Two-Equation Method
- ETC.

Finite Difference Solution



- Explicit Dufort-Frankel Scheme
- Implicit Crank-Nicolson Scheme
- Implicit Keller-Box Scheme
- ETC.

Boundary Layer Solution. Direct or Inverse?

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = u_e\frac{du_e}{dx} + \frac{1}{Re}\frac{\partial^2 u}{\partial y^2} \qquad \begin{cases} v(x,0) = v_w(x) \\ u(x,0) = u_s(x) \\ u(x,\infty) = u_e(x) \end{cases}$$

Direct or Standard Form

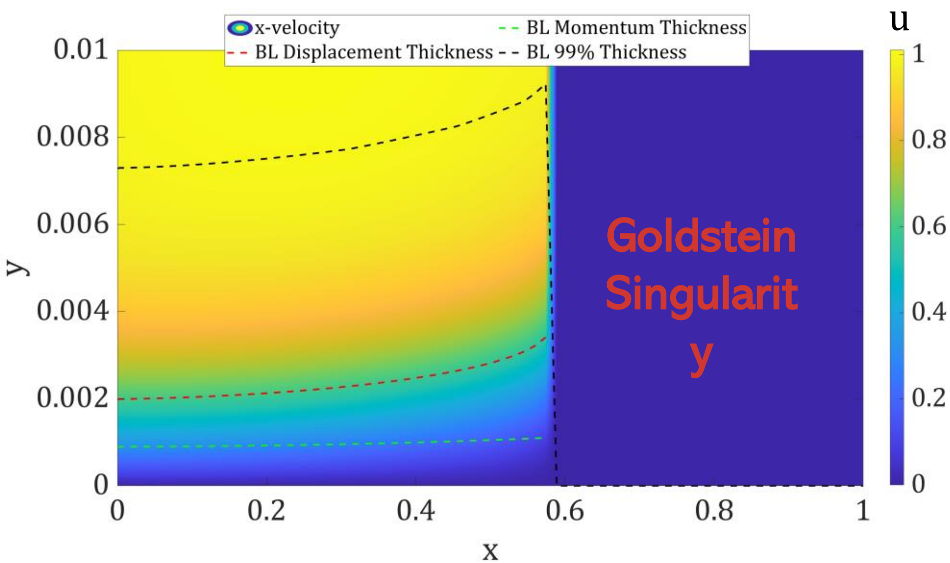
Breaks Down for Separated Flow

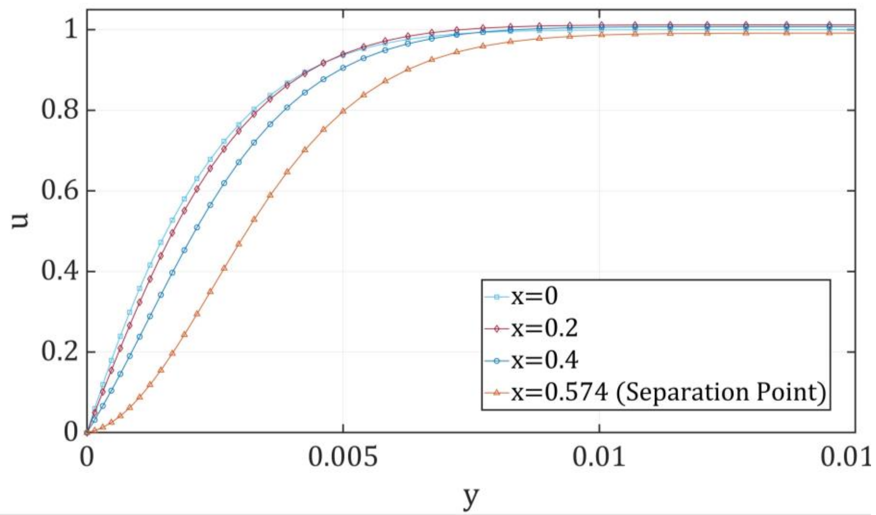
Goldstein Singularity

(Goldstein, 1948)

Separated Boundary Layer Example-Direct Solution

Test Case	Re	$u_e(x)$	$v_w(x)$	$u_s(x)$	$u(0,y)$
Separated Flow- Direct	10^6	$1+0.1x-0.2x^2$	0	0	Blasius Profile





Boundary Layer Solution. Direct or Inverse?

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = u_e\frac{du_e}{dx} + \frac{1}{Re}\frac{\partial^2 u}{\partial y^2}$$

$$\begin{cases} v(x,0) = v_w(x) \\ u(x,0) = u_s(x) \\ \int_0^\infty \left(1 - \frac{u(x,y)}{u_e(x)}\right) dy = \delta^*(x) \end{cases}$$

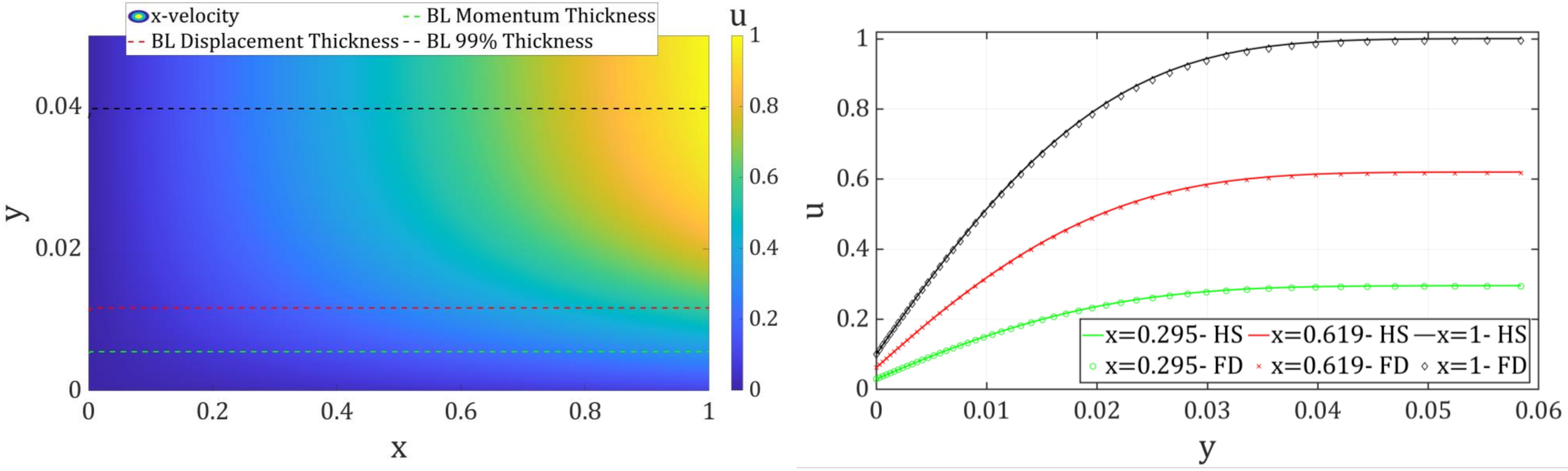
Inverse Form

Idea by (Catherall and Mangler, 1966)

Separated Solution

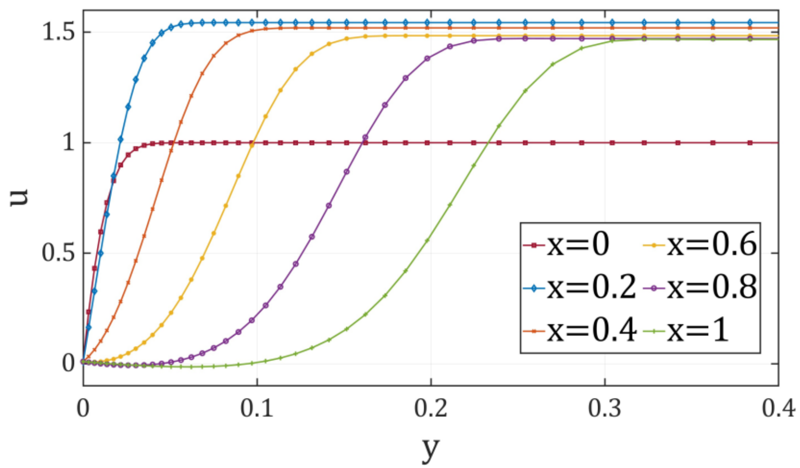
Verification-Stagnation Point Flow- Inverse Solution

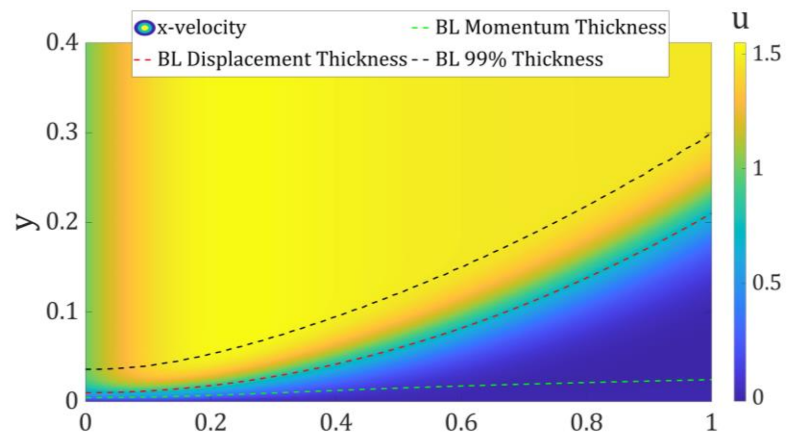
Test case	Re	$\delta^*(x)$	$v_w(x)$	$u_s(x)$	$u(0,y)$	$u(0,y_\infty)$
Stagnation flow	10000	0.01168	0.02	$0.001+0.1x$	Hiemenz Flow Profile	0.001

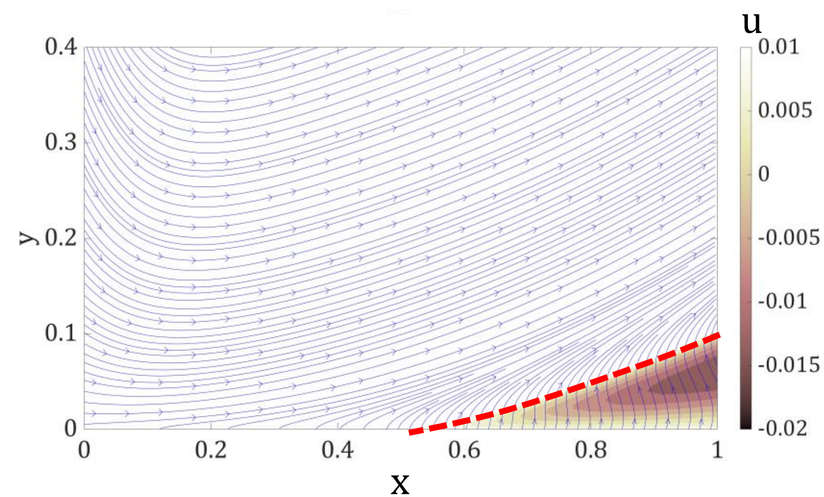


Separated Boundary Layer Example-Inverse Example

Test case	Re	$\delta^*(x)$	$v_w(x)$	$u_s(x)$	$u(0, y)$	$u(0, y_\infty)$
Separated flow	10000	$0.01 + 0.2x^2$	0.02	0.01	Pohlhausen Polynomial	1



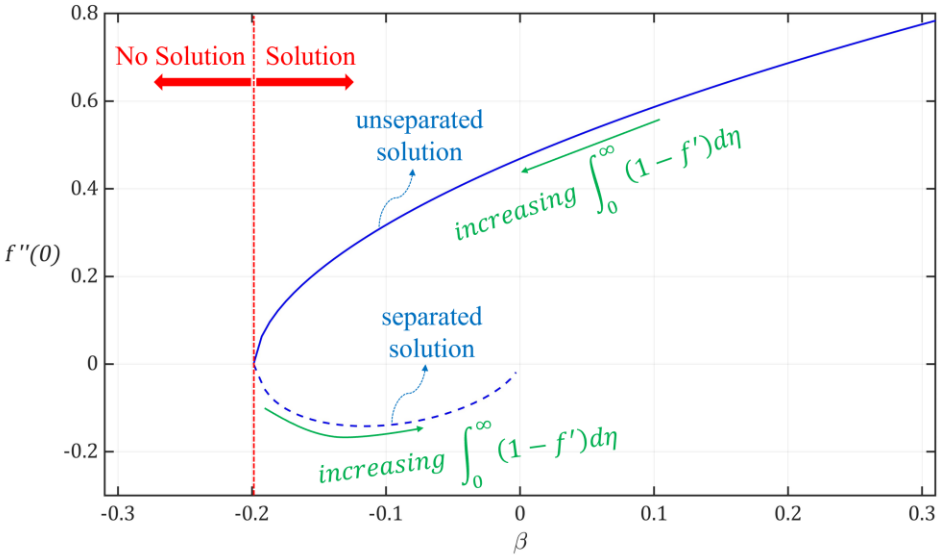


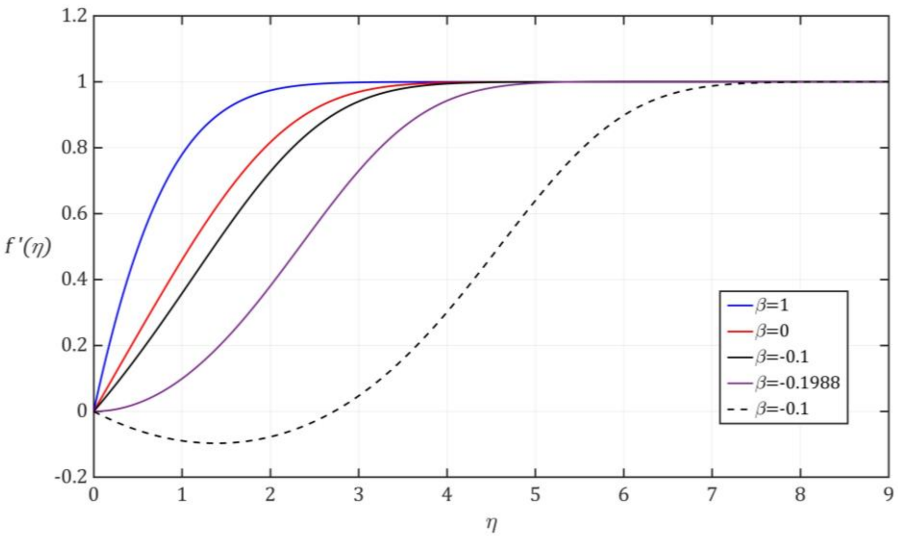


Boundary Layer Solution. Direct or Inverse?

Falkner-Skan Example

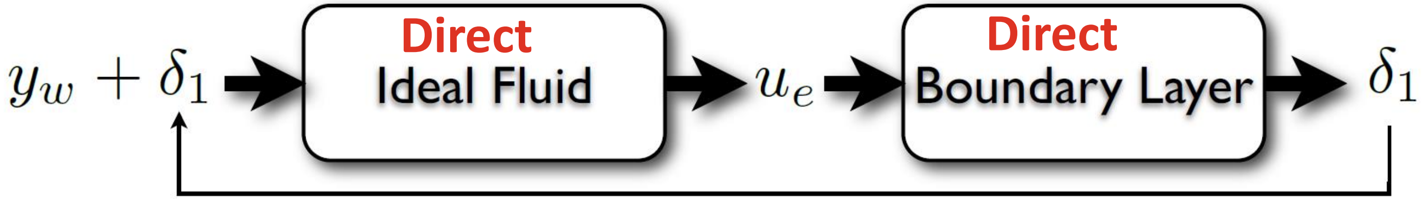
$$f''' + ff'' + \beta(1 - f'^2) = 0 \qquad u_e(x) = cx^{\frac{\beta}{2-\beta}}$$





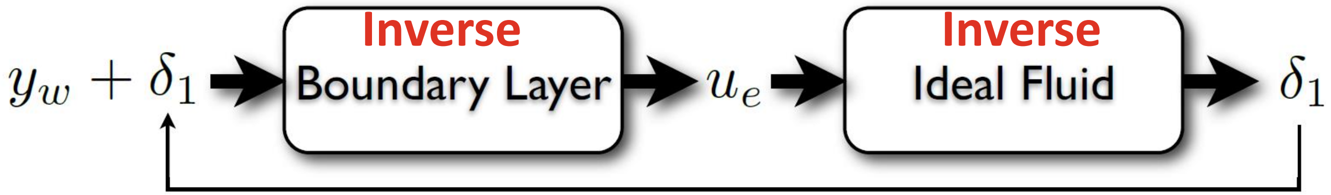
IBL Coupling Algorithms: Direct Coupling

->Natural but unstable! [does not overcome separation]



(Lagrée, 2010)

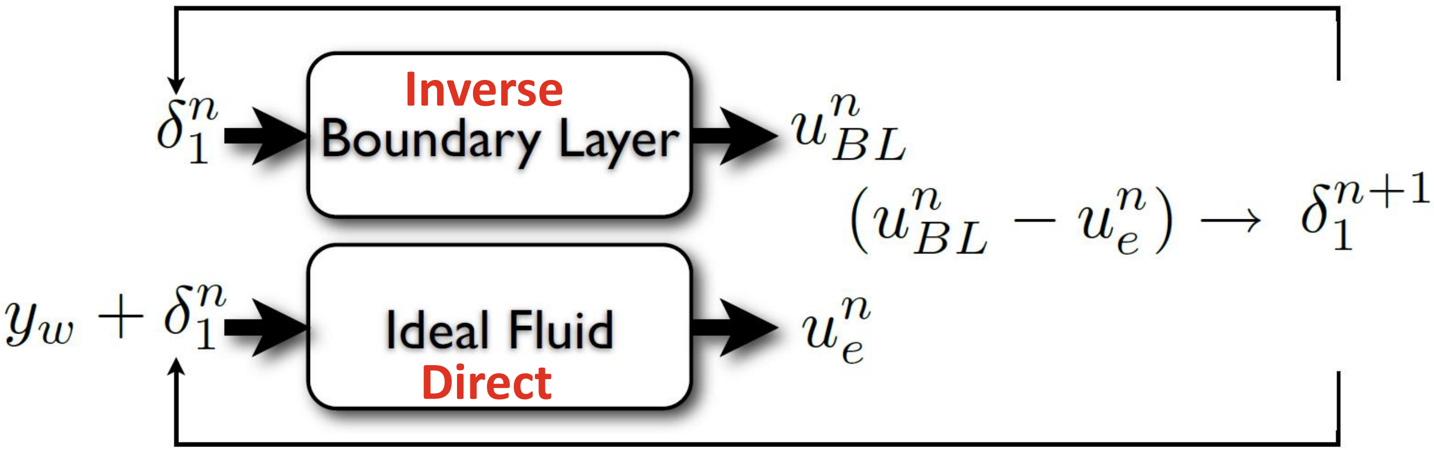
IBL Coupling Algorithms: Inverse Coupling
->Impractical



(Lagrée, 2010)

IBL Coupling Algorithms: Semi-Inverse Coupling

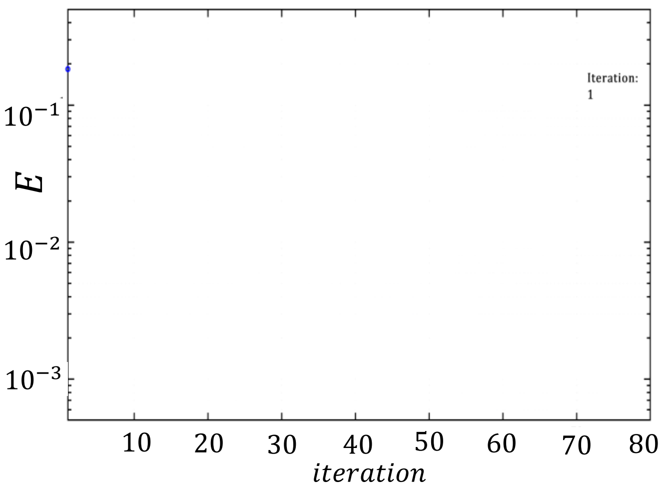
-> the way to go!

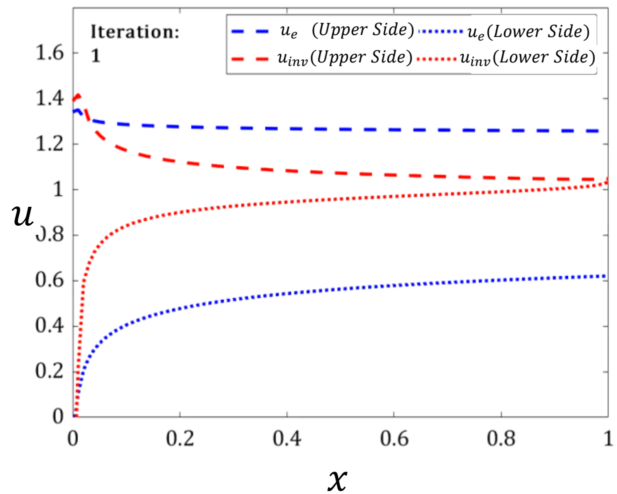


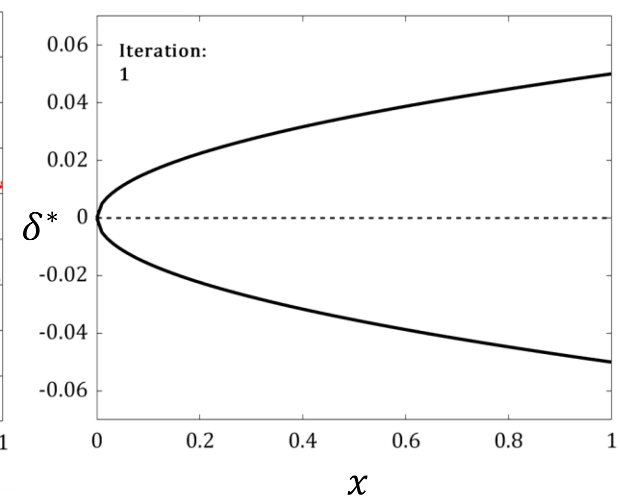
(Lagrée, 2010)

Convergence of IBL Algorithm

Porous Membrane with Porosity 0.9 and $Re_p = 1$ Test Case, Angle of Attack: 3°







$$(\delta_i^*)_{k+1}^{+} = (\delta_i^*)_k^{+} + \lambda\left[\left(u_e^k\right)_i^{+} - \left(u_{inv}^k\right)_i^{+}\right]$$

$$(\delta_i^*)_{k+1}^{-} = (\delta_i^*)_k^{-} + \lambda\left[\left(u_e^k\right)_i^{-} - \left(u_{inv}^k\right)_i^{-}\right]$$

$$E = \sqrt{\frac{1}{I}\sum_{i=1}^{I}\left(\frac{(u_{inv}^k)_i^{+} - (u_e^k)_i^{+}}{(u_{inv}^k)_i^{+} + (u_e^k)_i^{+}}\right)^2 + \left(\frac{(u_{inv}^k)_i^{-} - (u_e^k)_i^{-}}{(u_{inv}^k)_i^{-} + (u_e^k)_i^{-}}\right)^2}$$

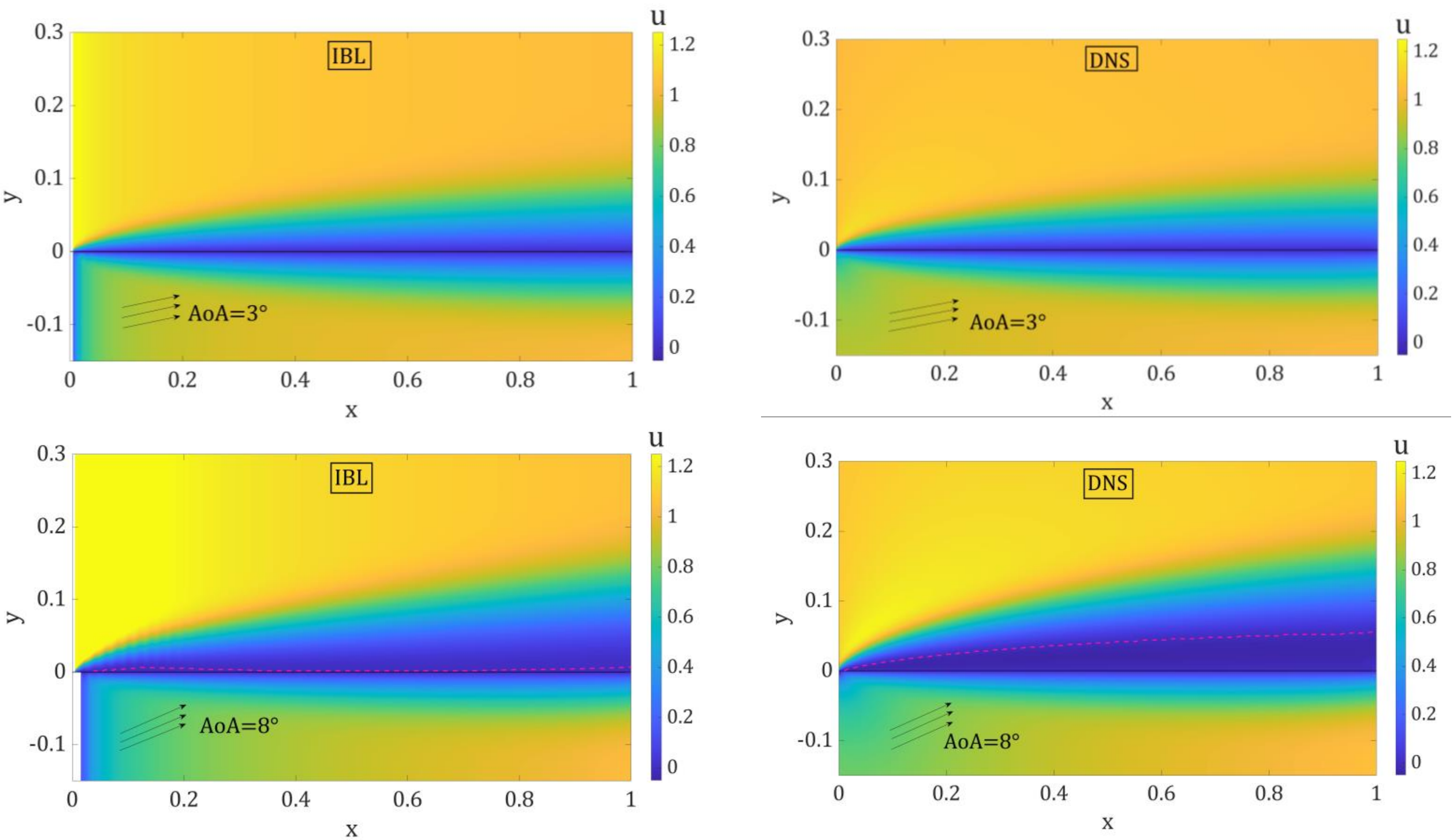
Comparison IBL-DNS

$Re = 1000,$

Angle of attack 3°

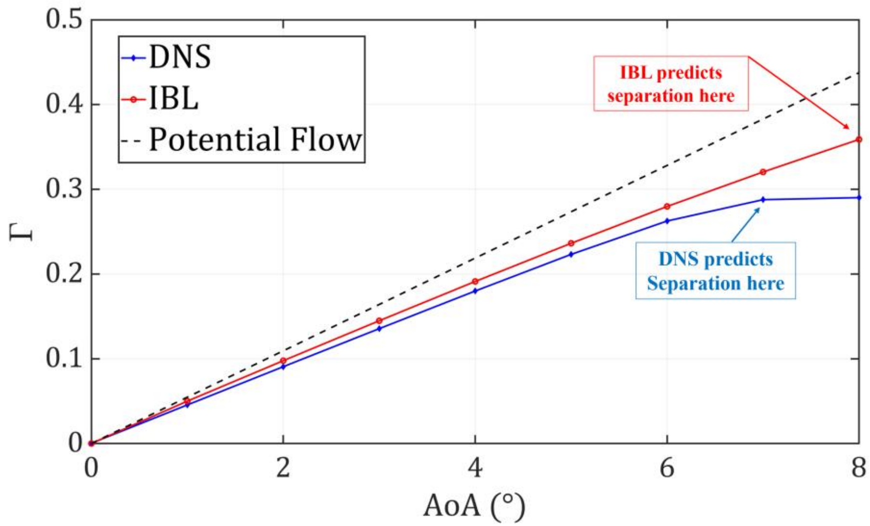
$Re = 1000,$

Angle of attack 8°



Circulation

$Re = 1000,$ Impermeable



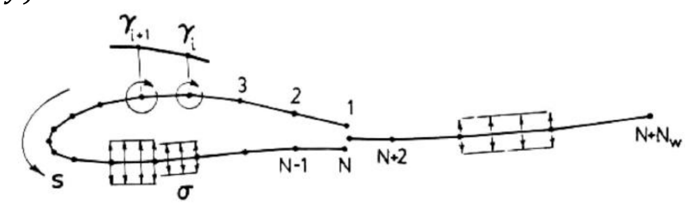
IBL is the machinery behind XFOIL

(Drela, 1989)

BL Mass Defect

Inviscid Analysis

$$\begin{cases} \Psi(x,y) = u_\infty y - v_\infty x + \dfrac{1}{2\pi}\displaystyle\int \gamma(s)\ln r(s;x,y)\,ds + \underbrace{\dfrac{1}{2\pi}\displaystyle\int \sigma(s)\ln r(s;x,y)\,ds}_{\text{BL Mass Defect}} \\ \displaystyle\sum_{j=1}^{N} a_{ij}\gamma_j - \Psi_0 = -u_\infty y_i + v_\infty x_i - \sum_{j=1}^{N+N_w-1} b_{ij}\sigma_j \end{cases}$$



Boundary Layer Analysis

$$\begin{cases} \dfrac{d\theta}{d\xi} + (H + 2 - M_e)\dfrac{\theta}{u_e}\dfrac{du_e}{d\xi} = \dfrac{C_f}{2} \\ \theta\dfrac{dH^*}{d\xi} + [2H^{**} + H^*(1-H)]\dfrac{\theta}{u_e}\dfrac{du_e}{d\xi} = 2C_D - H^*\dfrac{C_f}{2} \\ H^* = H^*(H_k, M_e, Re_\theta), H^{**} = H^{**}(H_k, M_e), C_f = C_f(H_k, M_e, Re_\theta), etc. \\ \dfrac{d\tilde{n}}{d\xi} = \dfrac{d\tilde{n}}{dRe_\theta}(H_k)\dfrac{dRe_\theta}{d\xi}(H_k, \theta) \quad \text{(In Laminar Regime)} \end{cases}$$

Inverse Viscous-Inviscid Coupling By Newton's Iteration

IBL is the machinery behind XFOIL

(Drela, 1989)

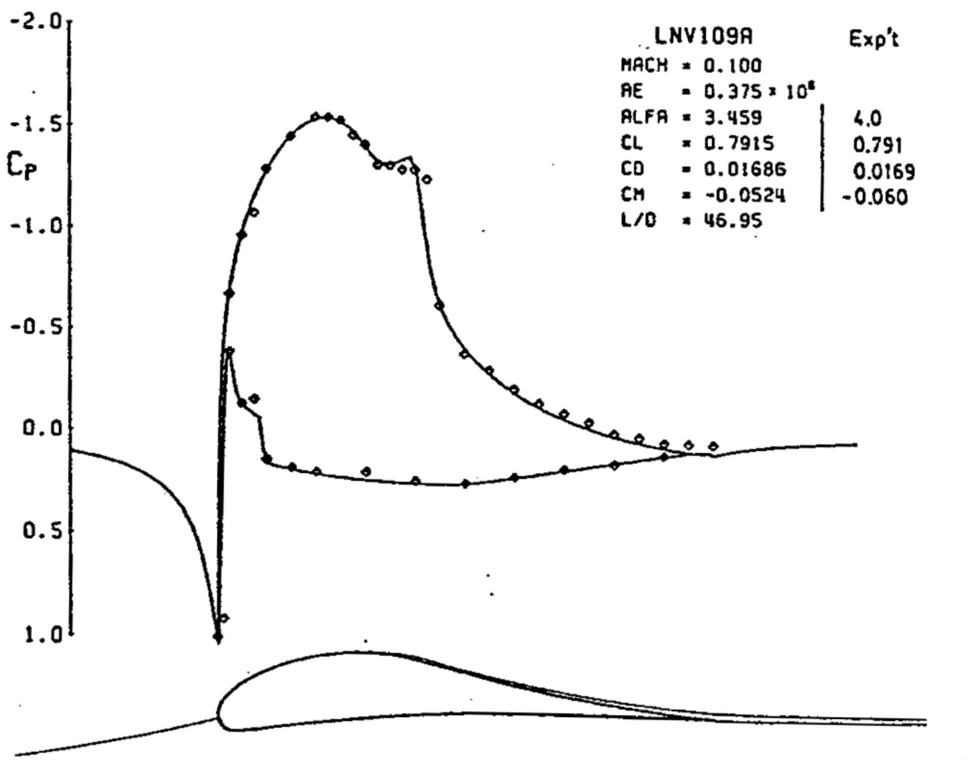
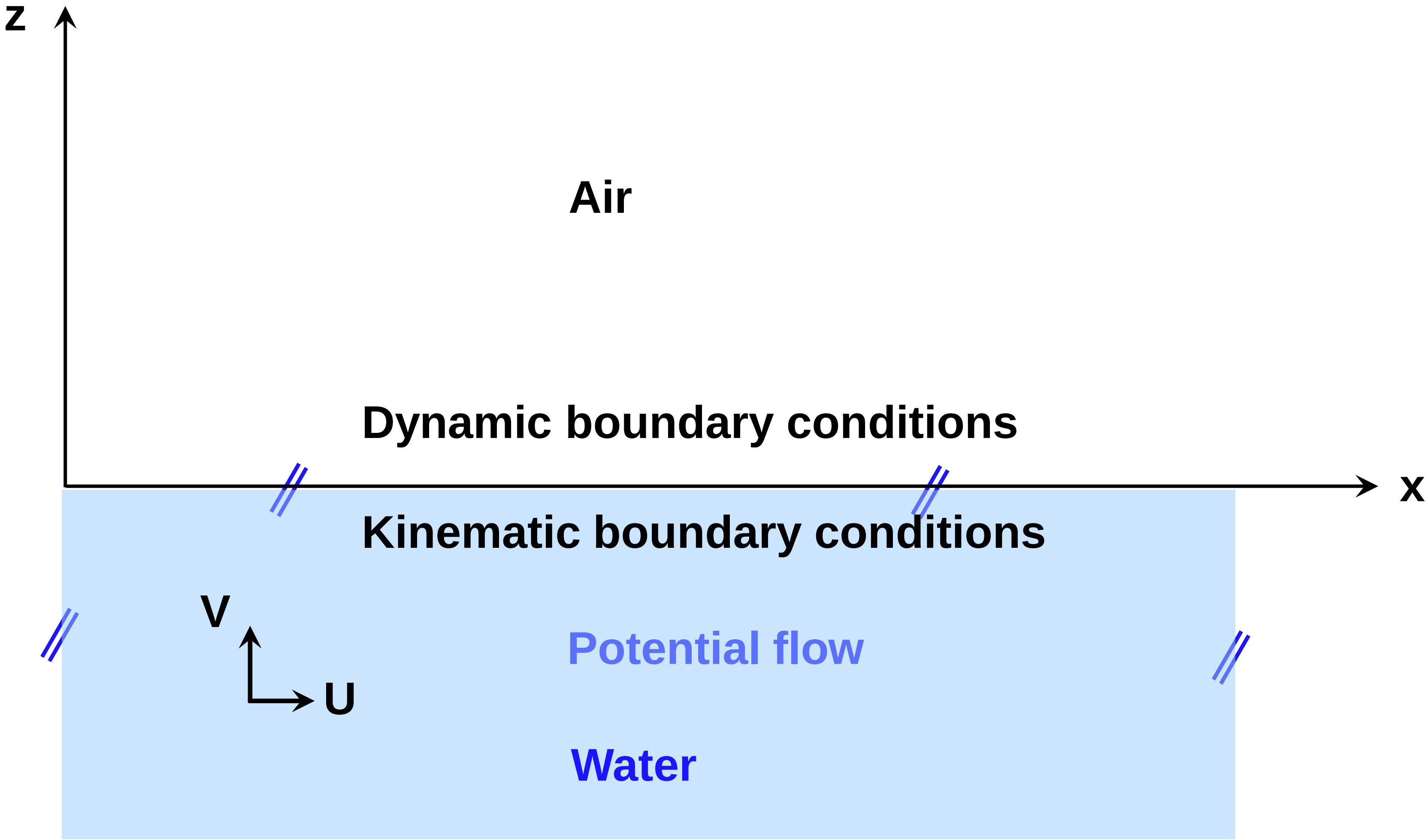


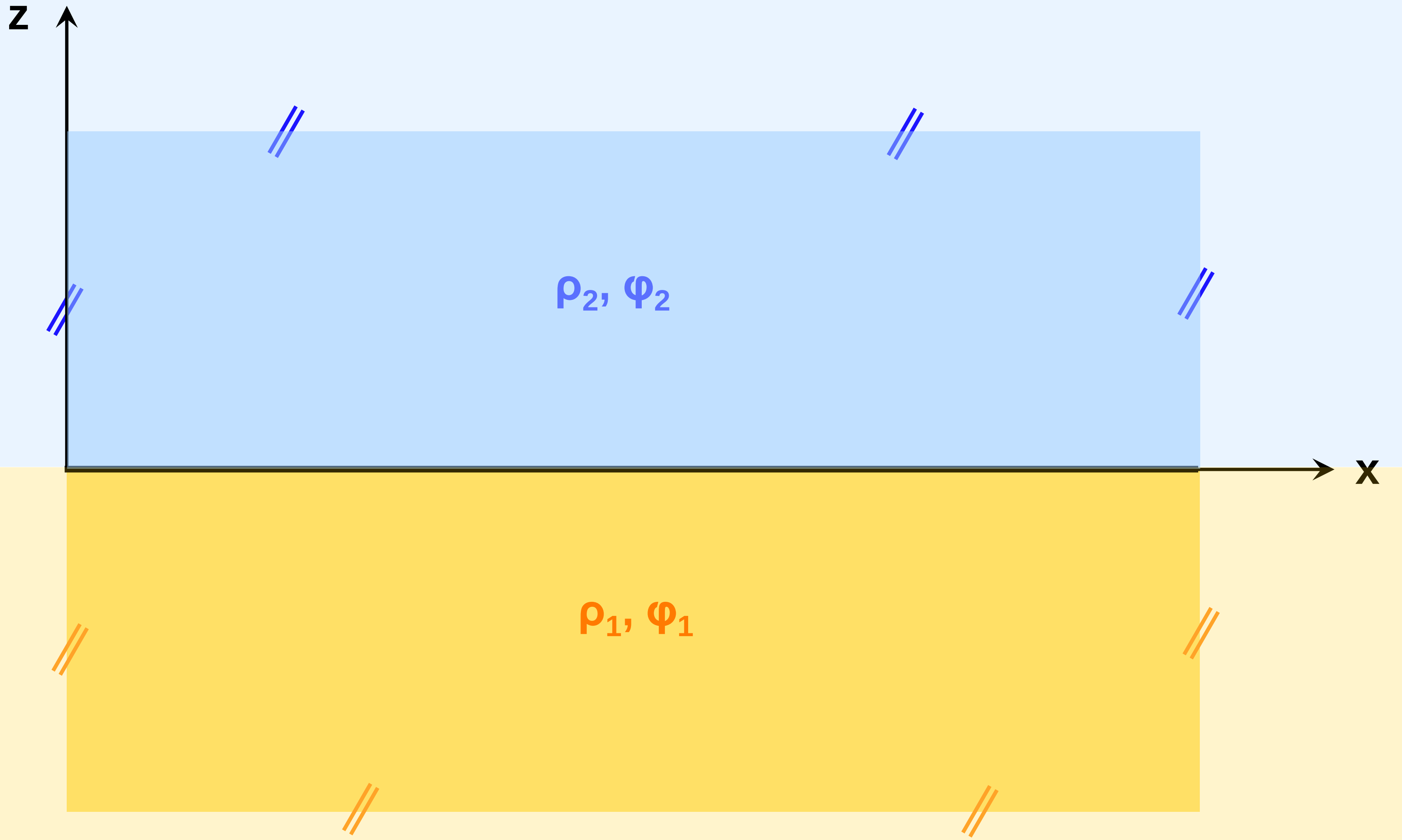
Fig. 9 LNV109A calculated and experimental pressure distributions.

Hydrodynamics 13
Waves

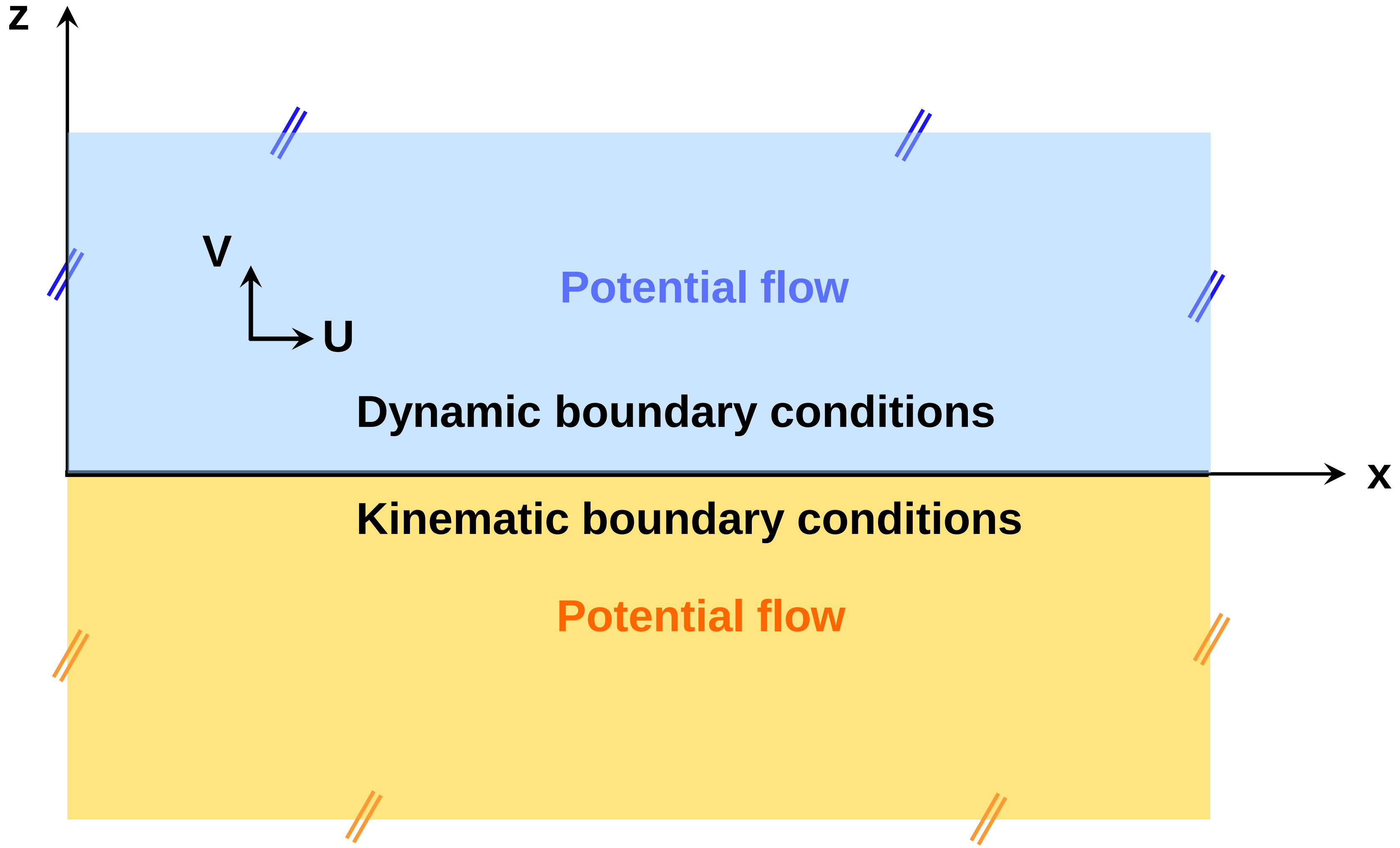
Waves


z
Air
Dynamic boundary conditions
x
Kinematic boundary conditions
V
Potential flow
U
Water

General case: two fluids



General case: two fluids



Linear waves dispersion relation

1. **Equations and boundary conditions**
2. **Base state**
3. **Linearized equations**
4. **Normal mode expansion**
5. **Dispersion relation**
6. **Analysis of the dispersion relation**

1. Equations

$$\begin{aligned}\Delta\Phi_1 &= 0\\ \Delta\Phi_2 &= 0\end{aligned}$$

Potential flow

$$U_1 = \frac{\partial\Phi_1}{\partial x}, \qquad V_1 = \frac{\partial\Phi_1}{\partial z}$$

$$U_2 = \frac{\partial\Phi_2}{\partial x}, \qquad V_2 = \frac{\partial\Phi_2}{\partial z}$$

Velocity field

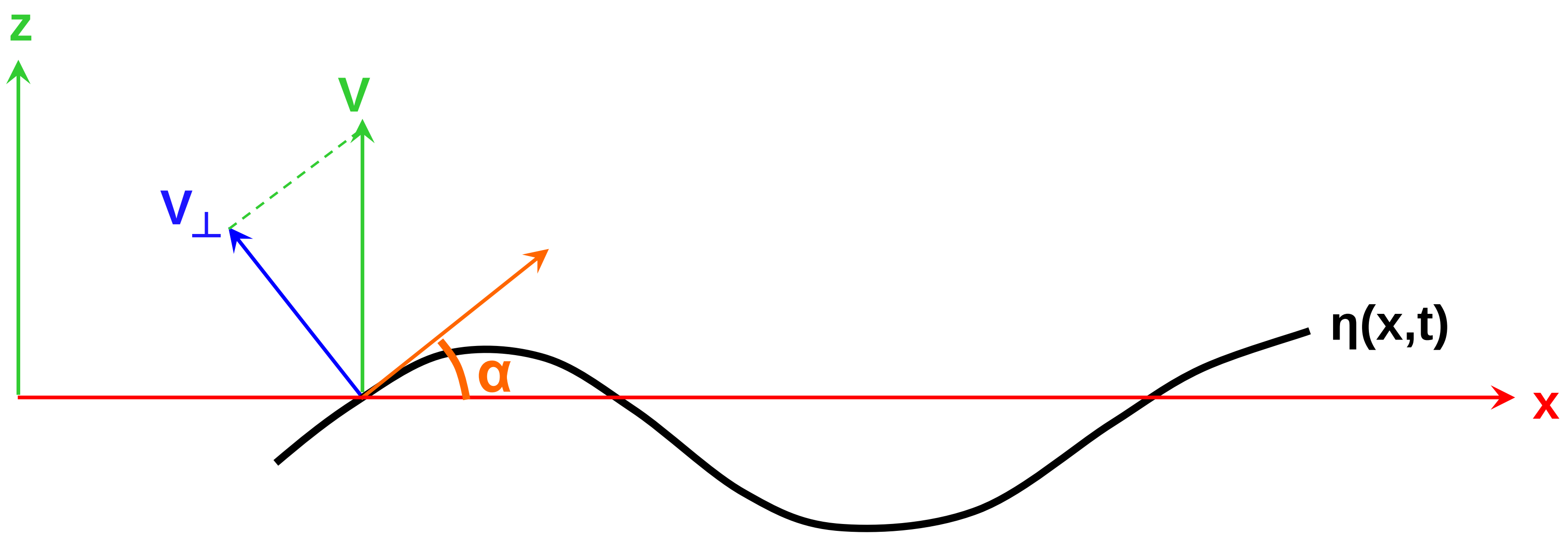
1. Boundary conditions

$$\Phi_1 = 0 \text{ at } z = -\infty$$
$$\Phi_2 = 0 \text{ at } z = +\infty$$

far-field

$\text{at } z = \eta$ **?**

1. Kinematic boundary condition

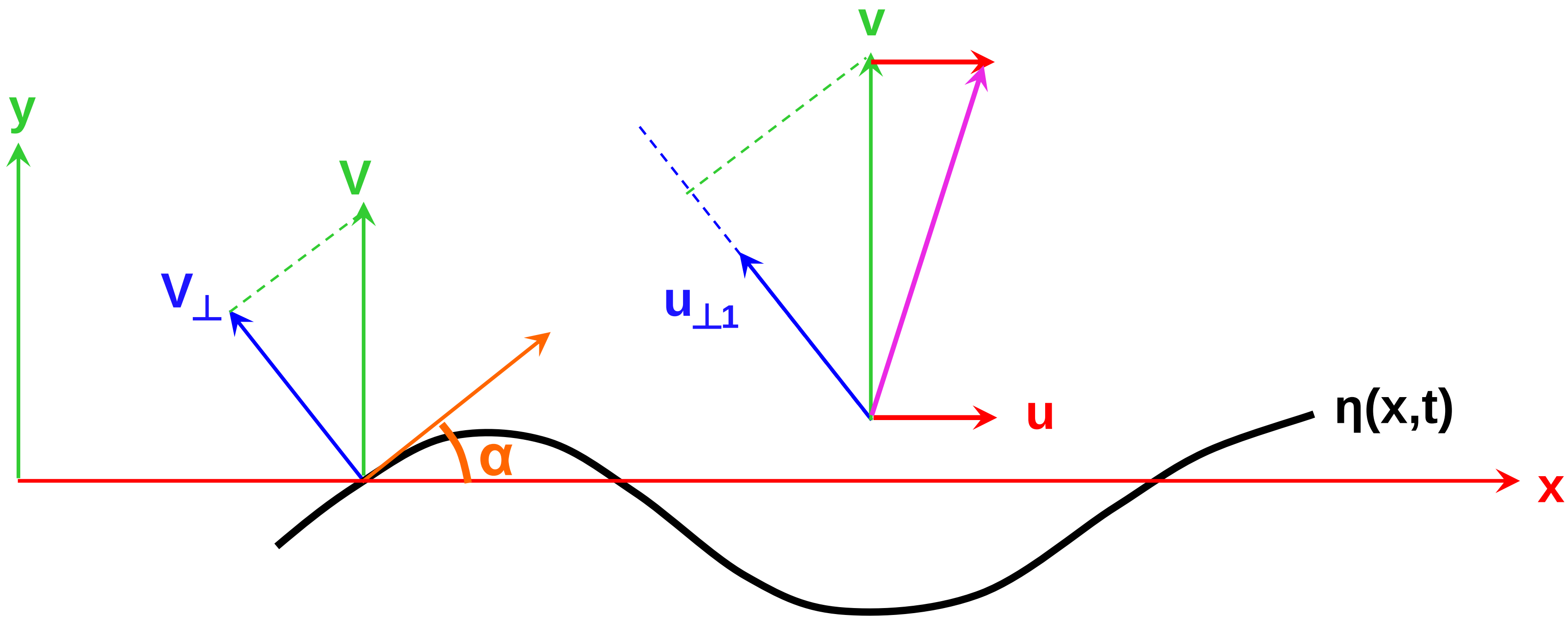


Kinematic condition : impermeability (no penetration)

No fluid particles going across the interface through the normal direction

$$V_{\perp} = \partial\eta/\partial t \cos(\alpha)$$

1. Kinematic boundary condition



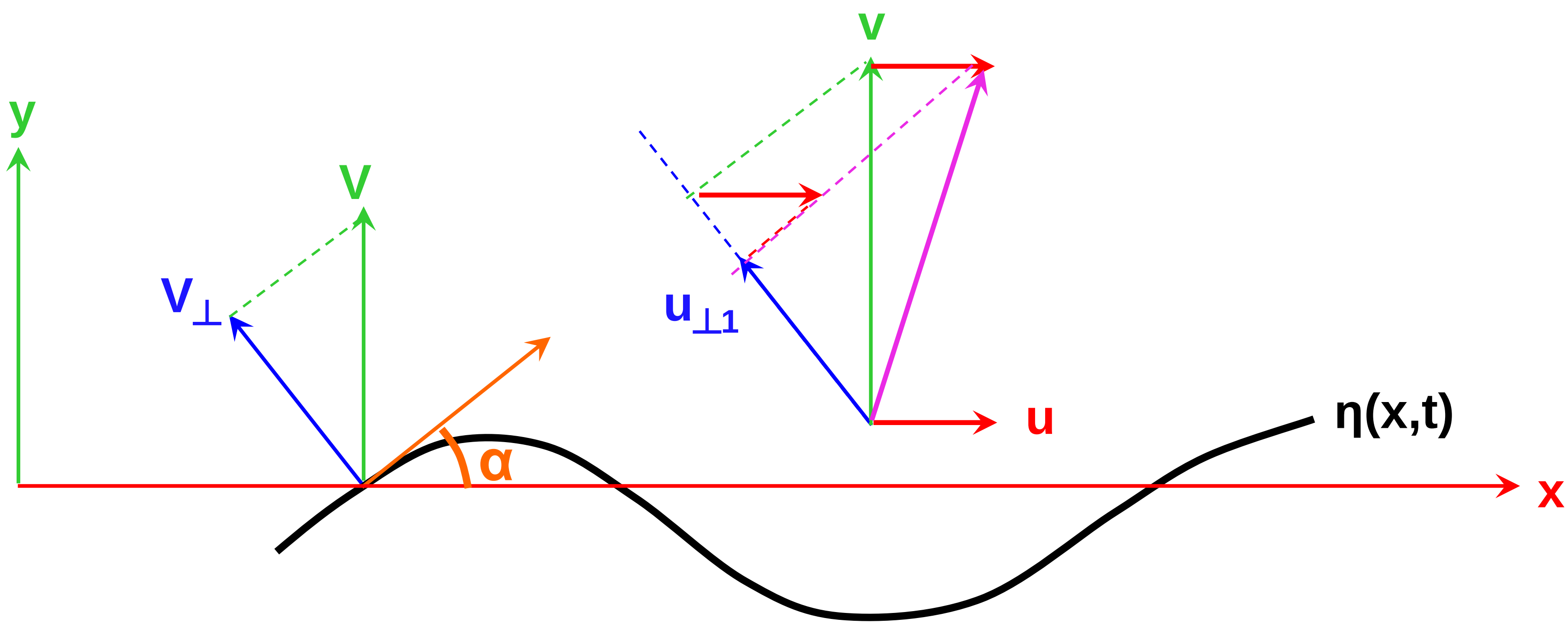
Kinematic condition : impermeability (no penetration)

No fluid particles going across the interface through the normal direction

$V_{\perp} = \partial\eta/\partial t \cos(\alpha)$

$u_{\perp 1} = v_1 \cos(\alpha) +$

1. Kinematic boundary condition



Kinematic condition : impermeability (no penetration)

No fluid particles going across the interface through the normal direction

$$V_\perp = \partial\eta/\partial t \cos(\alpha)$$

$$u_{\perp 1} = v_1 \cos(\alpha) - u_1 \sin(\alpha)$$

$$\partial\eta/\partial t = v_1 - u_1 \tan(\alpha) \;\Rightarrow\; \boxed{\partial\eta/\partial t = v_1 - u_1 \partial\eta/\partial x}$$

1. Kinematic boundary conditions

$$\Phi_1 = 0 \text{ at } z = -\infty$$
$$\Phi_2 = 0 \text{ at } z = +\infty$$

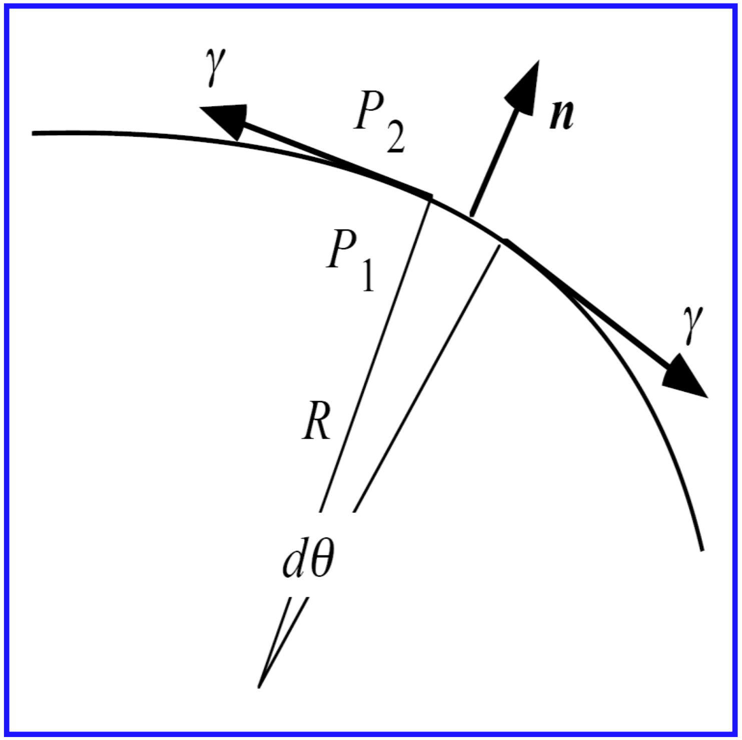
far-field

$$U_1 \frac{\partial \eta}{\partial x} - V_1 = -\frac{\partial \eta}{\partial t}$$
$$U_2 \frac{\partial \eta}{\partial x} - V_2 = -\frac{\partial \eta}{\partial t}$$

$$\text{at } z = \eta$$

1. Dynamic boundary conditions

$$P_1 - P_2 = -\gamma \frac{\dfrac{\partial^2 \eta}{\partial x^2}}{\left(1 + \dfrac{\partial \eta}{\partial x}^2\right)^{3/2}} \text{ at } z = \eta$$



$$\mathbf{n} = \frac{(-\partial_x \eta, 1)}{\sqrt{1 + \partial_x^2 \eta}}$$

$$\mathcal{C} = \nabla . \mathbf{n}$$

1. More equations

$$\frac{\partial \Phi_1}{\partial t} + \frac{U_1^2 + V_1^2}{2} + \frac{P_1}{\rho_1} + gz = C_1(t) \quad = 0$$

$$\frac{\partial \Phi_2}{\partial t} + \frac{U_2^2 + V_2^2}{2} + \frac{P_2}{\rho_2} + gz = C_2(t) \quad = 0$$

2nd Bernouilli relations

2. Base state

$$\Phi_1 = 0,$$

$$\Phi_2 = 0$$

$$\eta = 0$$

$$P_1 = -\rho_1 g z$$

3. Perturb and linearize perturbation expansion

$$
\begin{aligned}
\Phi_1 &= 0 &+\epsilon\phi_1 \\
\Phi_2 &= 0 &+\epsilon\phi_2 \\
U_1 &= 0 &+\epsilon u_1 \\
V_1 &= 0 &+\epsilon v_1 \\
U_2 &= 0 &+\epsilon u_2 \\
V_2 &= 0 &+\epsilon v_2 \\
P_1 &= -\rho_1 g z &+\epsilon p_1 \\
P_2 &= -\rho_2 g z &+\epsilon p_2 \\
\eta &= 0 &+\epsilon\sigma
\end{aligned}
$$

$$\epsilon \ll 1$$

Variables **Base state** **Small perturbation**

3. Linearized equations

$$
\begin{aligned}
\Delta\phi_1 &= 0 \\
\Delta\phi_2 &= 0
\end{aligned}
$$

perturbed potential flow

$$
u_1 = \frac{\partial \phi_1}{\partial x}, \qquad v_1 = \frac{\partial \phi_1}{\partial z}
$$

$$
u_2 = \frac{\partial \phi_2}{\partial x}, \qquad v_2 = \frac{\partial \phi_2}{\partial z}
$$

3. Perturbed kinematic boundary conditions

$$\phi_1 = 0 \text{ at } z = -\infty$$

$$\phi_2 = 0 \text{ at } z = +\infty$$

$$-\epsilon^2 u_1 \frac{\partial \sigma}{\partial x} + \epsilon v_1 = \epsilon \frac{\partial \sigma}{\partial t} \text{ at } z = \epsilon\sigma$$

$$-\epsilon^2 u_2 \frac{\partial \sigma}{\partial x} + \epsilon v_2 = \epsilon \frac{\partial \sigma}{\partial t} \text{ at } z = \epsilon\sigma$$

3. Perturbed kinematic boundary conditions

$$\phi_1 = 0 \text{ at } z = -\infty$$

$$\phi_2 = 0 \text{ at } z = +\infty$$

$$-\epsilon^2 u_1 \frac{\partial \sigma}{\partial x} + \epsilon v_1 = \epsilon \frac{\partial \sigma}{\partial t} \text{ at } z = \epsilon\sigma$$

$$-\epsilon^2 u_2 \frac{\partial \sigma}{\partial x} + \epsilon v_2 = \epsilon \frac{\partial \sigma}{\partial t} \text{ at } z = \epsilon\sigma$$

$$v_1 = \frac{\partial \sigma}{\partial t} \text{ at } z = \epsilon\sigma$$

$$v_2 = \frac{\partial \sigma}{\partial t} \text{ at } z = \epsilon\sigma$$

3. Flattened kinematic boundary conditions

$$\frac{\partial \phi_1}{\partial z} = \frac{\partial \sigma}{\partial t} \text{ at } z = \epsilon\sigma$$
$$\frac{\partial \phi_2}{\partial z} = \frac{\partial \sigma}{\partial t} \text{ at } z = \epsilon\sigma$$

Taylor expansion around 0: $\phi(\epsilon\sigma) = \phi(0) + (\epsilon\sigma)\frac{\partial \phi}{\partial z}\Big|_0$

$$\frac{\partial \phi_1}{\partial z} = \frac{\partial \sigma}{\partial t} \text{ at } z = 0$$
$$\frac{\partial \phi_2}{\partial z} = \frac{\partial \sigma}{\partial t} \text{ at } z = 0$$

⇨transforms a b.c. at an unkwown interface into a fixed place!

3. Perturbed dynamic boundary conditions

$$\left(P_1 + \epsilon p_1 - P_2 - \epsilon p_2\right)\Big|_{\epsilon\sigma} = -\gamma\epsilon\frac{\partial^2\sigma}{\partial x^2}\left(1 - 3/2\epsilon^2\left(\frac{\partial\sigma}{\partial x}\right)^2\right)$$

Replace P_1=-gρ_1z, … **and linearize**

$$\mathbf{g}\left(\rho_2 - \rho_1\right)\sigma + \left(p_1 - p_2\right)\Big|_{\epsilon\sigma} = -\gamma\frac{\partial^2\sigma}{\partial x^2}$$

flatten

$$\left(\rho_2 - \rho_1\right)g\sigma + \left(p_1 - p_2\right)\Big|_0 = -\gamma\frac{\partial^2\sigma}{\partial x^2}$$

3. Perturbed and linearized Bernouilli

Perturbed 2nd Bernouilli relations

$$\epsilon \frac{\partial \phi_1}{\partial t} + \epsilon^2 \frac{u_1^2 + v_1^2}{2} + \epsilon \frac{p_1}{\rho_1} = 0$$

$$\epsilon \frac{\partial \phi_2}{\partial t} + \epsilon^2 \frac{u_2^2 + v_2^2}{2} + \epsilon \frac{p_2}{\rho_2} = 0$$

Linearized 2nd Bernouilli relations

$$\frac{\partial \phi_1}{\partial t} + \frac{p_1}{\rho_1} = 0$$

$$\frac{\partial \phi_2}{\partial t} + \frac{p_2}{\rho_2} = 0$$

4. Normal mode expansion

Fourier transform in x and t

$$\begin{aligned}\phi_1 &= f_1(z)\exp(i(kx-\omega t)),\\ \phi_2 &= f_2(z)\exp(i(kx-\omega t)),\\ \sigma &= C\exp(i(kx-\omega t)),\end{aligned}$$

k is the wavenumber and ω the frequency (in rad/s)

$$\lambda = 2\pi/k \qquad T = 2\pi/\omega$$

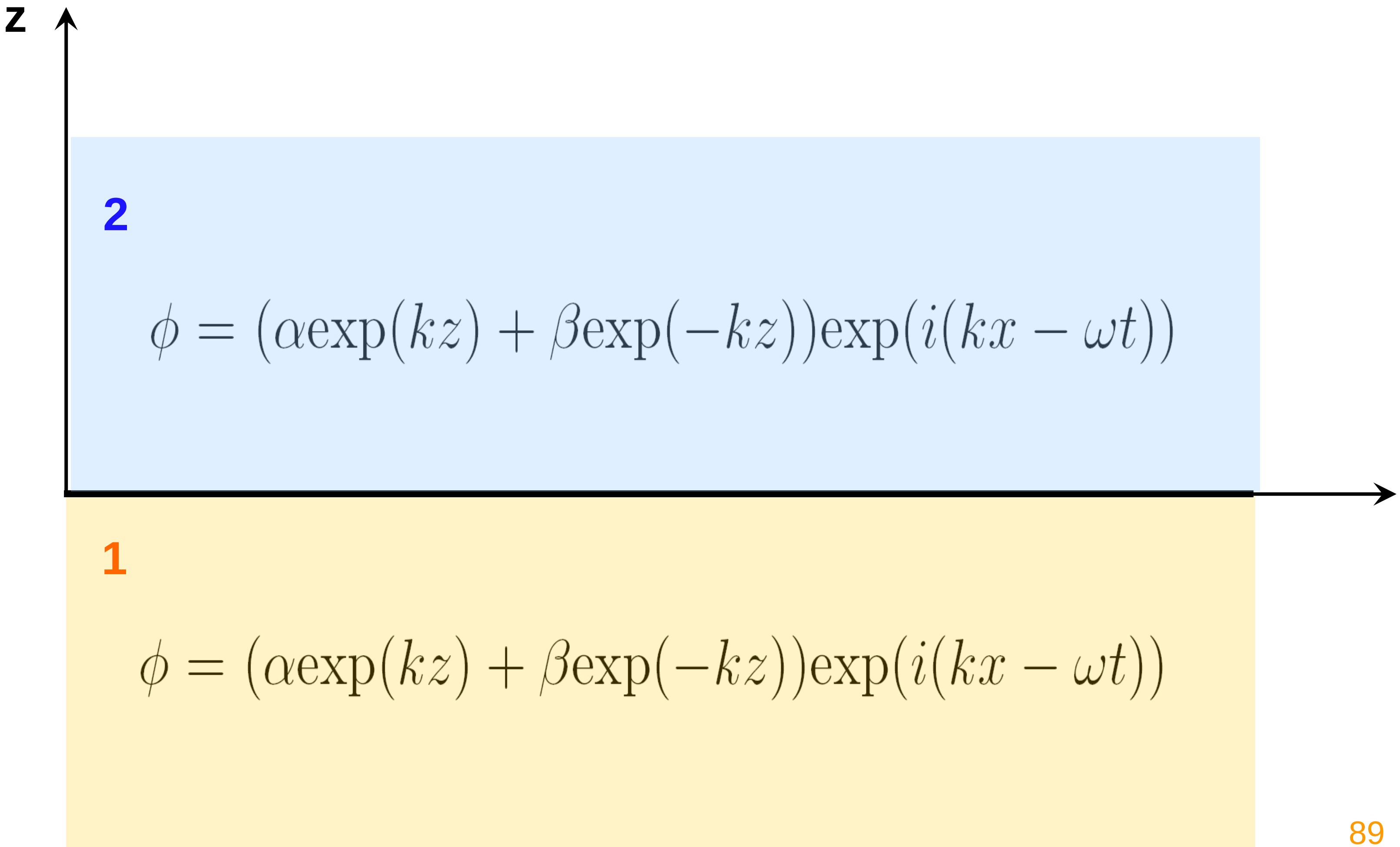
$$f = \omega/(2\pi)$$

4. Normal mode expansion

Solution to Laplace equation:

4. Normal mode expansion

Solution to Laplace equation:



4. Normal mode expansion

Solution to Laplace equation:

$$\begin{aligned}\phi_1 &= A\exp(kz)\exp(i(kx-\omega t)),\\ \phi_2 &= B\exp(-kz)\exp(i(kx-\omega t)),\\ \sigma &= C\exp(i(kx-\omega t)).\end{aligned}$$

4. Normal mode expansion

Replace in boundary conditions

$$\begin{aligned} g(\rho_2 - \rho_1)C + i\omega\rho_1 A - i\omega\rho_2 B &= \gamma k^2 C \\ kA &= -i\omega C \\ -kB &= -i\omega C \end{aligned}$$

This is an eigenvalue problem iωX=MX!

$$kg(\rho_2 - \rho_1)C + \omega^2\rho_1 C + \omega^2\rho_2 C = \gamma k^3 C$$

5. Dispersion relation

$$\omega^2 = \frac{-kg(\rho_2 - \rho_1) + \gamma k^3}{\rho_1 + \rho_2}$$

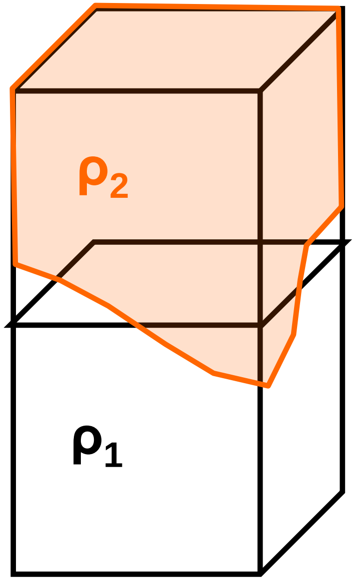
- **Unstable if there exists one ω, Im(ω)>0** $\rho_2 > \rho_1$
- **Neutral if for all ω, Im(ω)=0:** $\rho_1 > \rho_2$
- **Stable (or damped) if for all ω, Im(ω)<0:**

The flow considered is not damped, we have neglected dissipation by neglecting viscosity

Instability analysis:

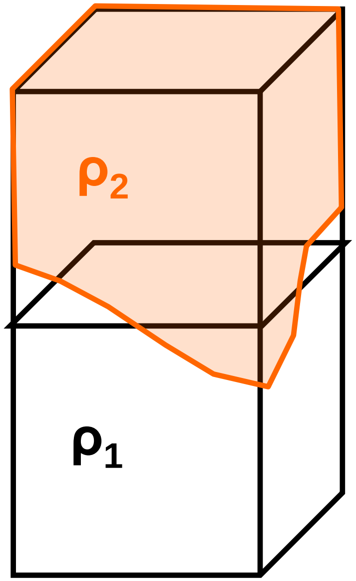
1. **Equations and boundary conditions**
2. **Base state**
3. **Linearized equations**
4. **Normal mode expansion**
5. **Dispersion relation**
6. **Analysis of the dispersion relation**

Dispersion relation



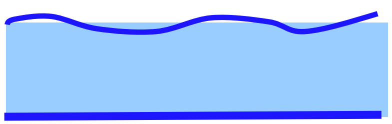
$$\omega^2 = \frac{-kg(\rho_2 - \rho_1) + \gamma k^3}{\rho_1 + \rho_2}$$

Dispersion relation



z

$$\omega^2 = \frac{-kg(\rho_2 - \rho_1) + \gamma k^3}{\rho_1 + \rho_2}$$



$$\omega^2 = \tanh(kH)\left(\frac{\gamma k^3}{\rho} + gk\right)$$

Dispersion relation

$$\omega^2 = \tanh(kH)\left(\frac{\gamma k^3}{\rho} + gk\right)$$

Capillary wavenumber: $k_c = \sqrt{\rho g/\gamma}$

Length scale: $\tilde{k} = k/k_c$

Time scale $\tilde{\omega} = \omega/\sqrt{gk_c}$

One single non-dimensional parameter $\tilde{H} = Hk_c$

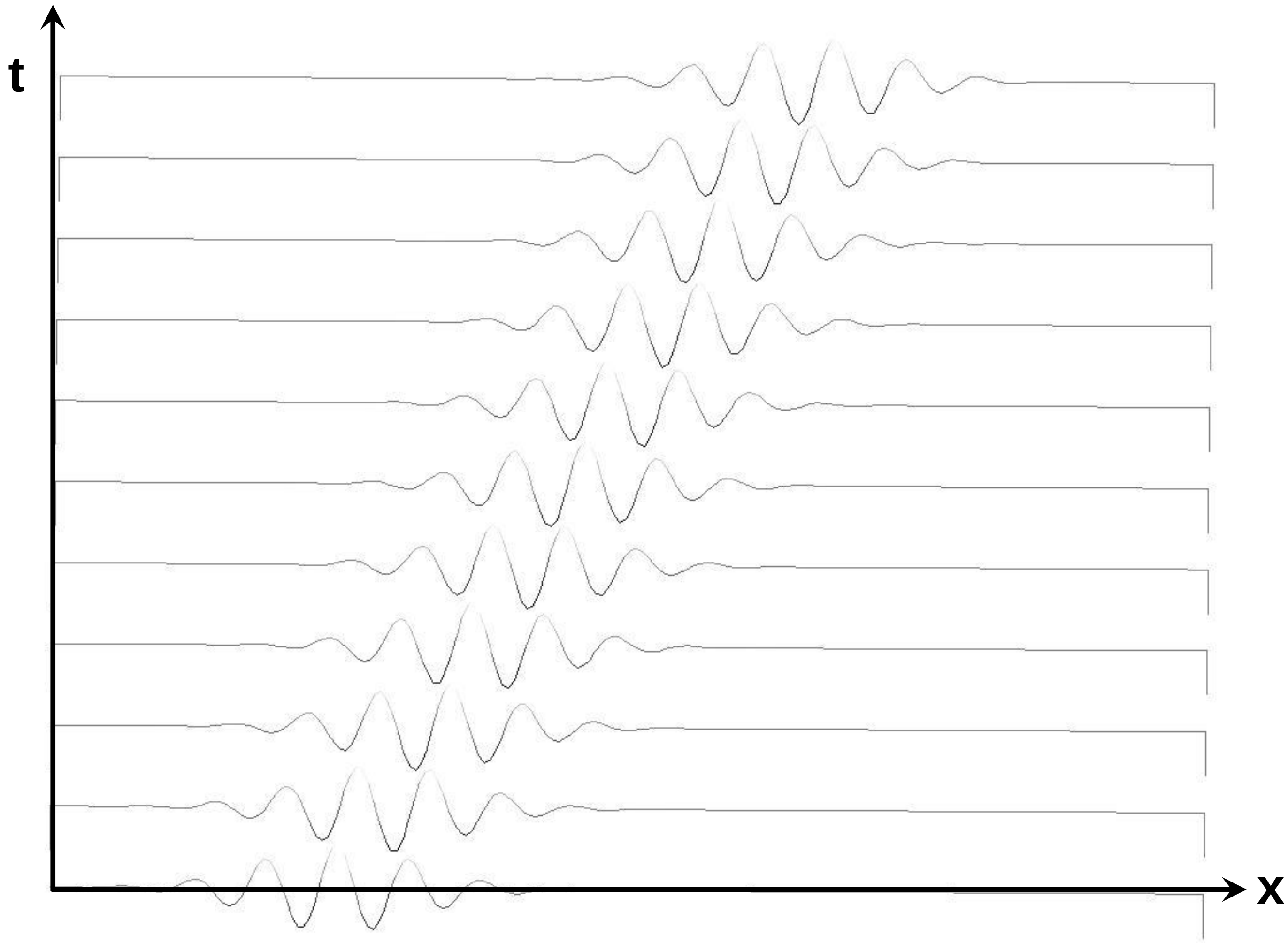
$$\tilde{\omega}^2 = \tanh(\tilde{k}\tilde{H})\left(\tilde{k}^3 + \tilde{k}\right)$$

Dispersion relation

$$\tilde{\omega}^2 = \tanh(\tilde{k}\tilde{H})\left(\tilde{k}^3 + \tilde{k}\right)$$

	gravity $\tilde{k} \ll 1$	capillary $\tilde{k} \gg 1$
shallow water $\tilde{k} \ll 1/\tilde{H}$	$\pm\tilde{k}$	$\pm\tilde{k}^2\sqrt{\tilde{H}}$
Deep water $\tilde{k} \gg 1/\tilde{H}$	$\pm\sqrt{\tilde{k}}$	$\pm\tilde{k}\sqrt{\tilde{k}}$

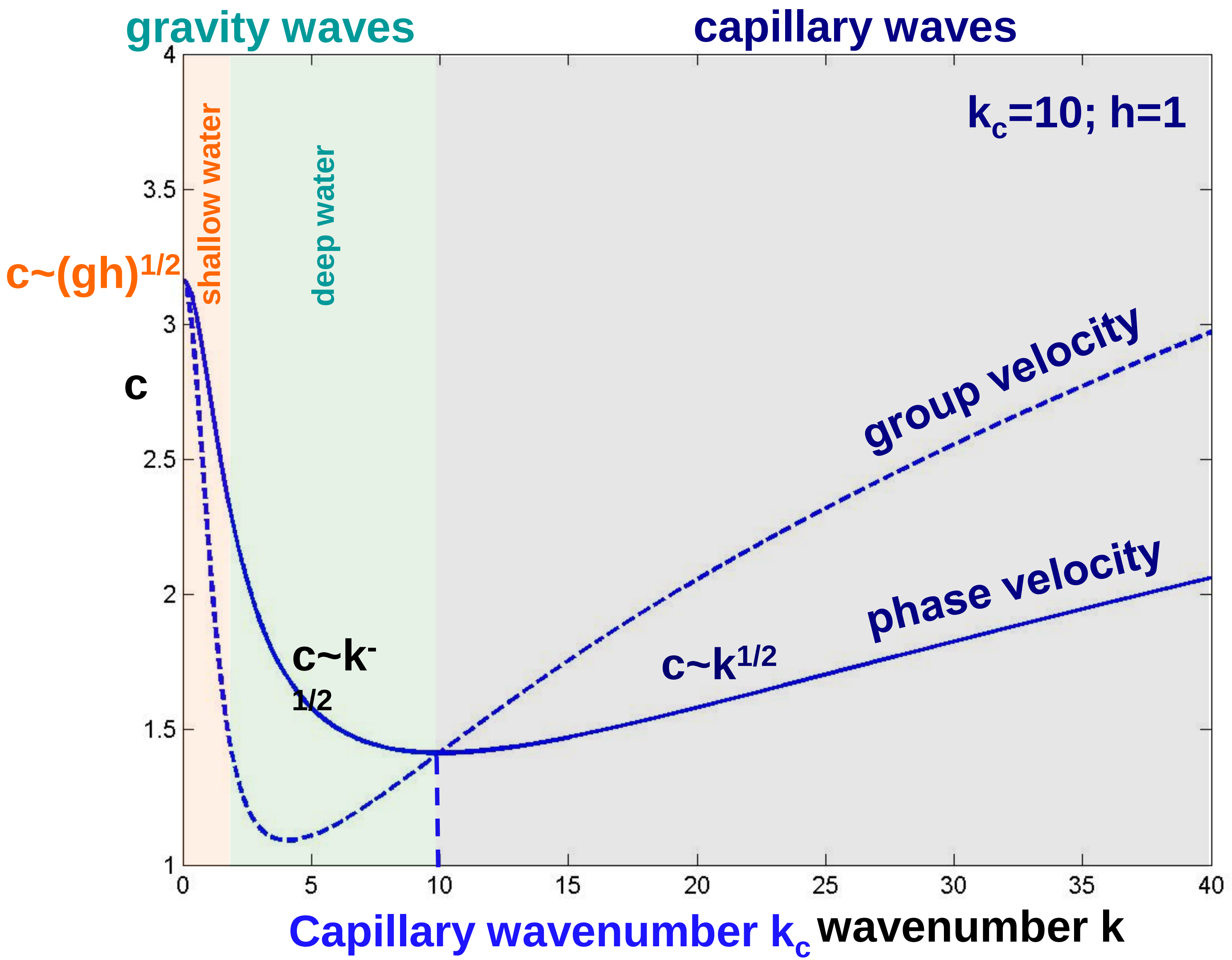
Difference between group velocity v and phase velocity c



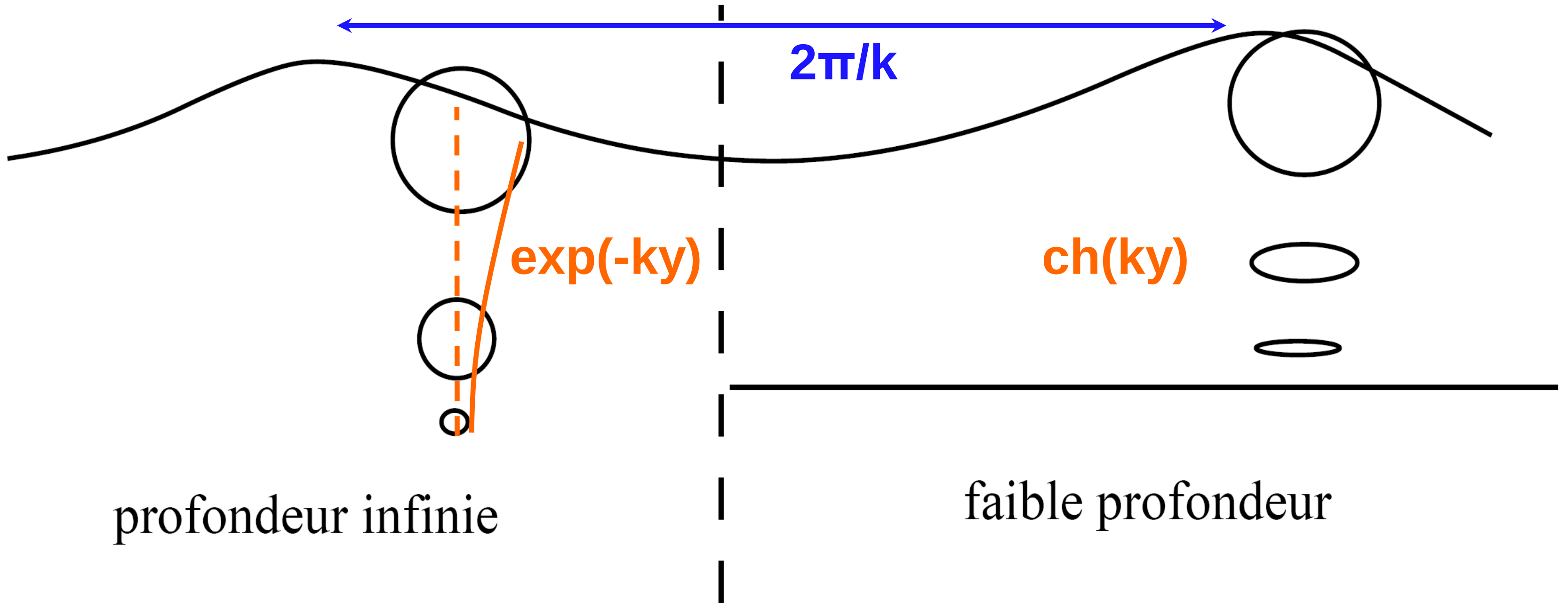
Dispersion relation

	gravity $\tilde{k} \ll 1$	capillary $\tilde{k} \gg 1$
shallow water $\tilde{k} \ll 1/\tilde{H}$	$\omega_{shallow/gravity} \sim \pm k\sqrt{gH}$ $c_{shallow/gravity} \sim \pm\sqrt{gH}$ $v_{shallow/gravity} \sim \pm\sqrt{gH}$	$\omega_{shallow/capillary} \sim \pm k^2\sqrt{\gamma H/\rho}$ $c_{shallow/capillary} \sim \pm k\sqrt{\gamma H/\rho}$ $v_{shallow/capillary} \sim \pm 2k\sqrt{\gamma H/\rho}$
Deep water $\tilde{k} \gg 1/\tilde{H}$	$\omega_{deep/gravity} \sim \pm\sqrt{gk}$ $c_{deep/gravity} \sim \pm\sqrt{\frac{g}{k}}$ $v_{deep/gravity} \sim \pm\frac{1}{2}\sqrt{\frac{g}{k}}$	$\omega_{deep/capillary} \sim \pm k^{3/2}\sqrt{\gamma/\rho}$ $c_{deep/capillary} \sim \pm k^{1/2}\sqrt{\gamma/\rho}$ $v_{deep/capillary} \sim \pm 3/2k^{1/2}\sqrt{\gamma/\rho}$

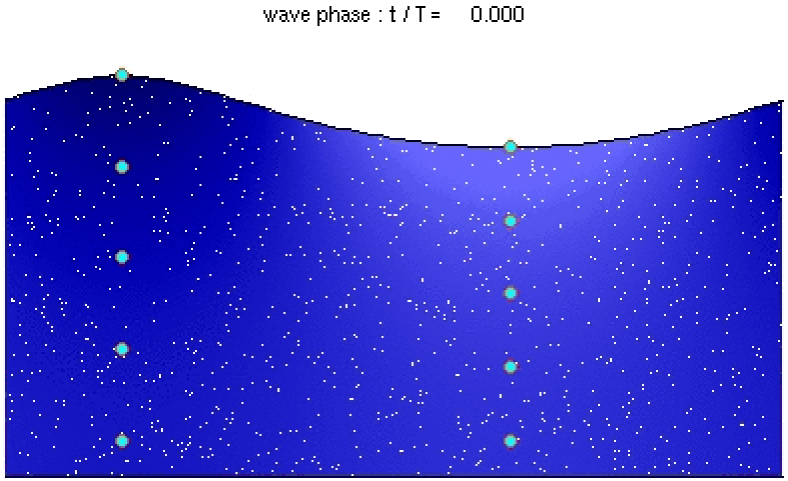
Dispersion relation



Trajectories below the waves



Stokes drift!

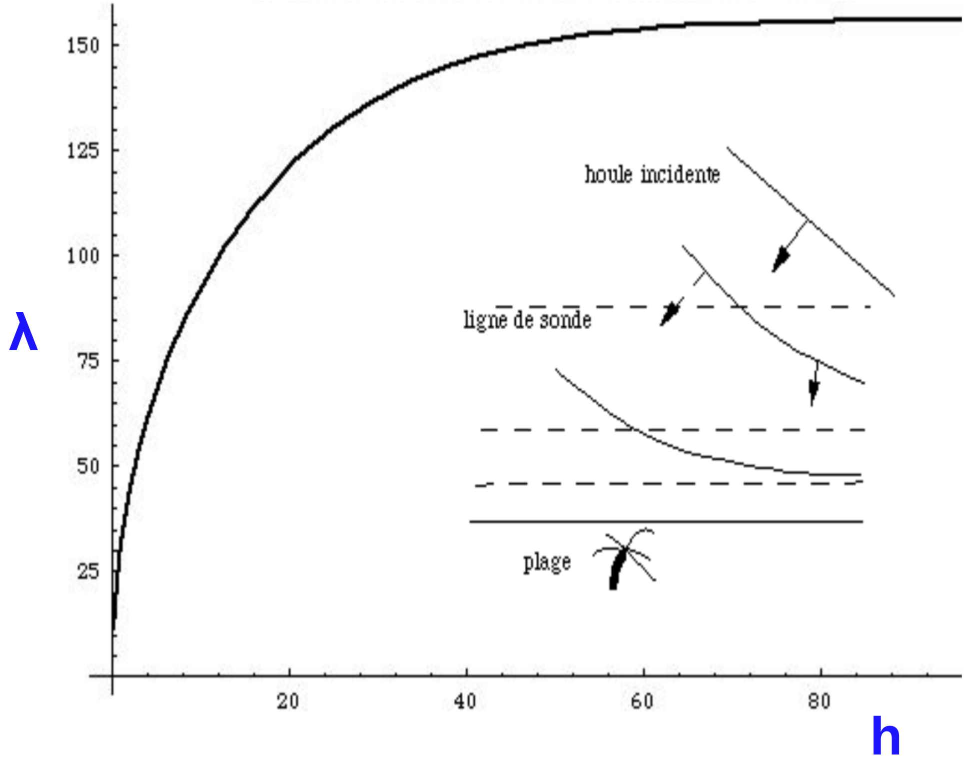


Why are the waves parallel to the shore?

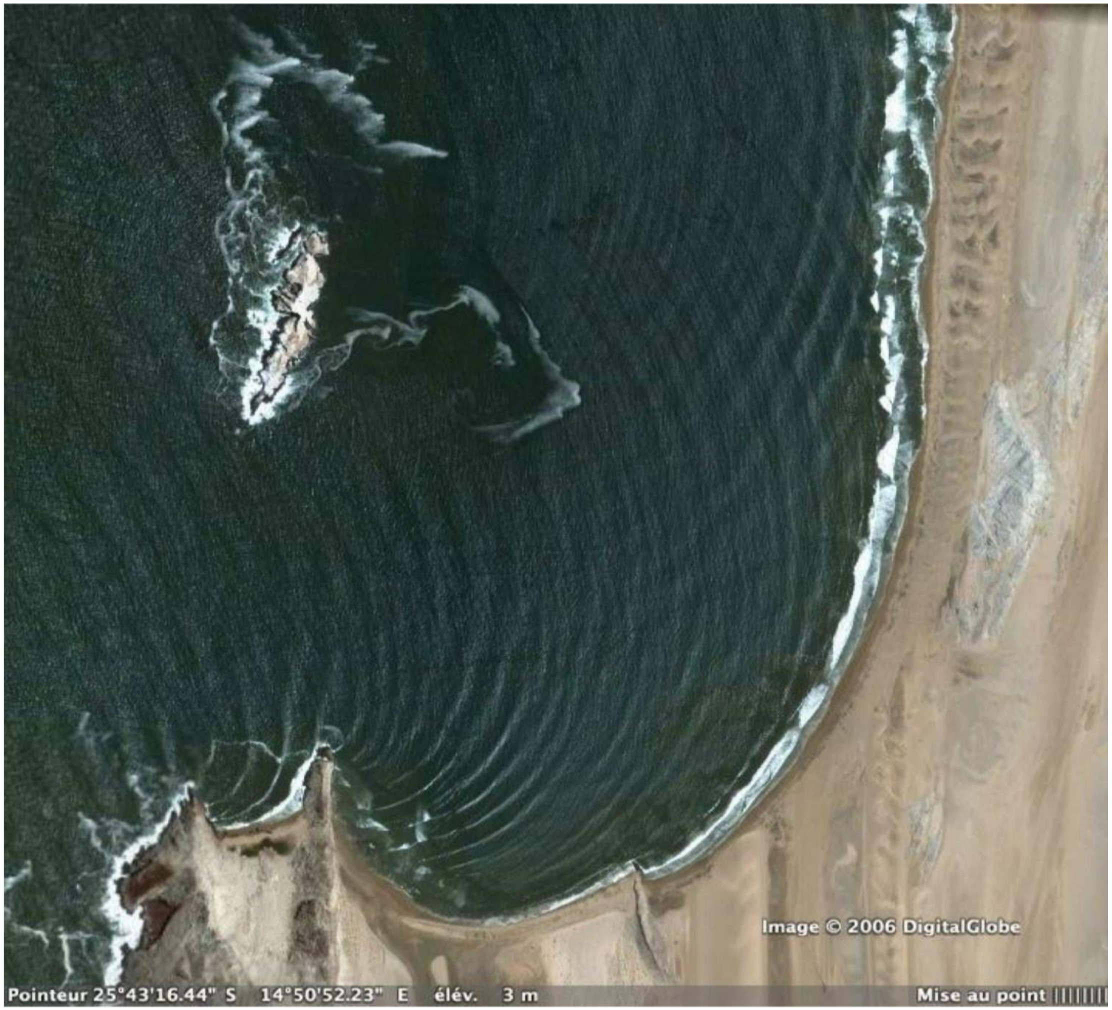
$c \sim (gh)^{1/2}$

$\lambda \sim T(gh)^{1/2}$

T=10s; ω=0.62

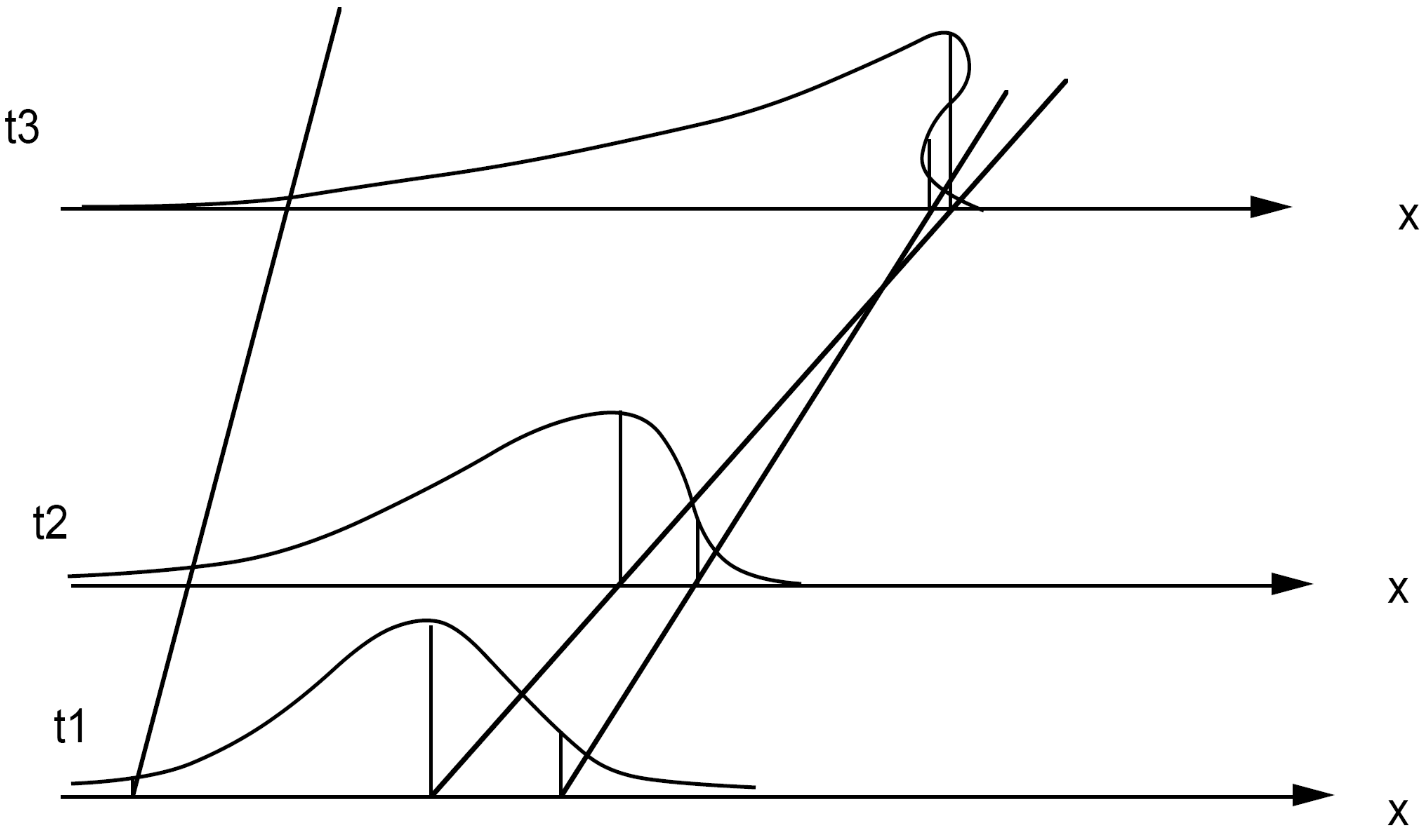


Refraction and diffraction of waves



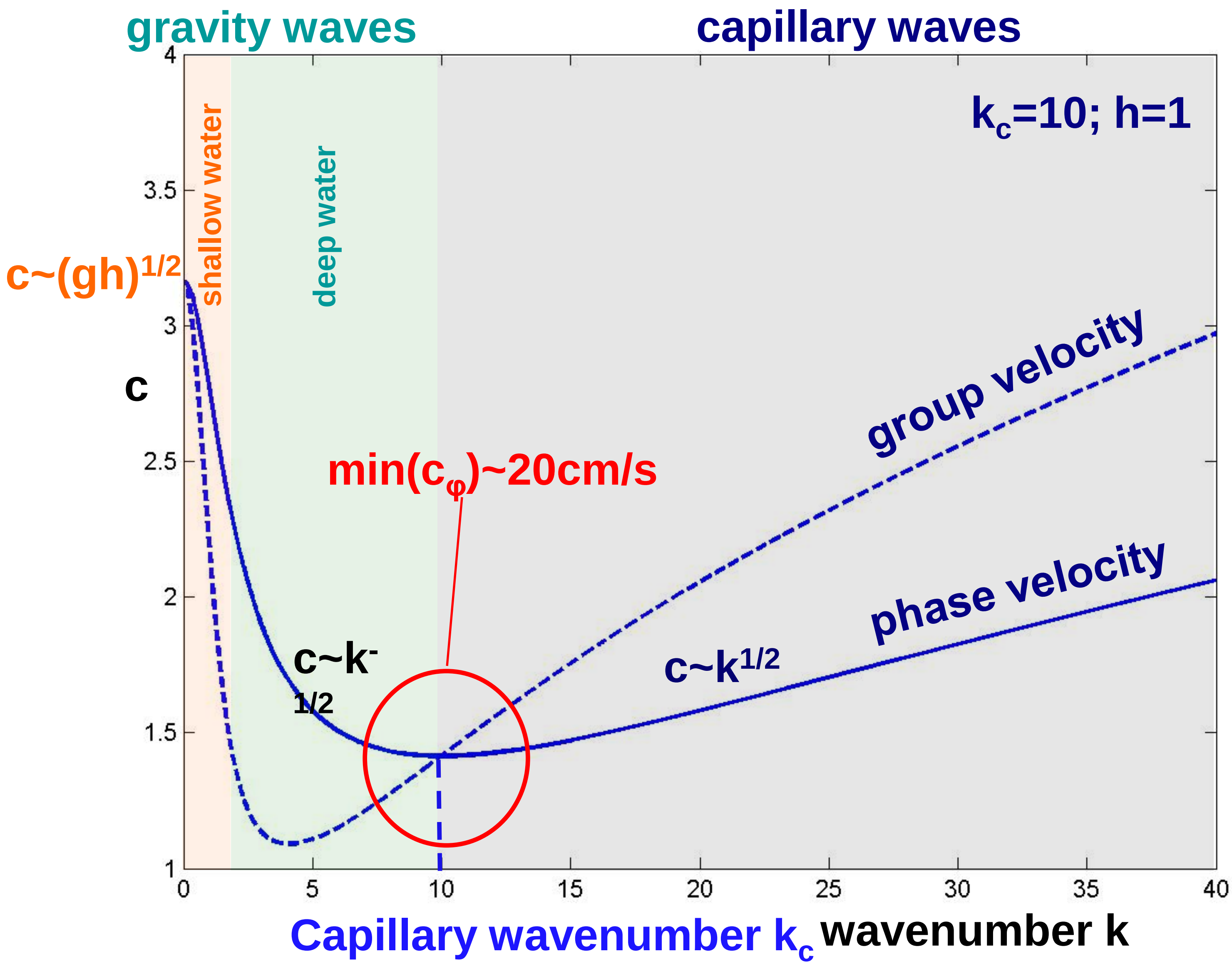
Satellite view Namibian coast

Nonlinear waves, wavebreaking



The celerity increases with the depth

Dispersion relation



Conditions for wave pattern formation?

$$V_{duck} \lessgtr c_{min} \quad ?$$