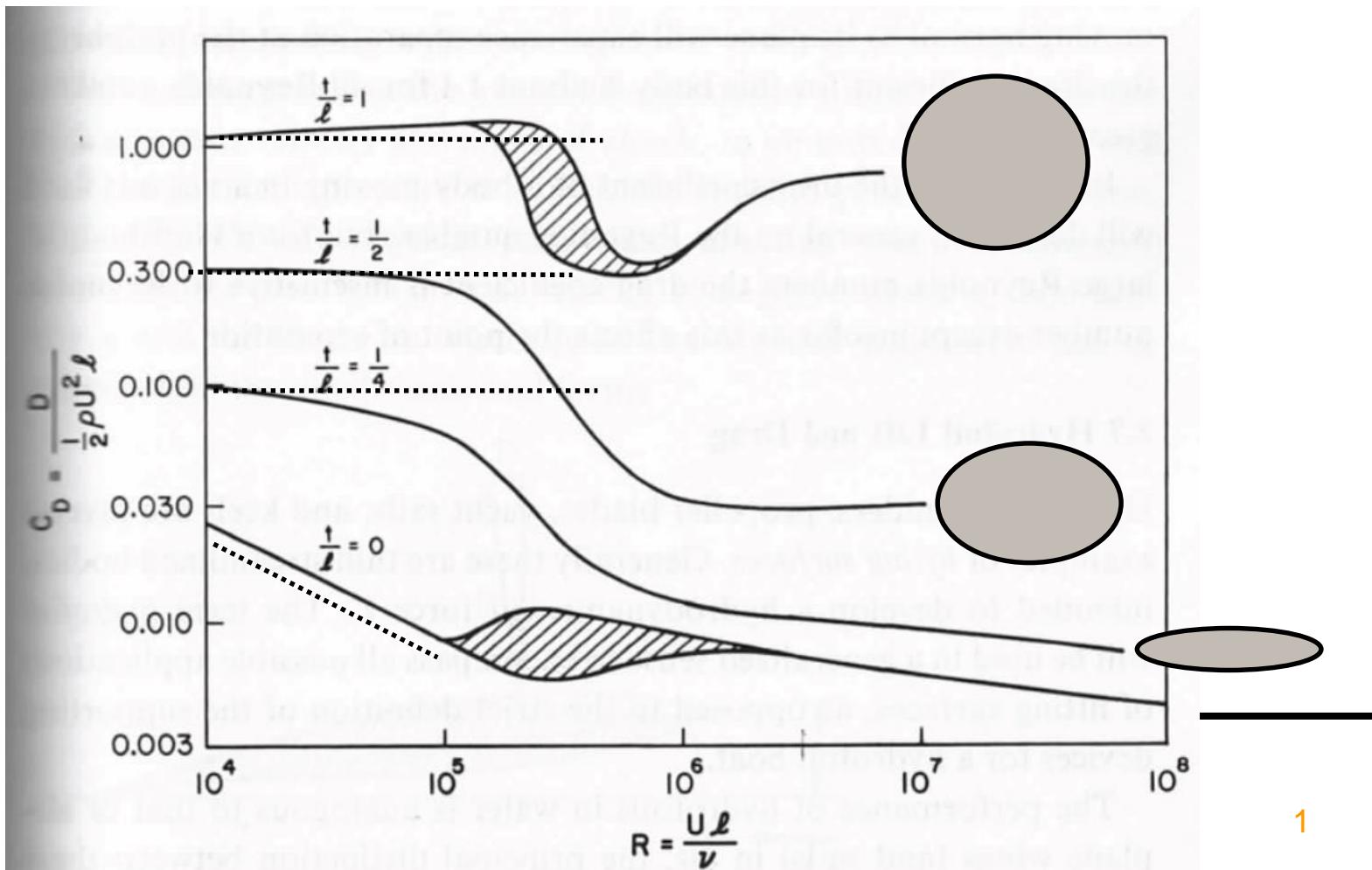


But this does still not explain the aerodynamic drag scaling:

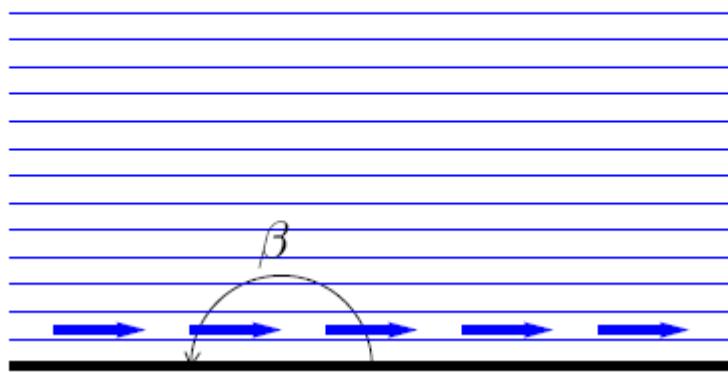
$$\frac{1}{2} \rho_\infty U_\infty^2 S$$



Pressure gradient effect

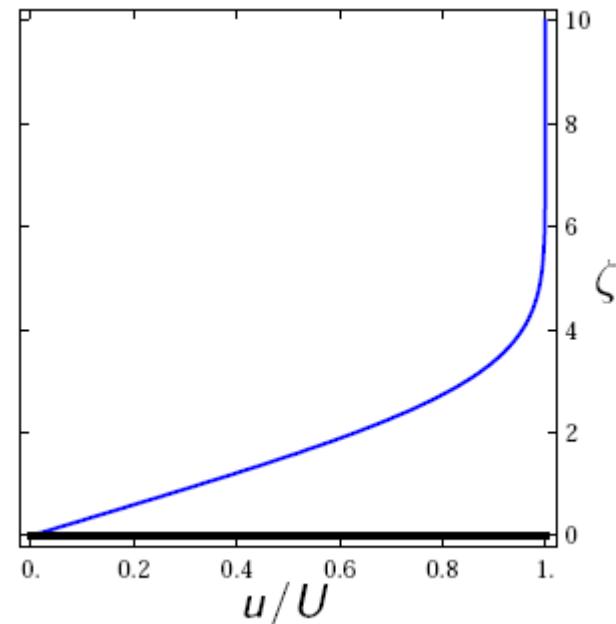
Consider the flow along an angle $\beta = \pi/(m+1) = \pi$ i.e. $m = 0$

This is the flow along a flat plate



Uniform flow- no pressure gradient

$$\nabla p = 0$$

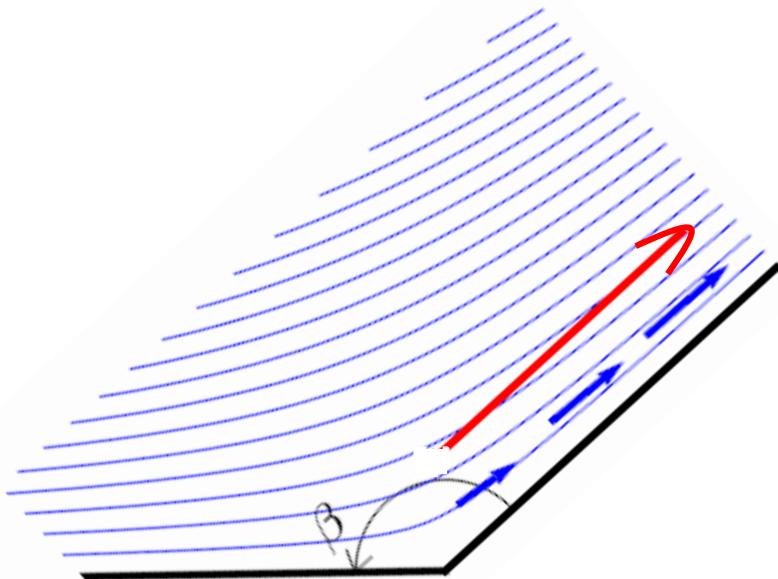


Blasius boundary layer

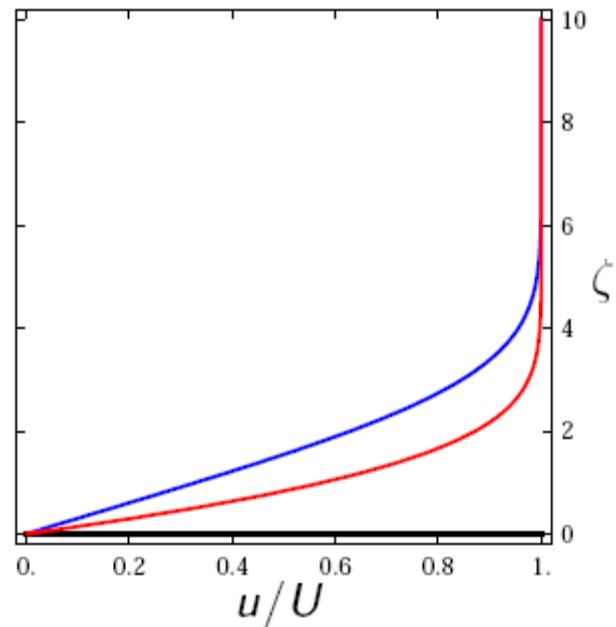
Pressure gradient effect

Consider the flow along an angle $\beta = \pi/(m + 1) < \pi$ i.e. $m > 0$

This is the flow along a « forward wedge »



Accelerated flow
favorable pressure gradient
 $\bar{\nabla}p > 0$

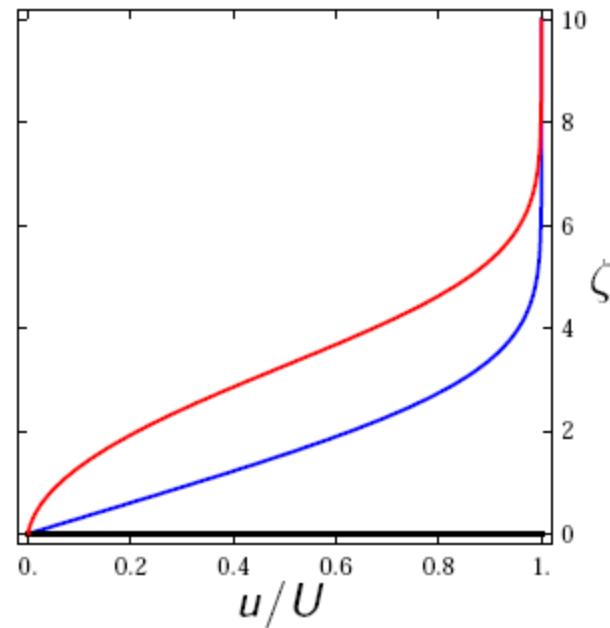
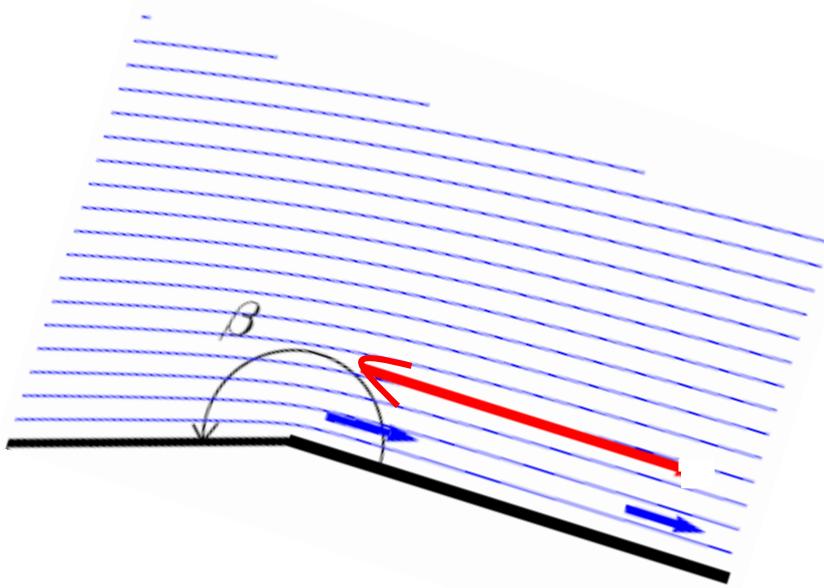


Thinner boundary layer

Pressure gradient effect

Consider the flow along an angle $\beta = \pi/(m + 1) > \pi$ i.e. $m < 0$

This is the flow along a « forward wedge »



Decelerated flow
unfavorable (adverse) pressure gradient

$$\nabla p < 0$$

Thicker boundary layer

Pressure gradient effect

Adverse pressure gradient :

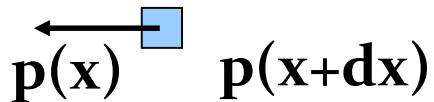
$$\frac{\partial p}{\partial x} > 0$$



$$p_1$$

$$p_2 > p_1$$

$$p_3 > p_2$$

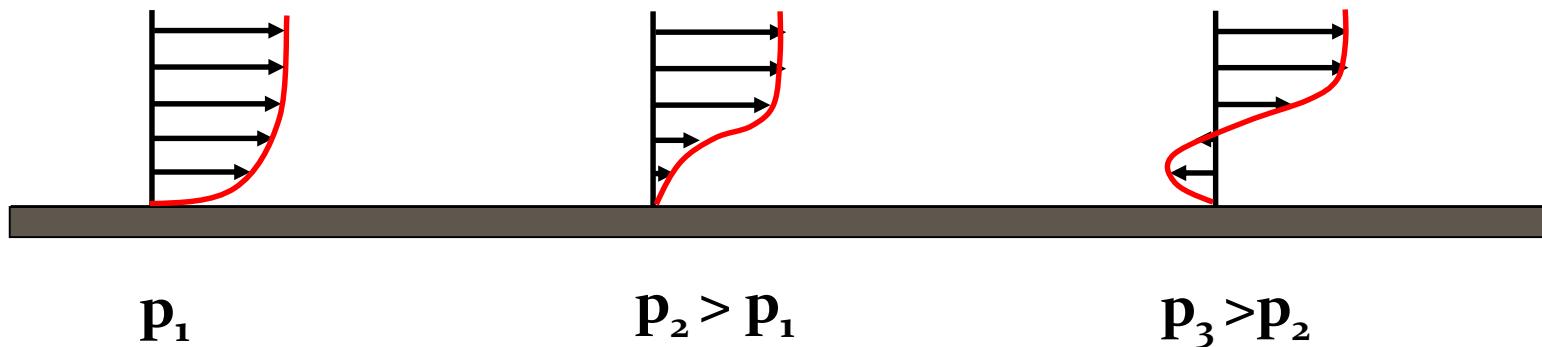


Resulting pressure force

Pressure gradient effect

Adverse pressure gradient :

$$\frac{\partial p}{\partial x} > 0$$



Close to the wall, the viscous effects dominate
The pressure gradient further decreases the velocity
⇒ Detachement

Pressure gradient effect

Favorable pressure gradient: $\frac{\partial p}{\partial x} < 0$



$$p_1$$

$$p_2 < p_1$$

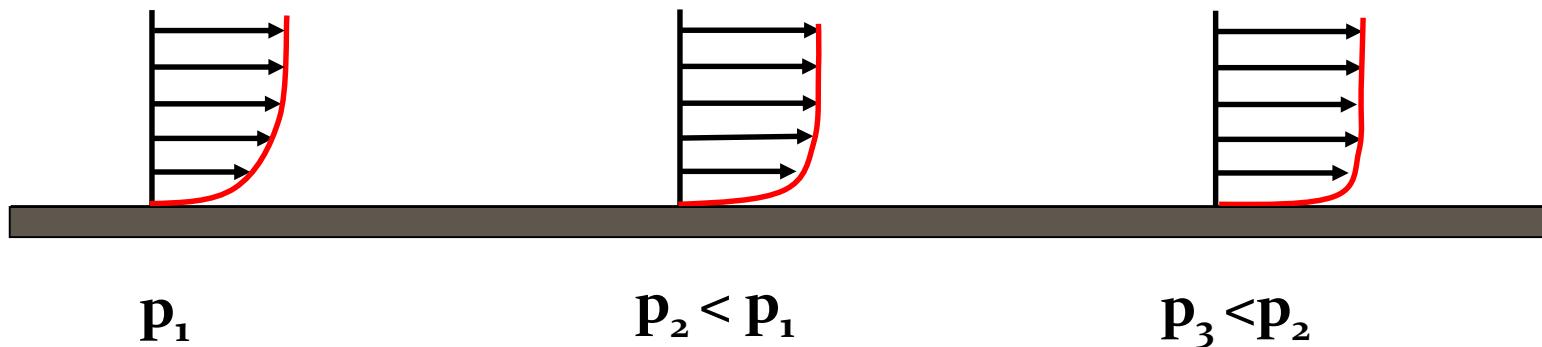
$$p_3 < p_2$$

A diagram showing a small rectangular element of width dx and height 1 . The left side is labeled $p(x)$ and the right side is labeled $p(x+dx)$. A horizontal arrow points from $p(x)$ to $p(x+dx)$, representing the pressure difference across the element.

Resulting pressure force

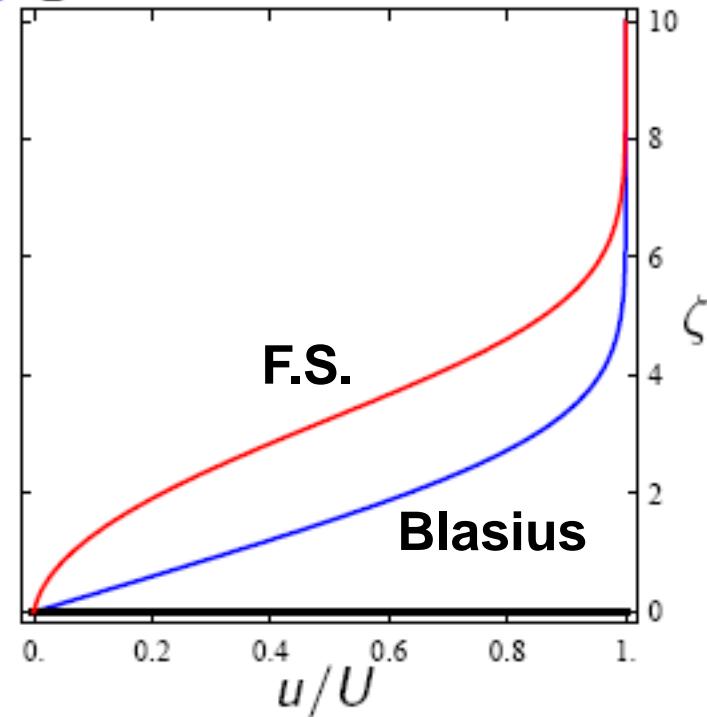
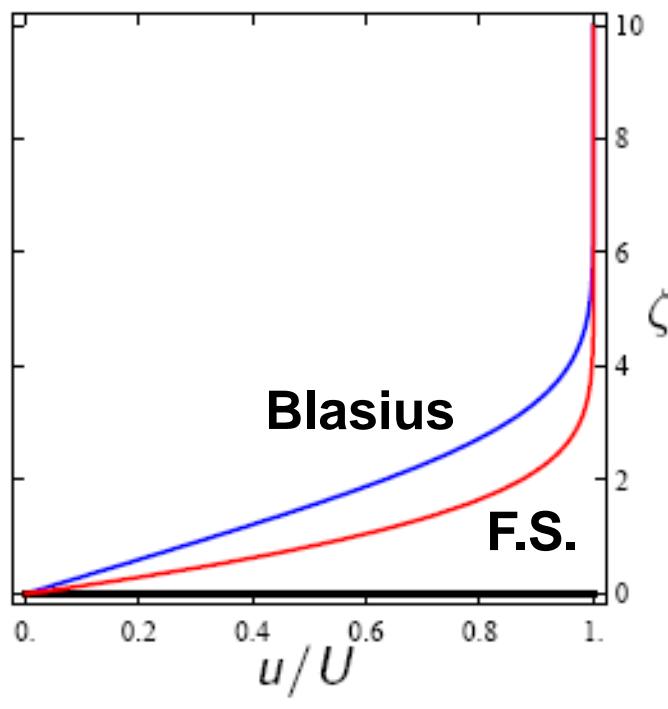
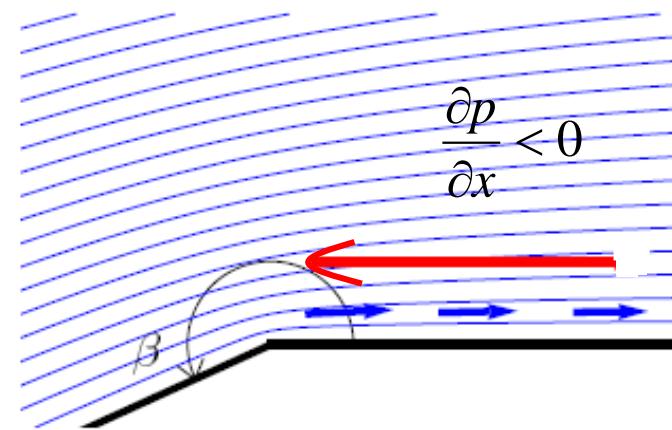
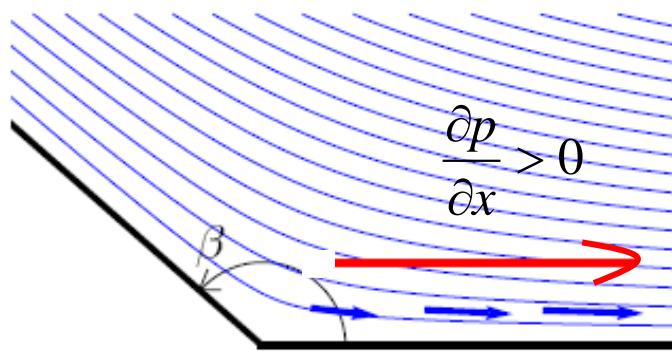
Pressure gradient effect

Favorable pressure gradient: $\frac{\partial p}{\partial x} < 0$



Close to the wall, the pressure gradient further increases the velocity of the flow \Rightarrow no detachment

Pressure gradient effect



Falkner-Skan solutions

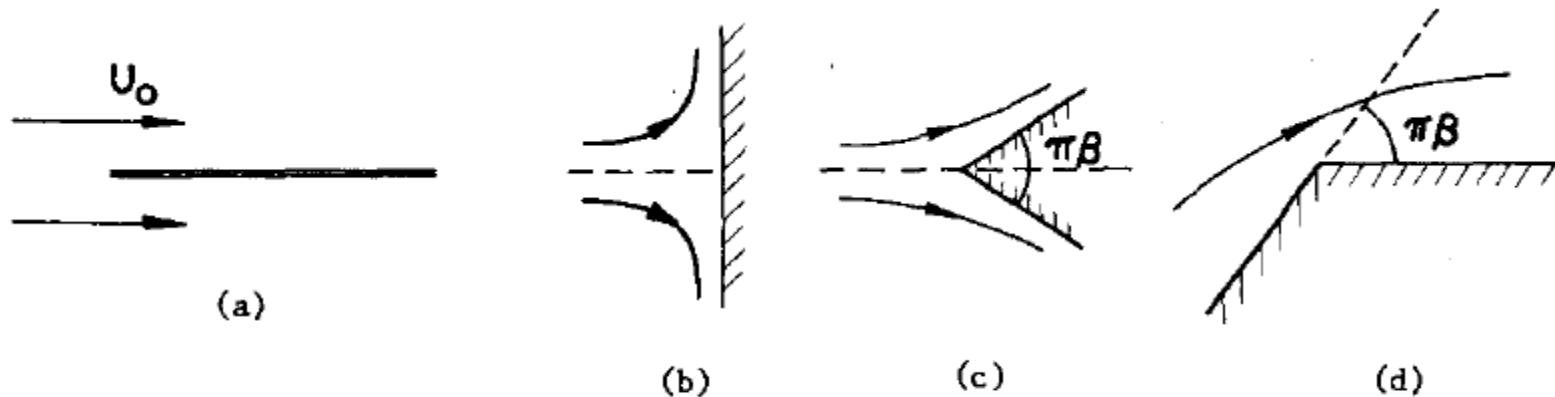
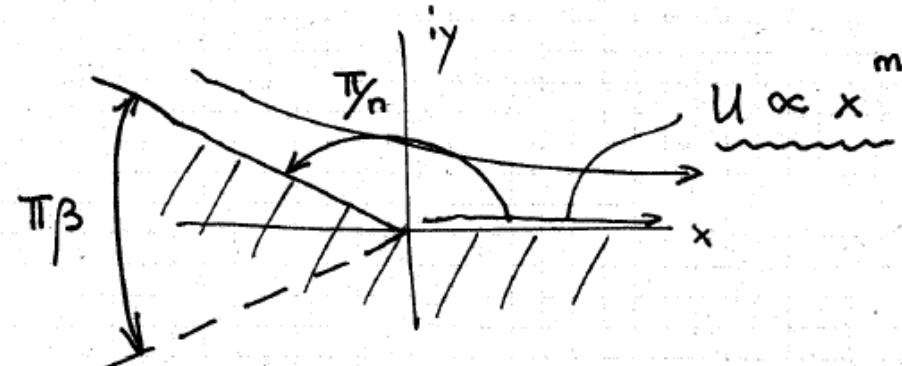


Figure 5.2 Boundary layer flows represented by solutions of the Falkner-Skan equation for different values of the parameter m : (a) $m = 0$; (b) $m = 1$; (c) $0 < m < 1$; (d) $-1/2 < m < 0$

Falkner-Skan far field solutions

pot. complexe

$$F(z) = C z^n$$



$$\frac{dF}{dz} = C n z^{n-1} = v_x - i v_y \rightarrow n = 1 + m$$

$$\frac{\pi}{n} + \frac{\pi \beta}{2} = \pi \rightarrow \frac{1}{1+m} + \frac{\beta}{2} = 1$$

$$\boxed{\beta = \frac{2m}{1+m}}$$

Falkner-Skan boundary layer equations

1. Prandtl equations

$$\hat{\psi}_{\hat{y}} \hat{\psi}_{x\hat{y}} - \hat{\psi}_x \hat{\psi}_{\hat{y}\hat{y}} = U \frac{dU}{dx} + \hat{\psi}_{\hat{y}\hat{y}\hat{y}},$$

$$\hat{\psi} = \hat{\psi}_{\hat{y}} = 0 \text{ on } \hat{y} = 0, \quad \hat{\psi}_{\hat{y}} \rightarrow U(x) \text{ as } \hat{y} \rightarrow \infty.$$

Falkner-Skan boundary layer equations

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2. Self-similar solution

$$\hat{\psi}(x, \hat{y}) = (Ax^{m+1})^{1/2} f(\eta) \text{ where } \eta = \hat{y}(Ax^{m-1})^{1/2}.$$

Falkner-Skan boundary layer equations

1. Prandtl equations

$$\hat{\psi}_{\hat{y}} \hat{\psi}_{x\hat{y}} - \hat{\psi}_x \hat{\psi}_{\hat{y}\hat{y}} = U \frac{dU}{dx} + \hat{\psi}_{\hat{y}\hat{y}\hat{y}},$$

$$\hat{\psi} = \hat{\psi}_{\hat{y}} = 0 \text{ on } \hat{y} = 0, \quad \hat{\psi}_{\hat{y}} \rightarrow U(x) \text{ as } \hat{y} \rightarrow \infty.$$

2. Self-similar solution

$$\hat{\psi}(x, \hat{y}) = (Ax^{m+1})^{1/2} f(\eta) \text{ where } \eta = \hat{y}(Ax^{m+1})^{1/2}.$$

3. Falkner-Skan equation

$$f''' + \frac{1}{2}(m+1) ff'' + m(1 - f'^2) = 0$$

$$f(0) = f'(0) = 0, \quad f'(\infty) = 1$$

Falkner-Skan boundary layer solutions

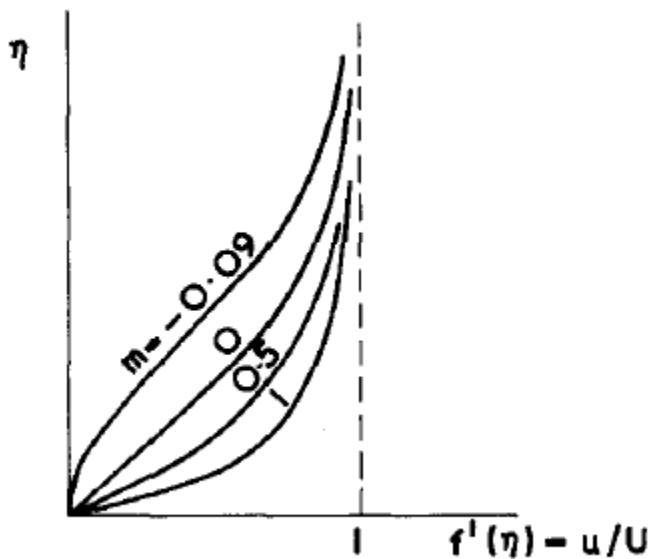
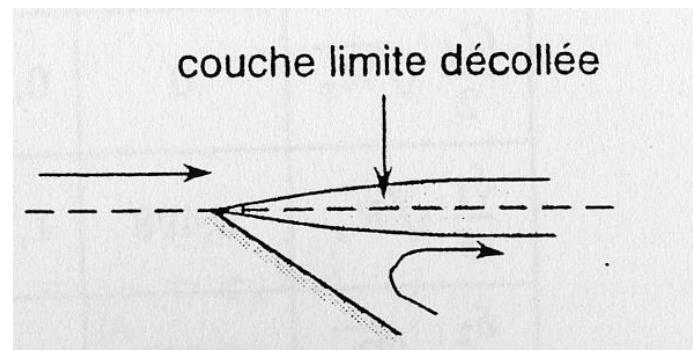
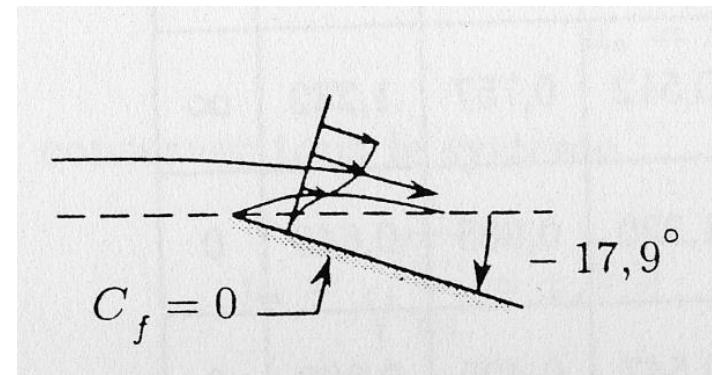
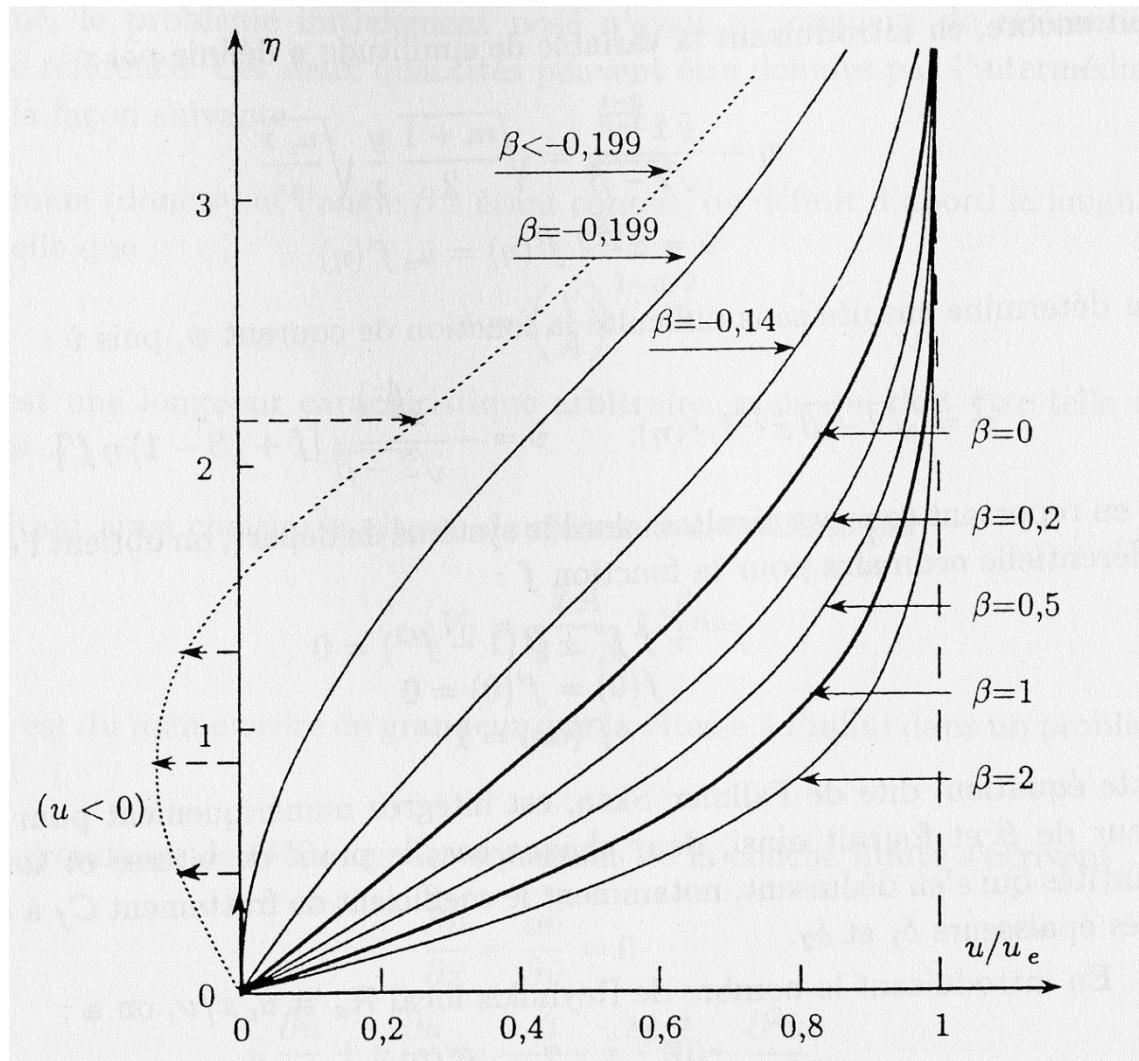


Figure 5.3 Sketch of velocity profiles given by solutions of the Falkner-Skan equation

Falkner-Skan boundary layer solutions

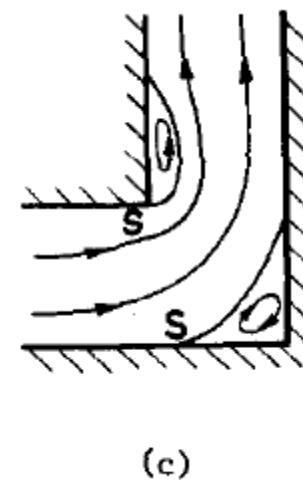
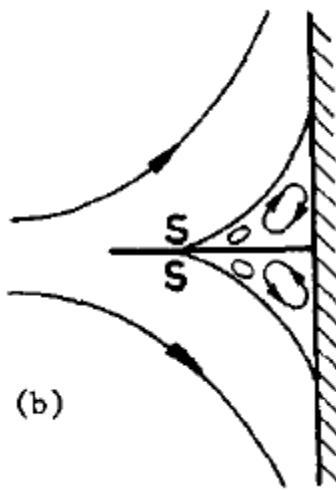
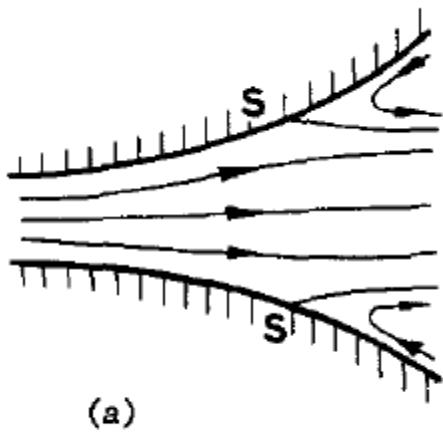


Boundary layer separation



$\hat{\psi} \sim (x - x_s)^{1/2}$, so that $\frac{\partial \hat{\psi}}{\partial x} \sim (x - x_s)^{-1/2}$ as $x \rightarrow x_s$.

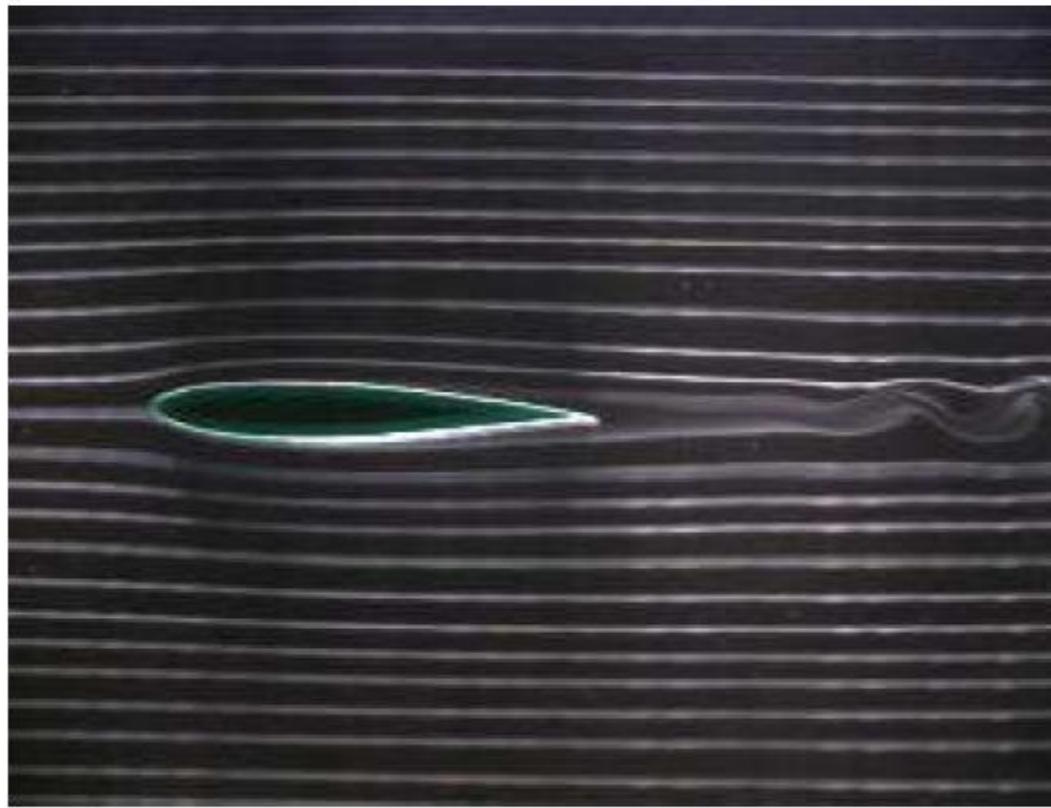
Boundary layer separation



Decollement sur un profil d'aile

Expériences en soufflerie menées à l'université de Stanford,
l'écoulement est visualisé grâce à des fumées :

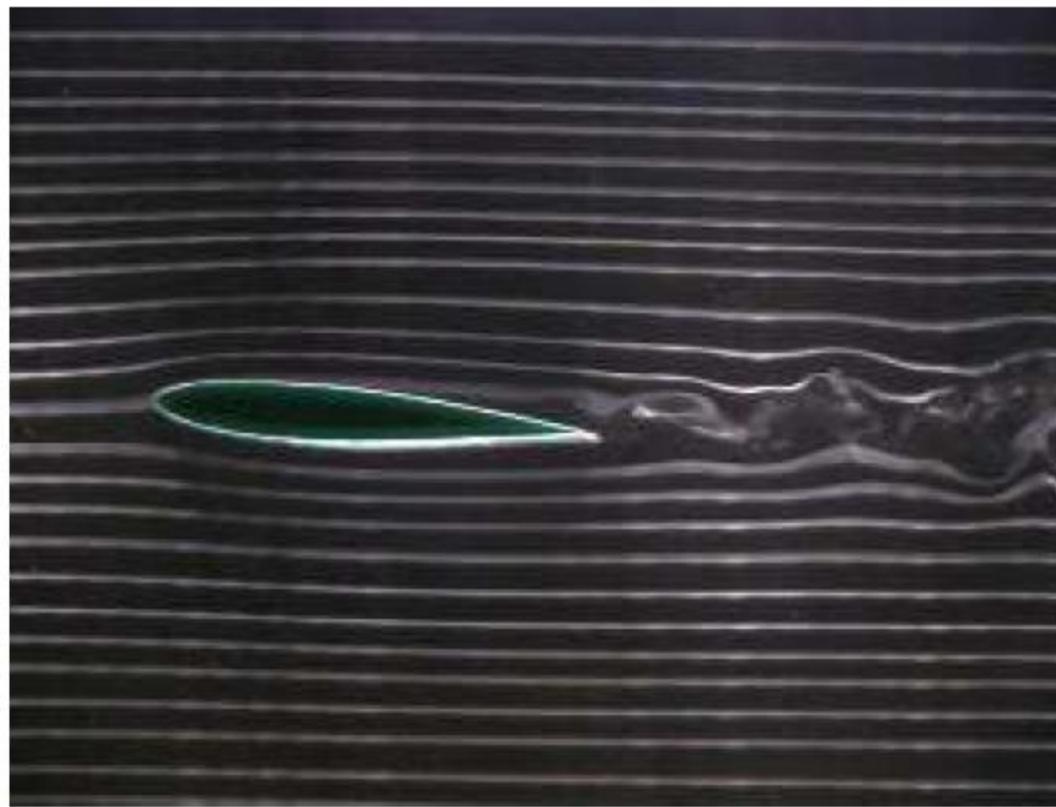
angle d'incidence $\gamma = 0^\circ$:



Décollement sur un profil d'aile

Expériences en soufflerie menées à l'université de Stanford,
l'écoulement est visualisé grâce à des fumées :

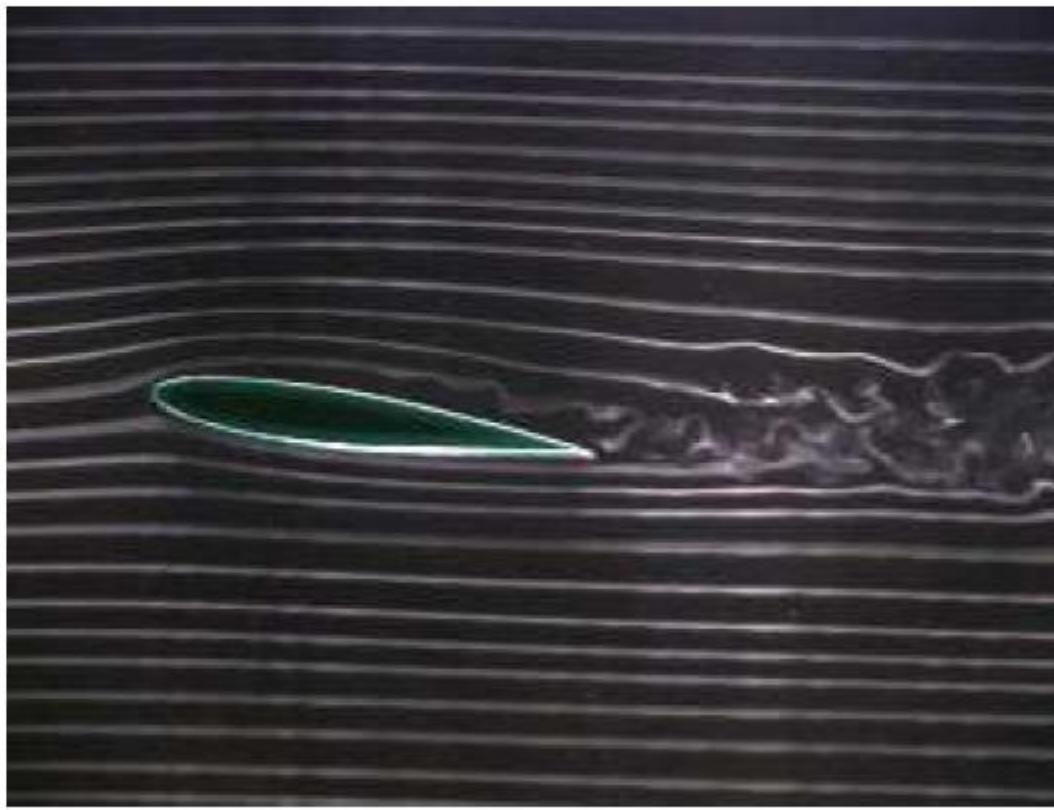
angle d'incidence $\gamma = 5^\circ$:



Décollement sur un profil d'aile

Expériences en soufflerie menées à l'université de Stanford,
l'écoulement est visualisé grâce à des fumées :

angle d'incidence $\gamma = 10^\circ$:



Décollement sur un profil d'aile

Expériences en soufflerie menées à l'université de Stanford,
l'écoulement est visualisé grâce à des fumées :

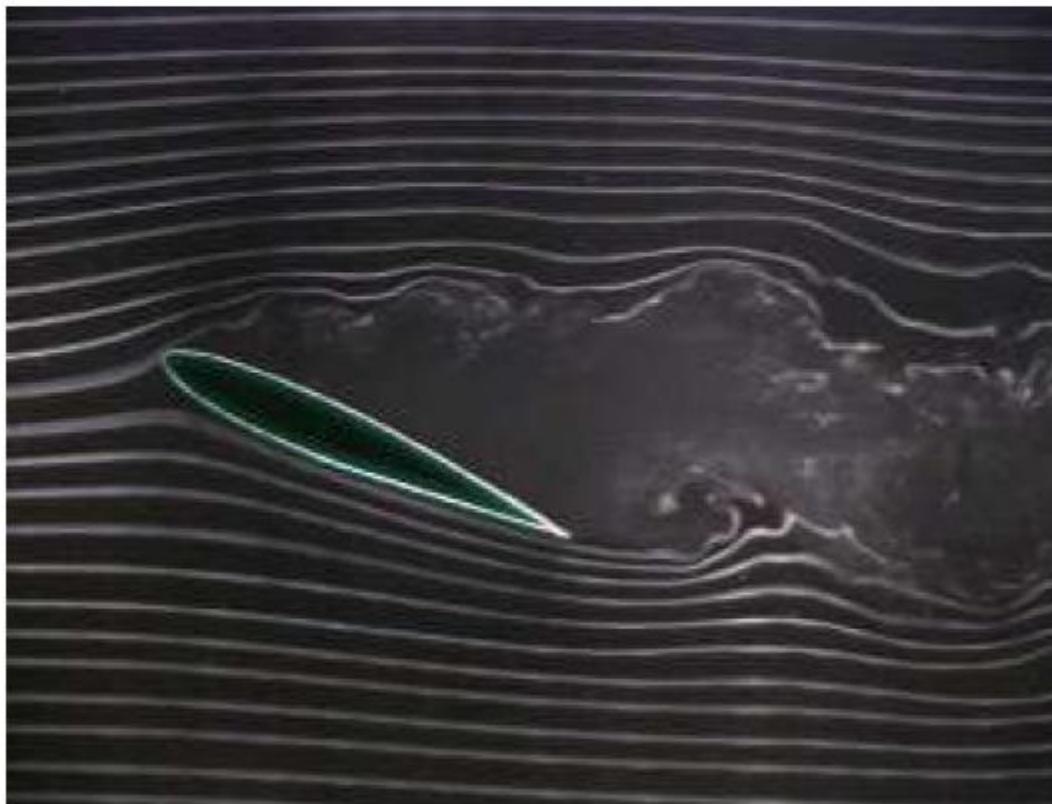
angle d'incidence $\gamma = 15^\circ$:



Décollement sur un profil d'aile

Expériences en soufflerie menées à l'université de Stanford,
l'écoulement est visualisé grâce à des fumées :

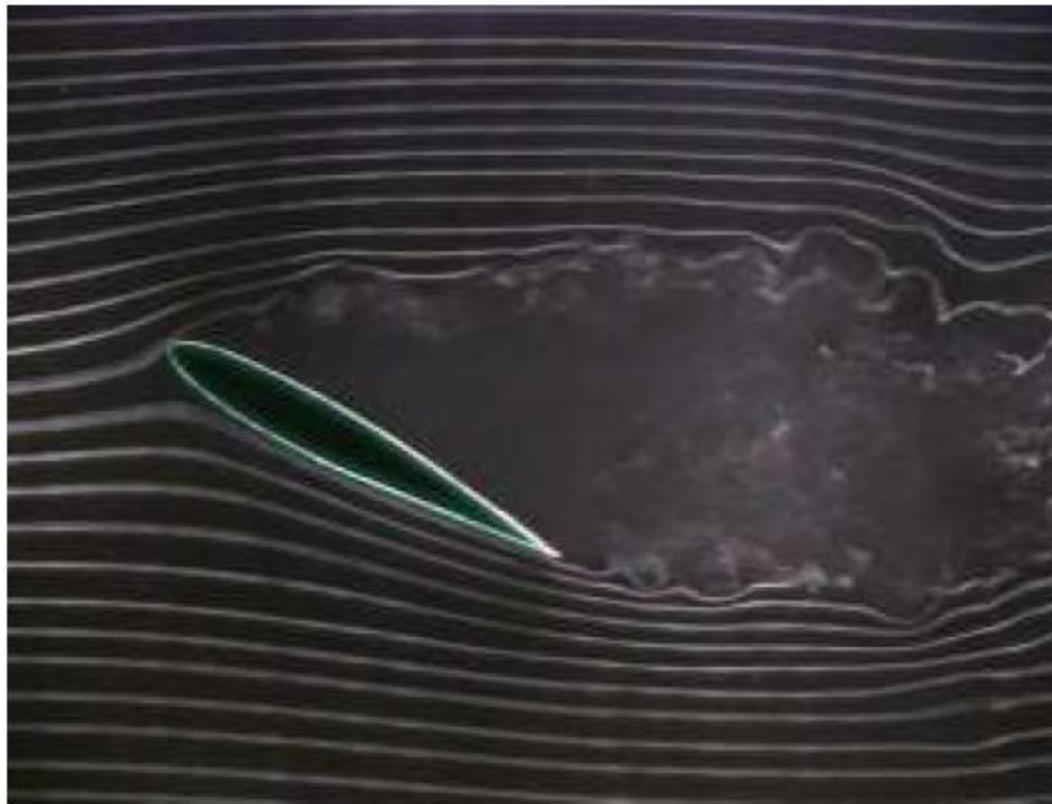
angle d'incidence $\gamma = 25^\circ$:



Décollement sur un profil d'aile

Expériences en soufflerie menées à l'université de Stanford,
l'écoulement est visualisé grâce à des fumées :

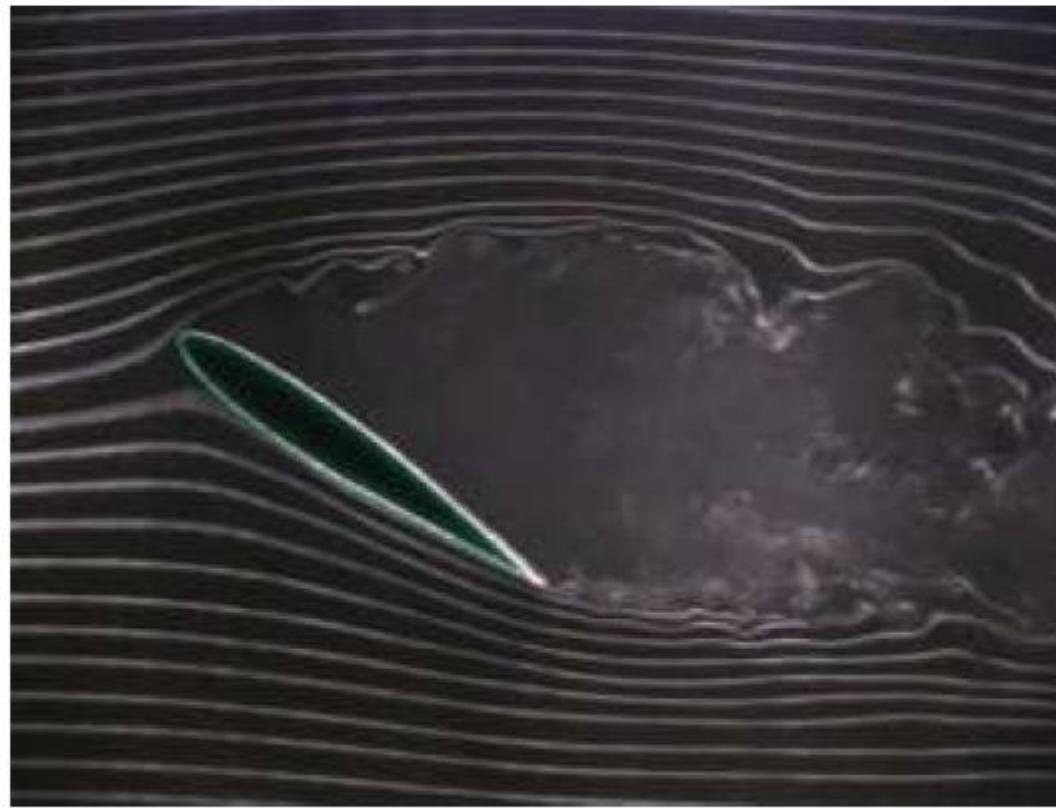
angle d'incidence $\gamma = 30^\circ$:



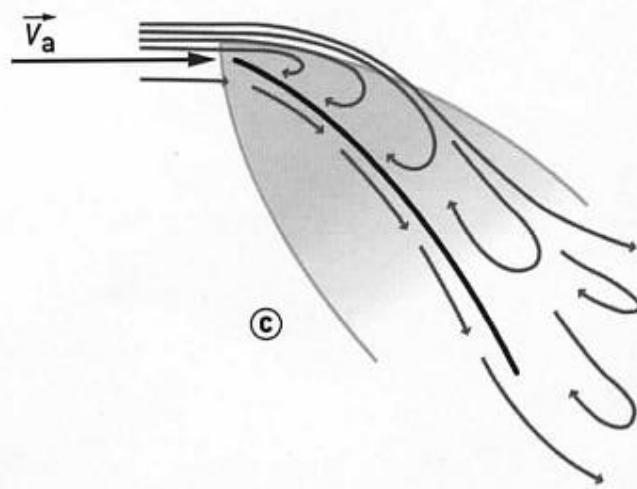
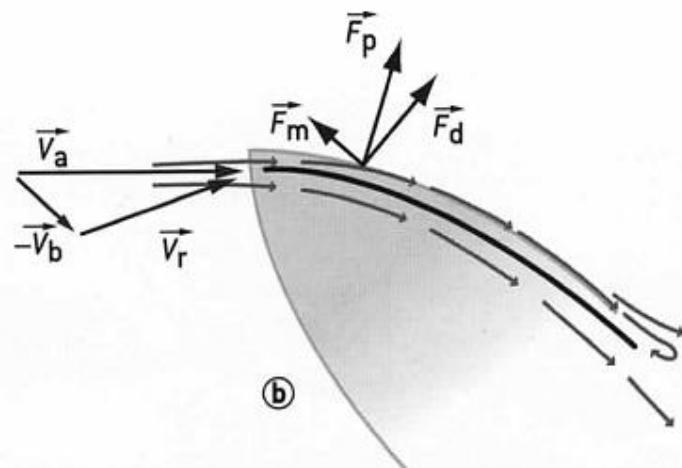
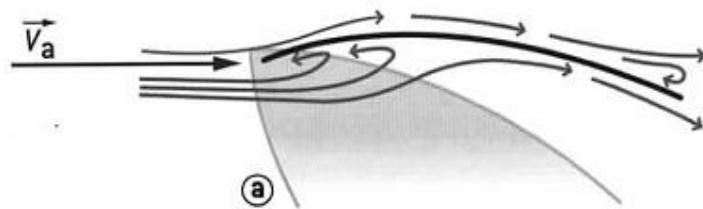
Décollement sur un profil d'aile

Expériences en soufflerie menées à l'université de Stanford,
l'écoulement est visualisé grâce à des fumées :

angle d'incidence $\gamma = 35^\circ$:



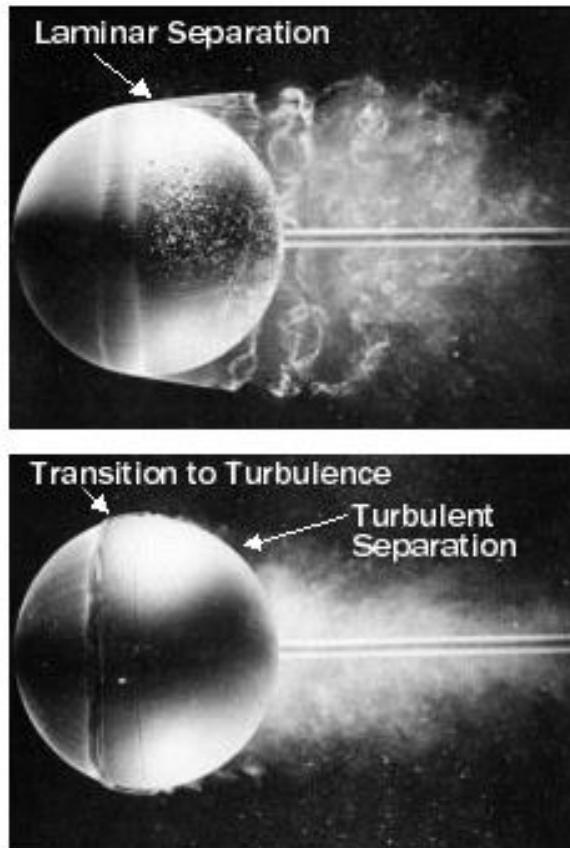
Application to sailing



Application to sailing

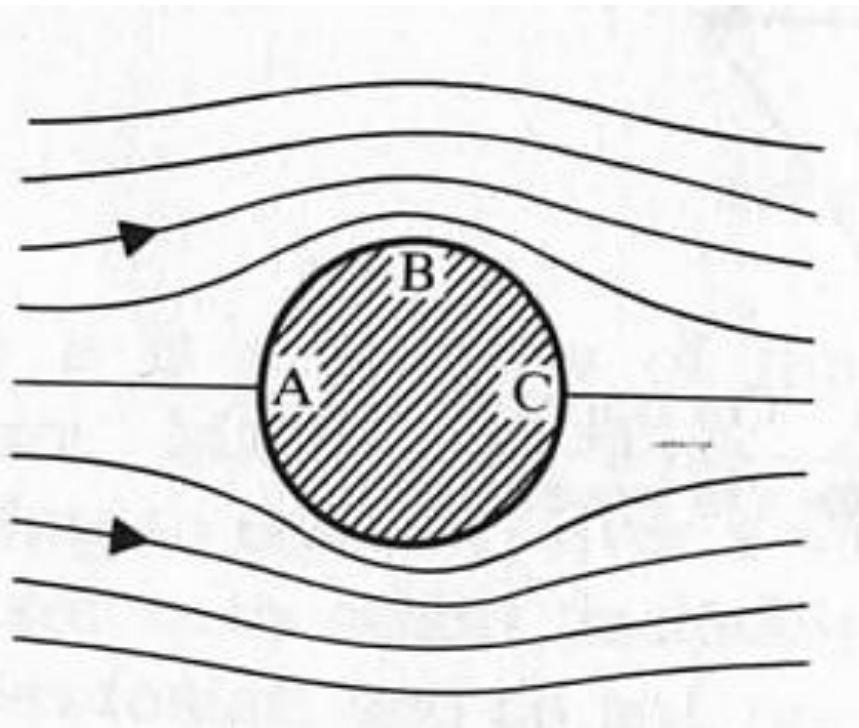


Example: Flow around a sphere

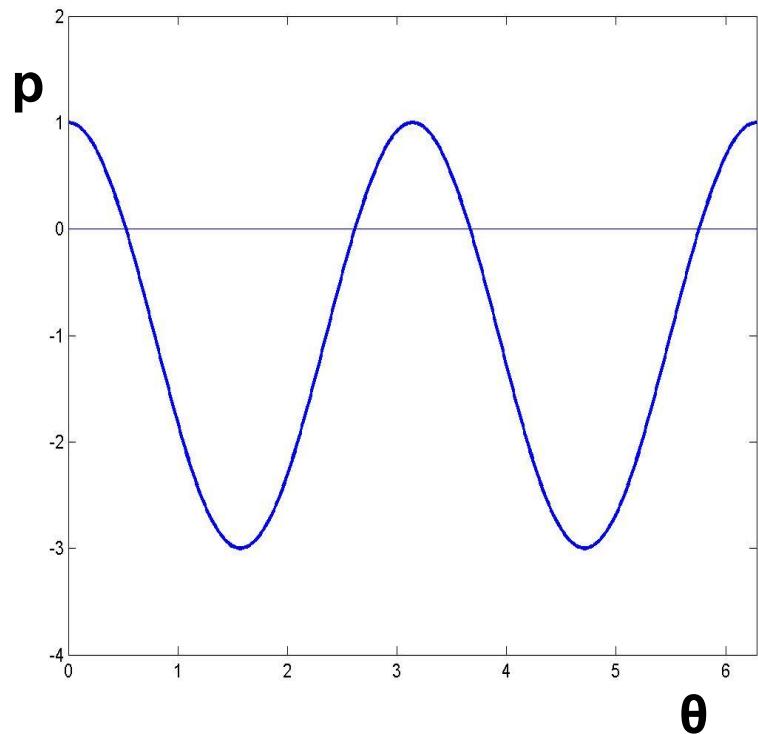


© ONERA

Flow around a cylinder

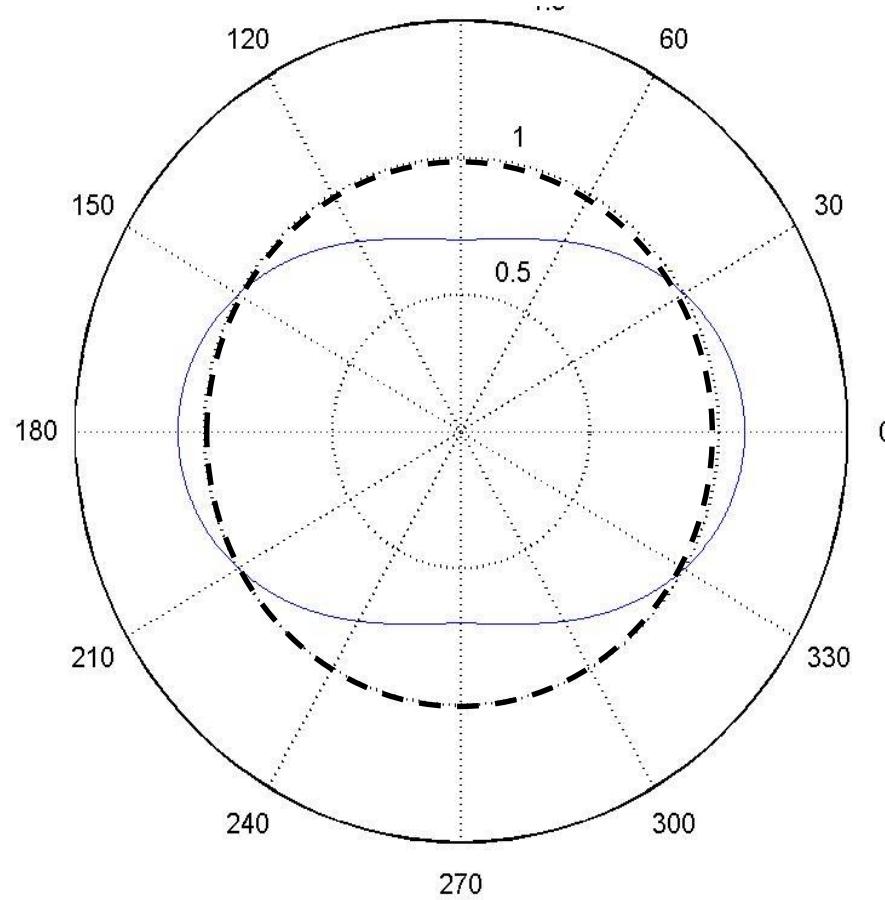


$$p(a, \theta) = \frac{1}{2} \rho U_{\infty}^2 (1 - 4 \sin \theta^2)$$

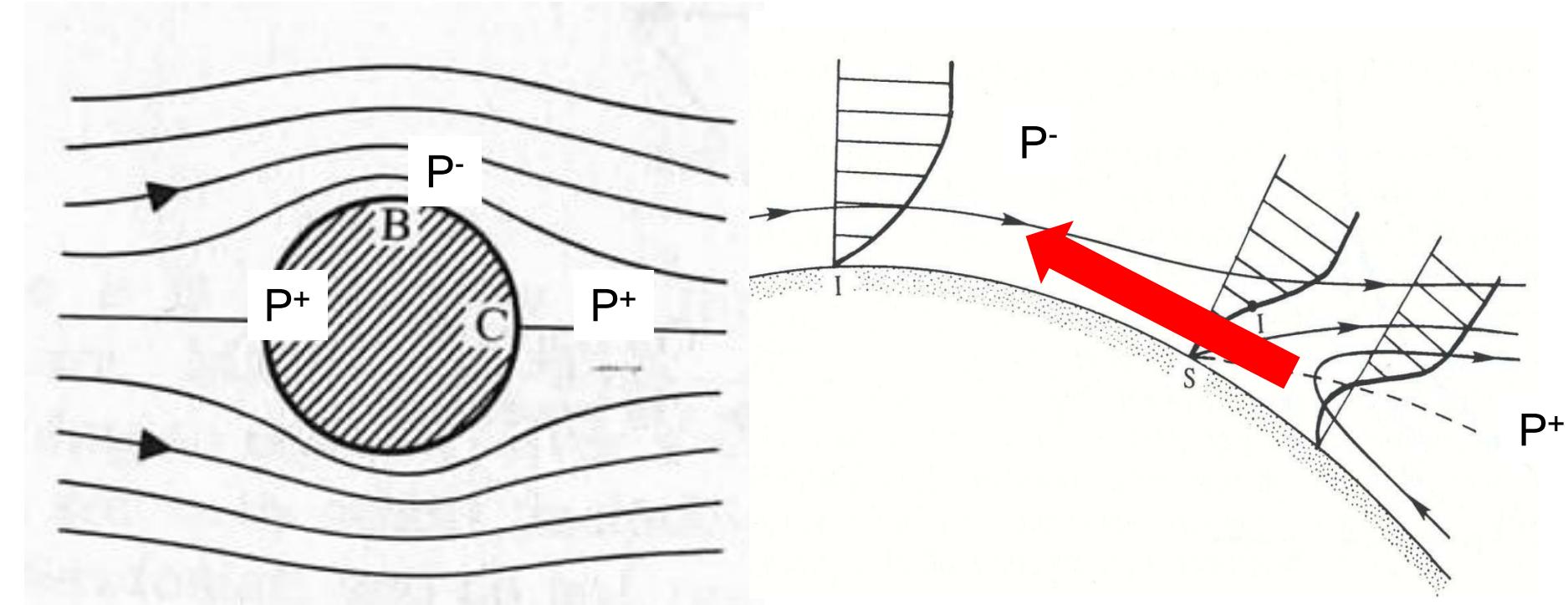


Flow around a cylinder

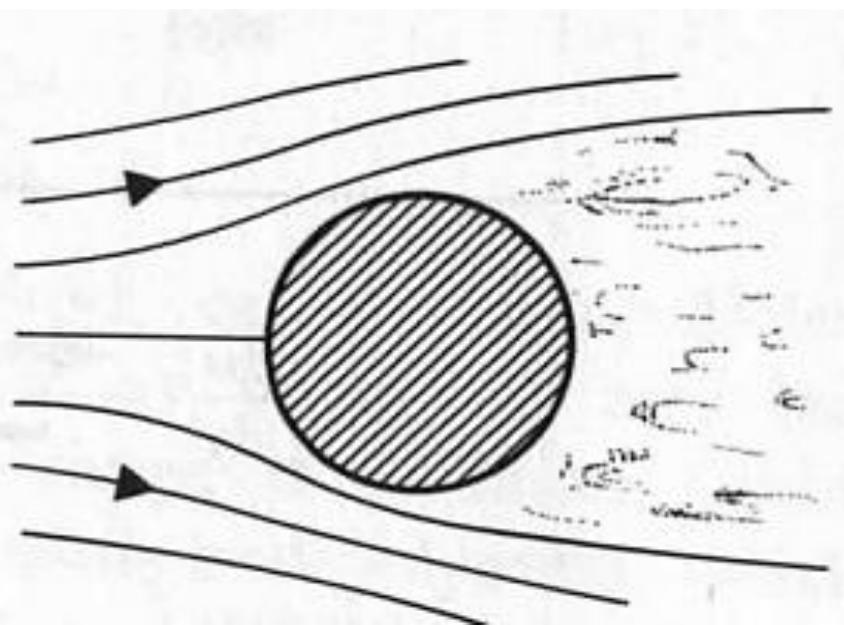
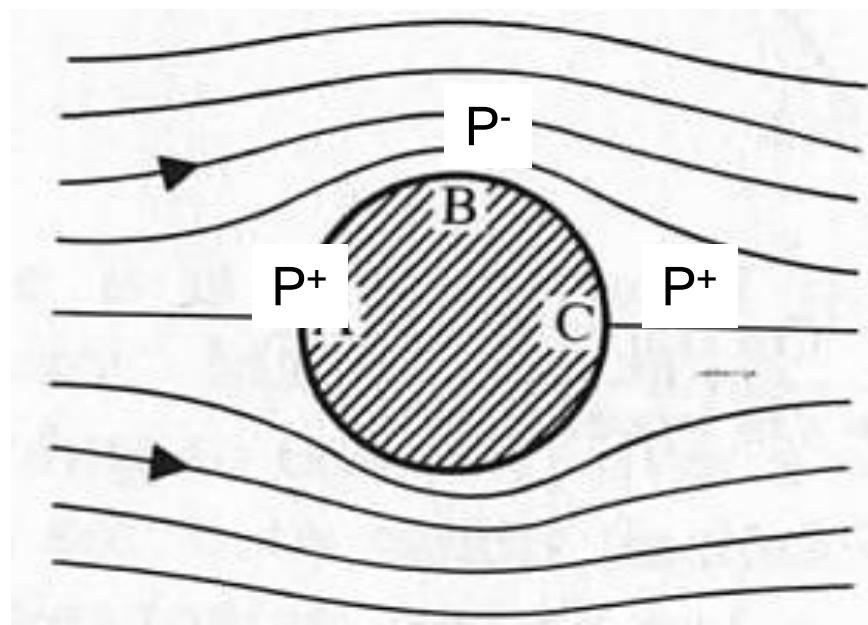
$$p(a, \theta) = \frac{1}{2} \rho U_\infty^2 (1 - 4 \sin \theta^2)$$



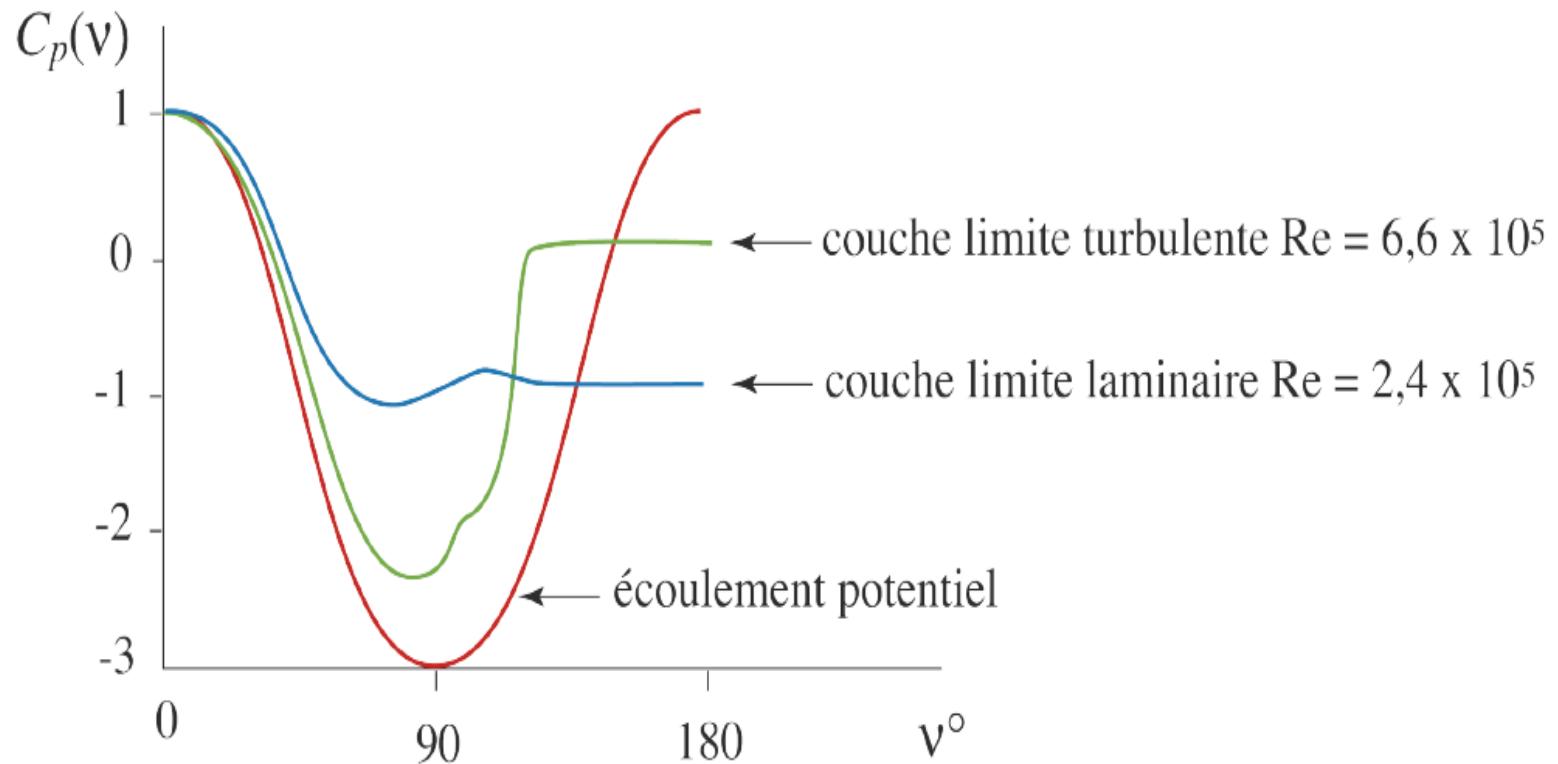
Origin of detachment: pressure gradient



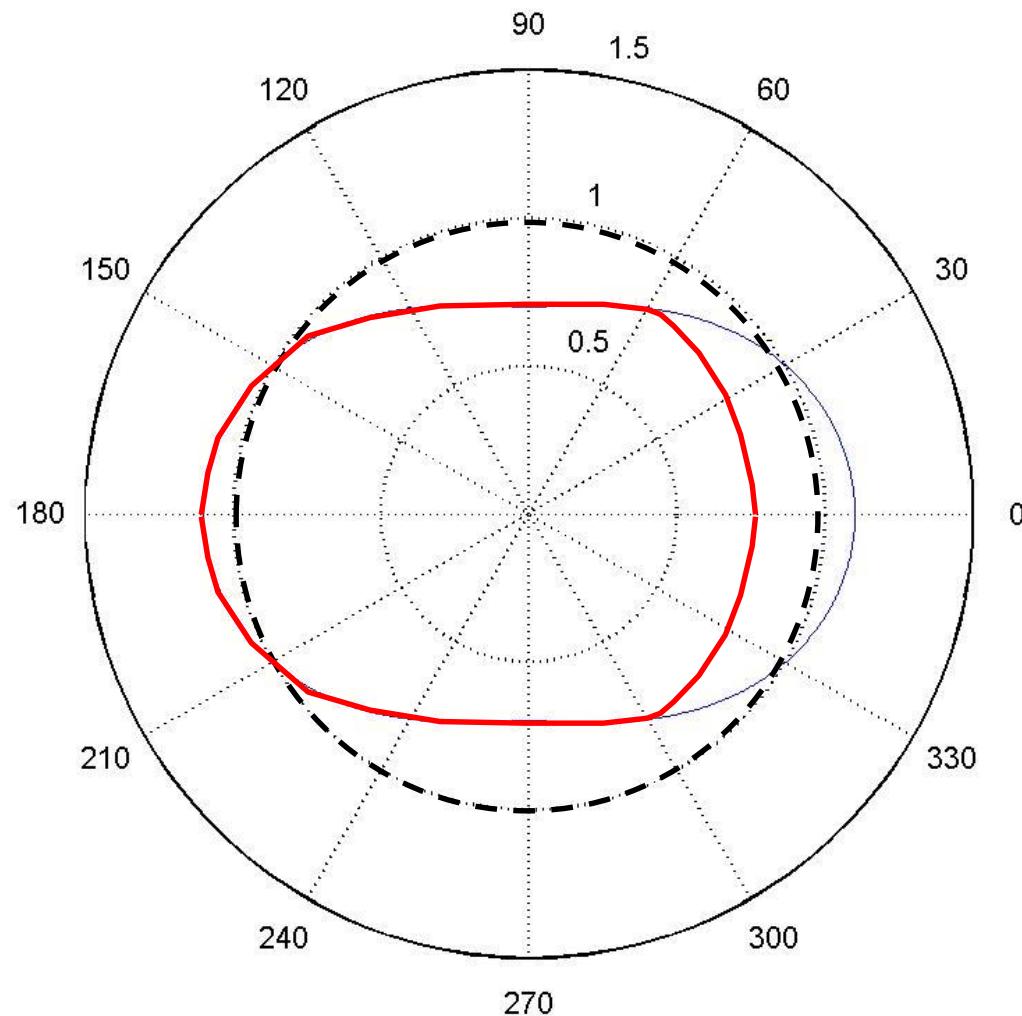
A viscous flow close to the wall opposes the free-stream



Pressure coefficient



Form drag



Drag coefficient

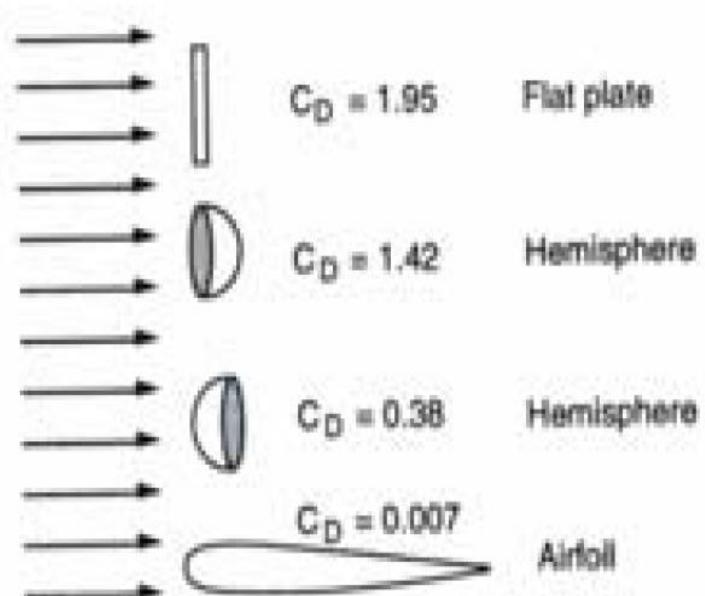
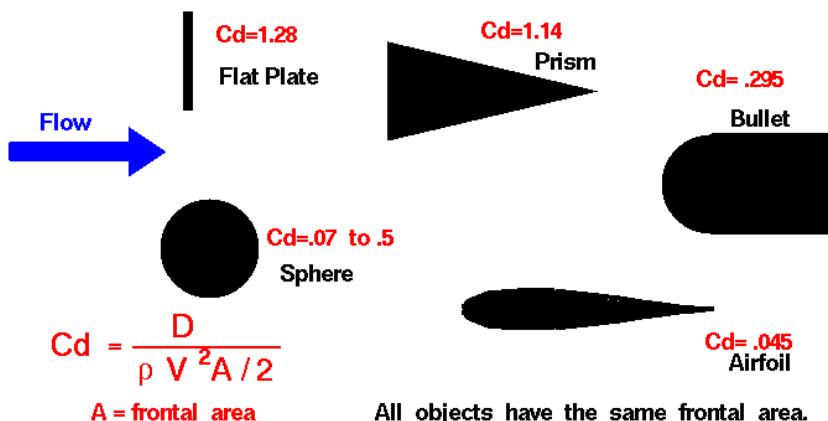
$$C_X = \frac{trainée}{\frac{1}{2} \rho U^2 A}$$



Shape Effects on Drag

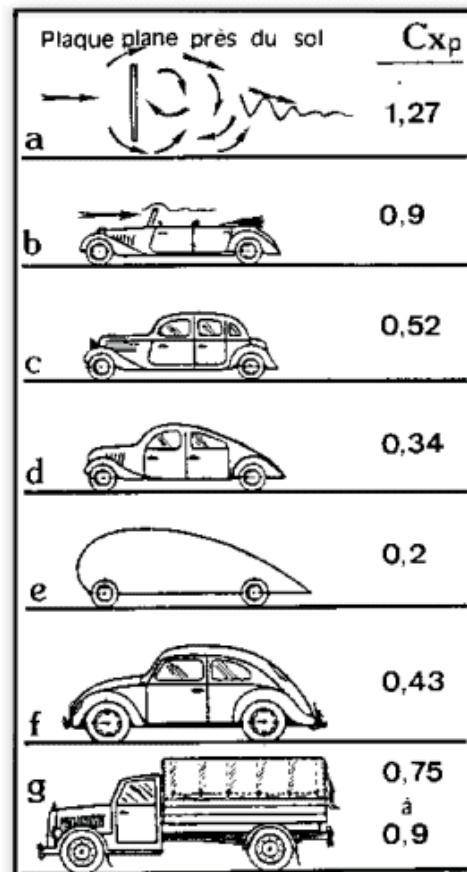
Glenn
Research
Center

The shape of an object has a very great effect on the amount of drag.

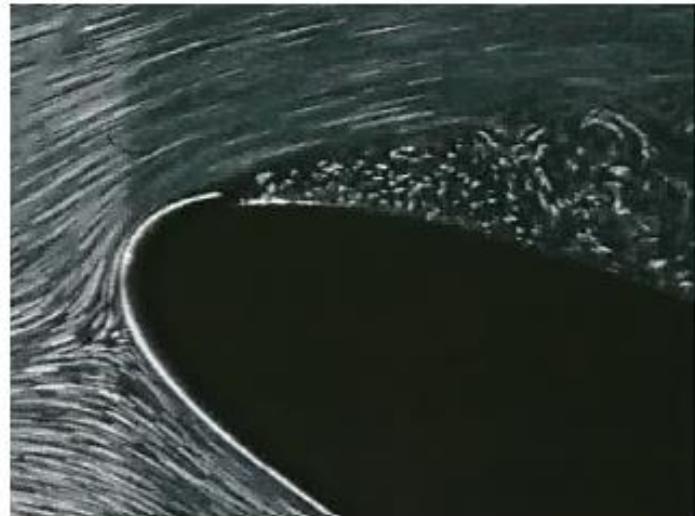


$$C_x = \frac{\text{drag}}{\frac{1}{2} \rho U^2 A}$$

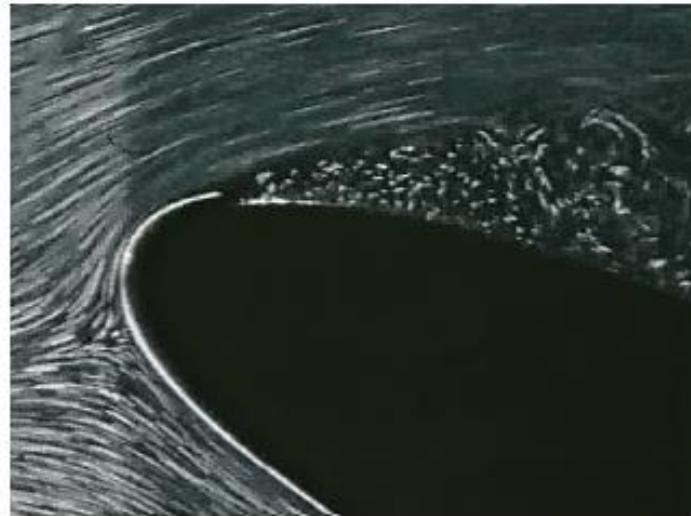
section, somewhat arbitrary...



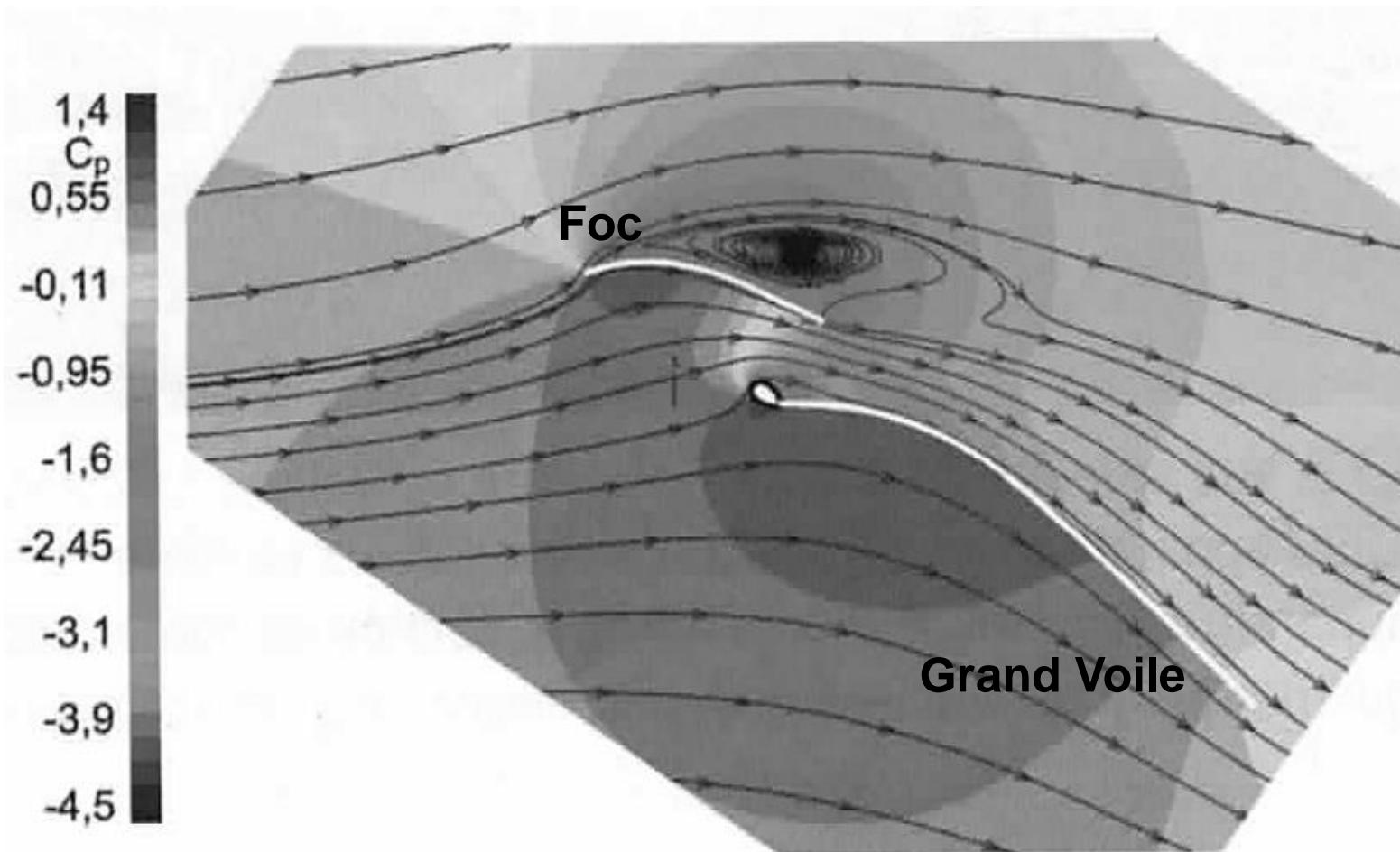
Separation control



Separation control



Application to sailing



Thickness effect

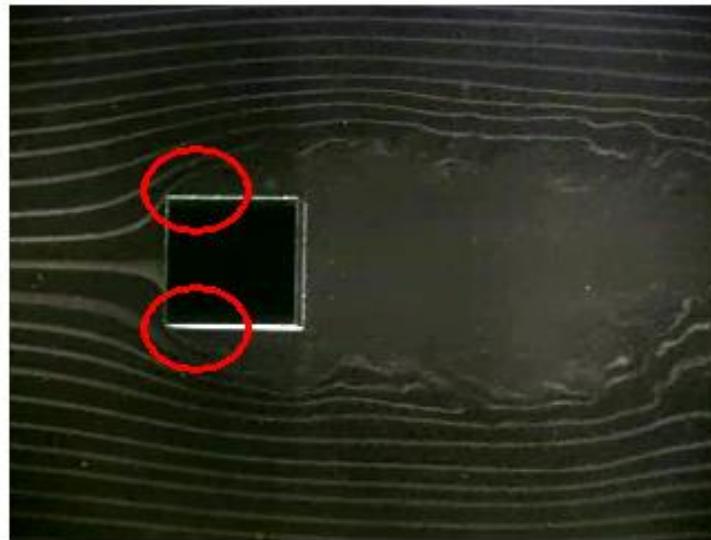


Attached

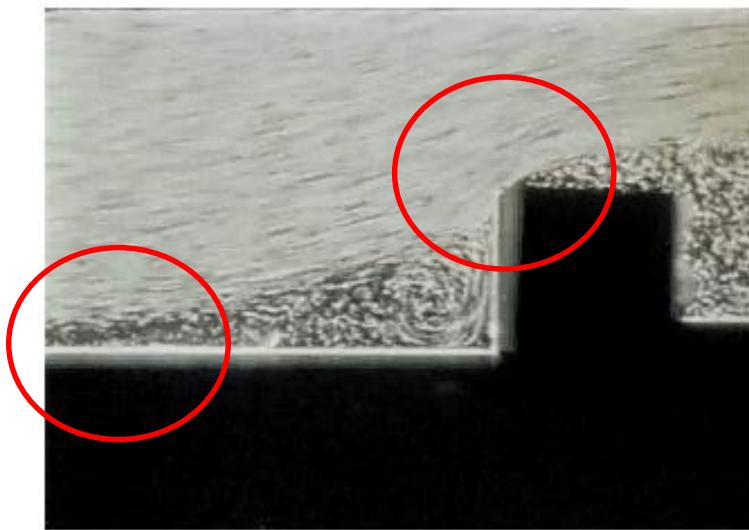
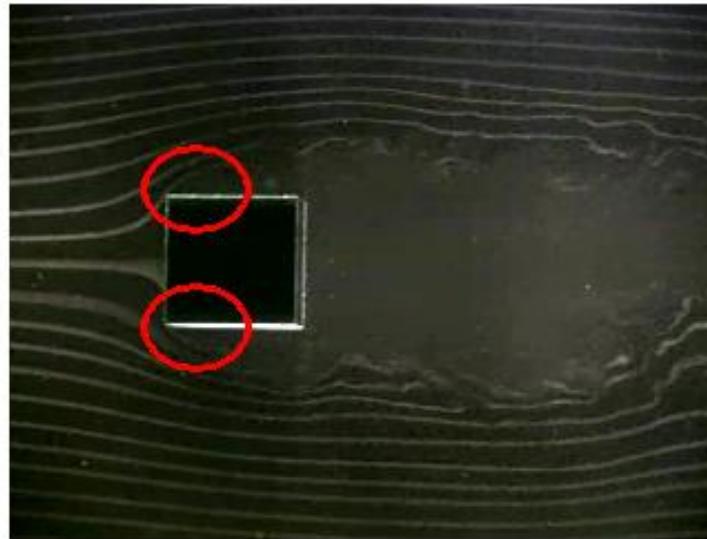


Detached

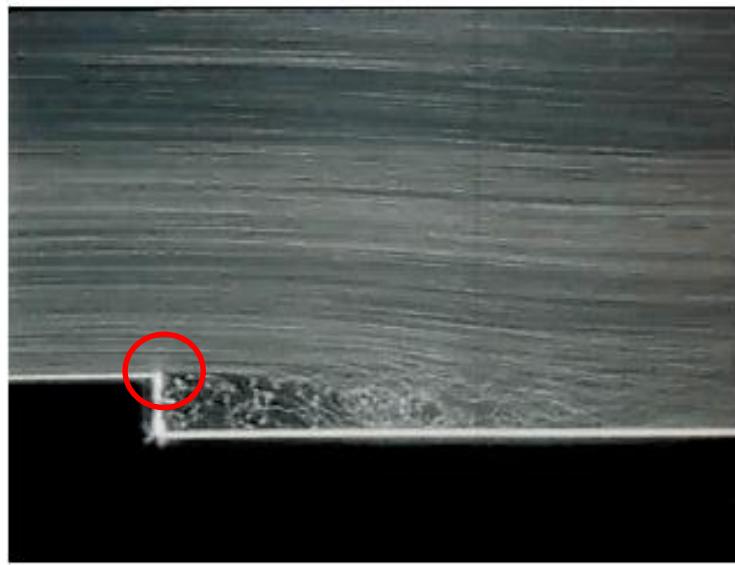
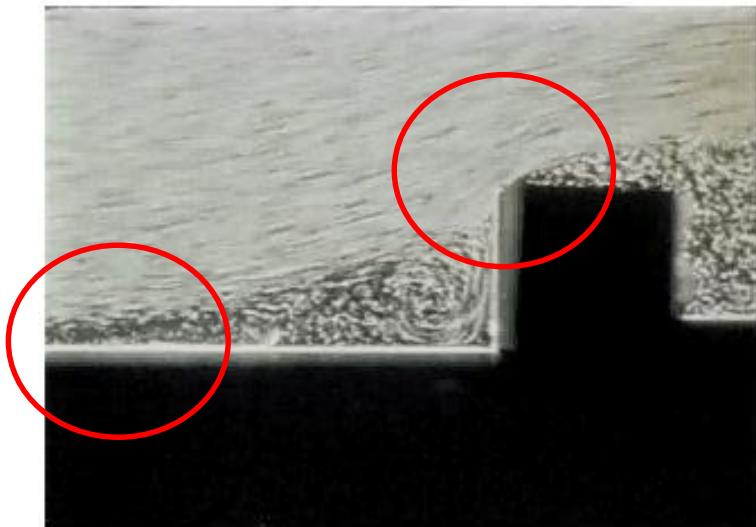
A gallery of detached flows



A gallery of detached flows



A gallery of detached flows

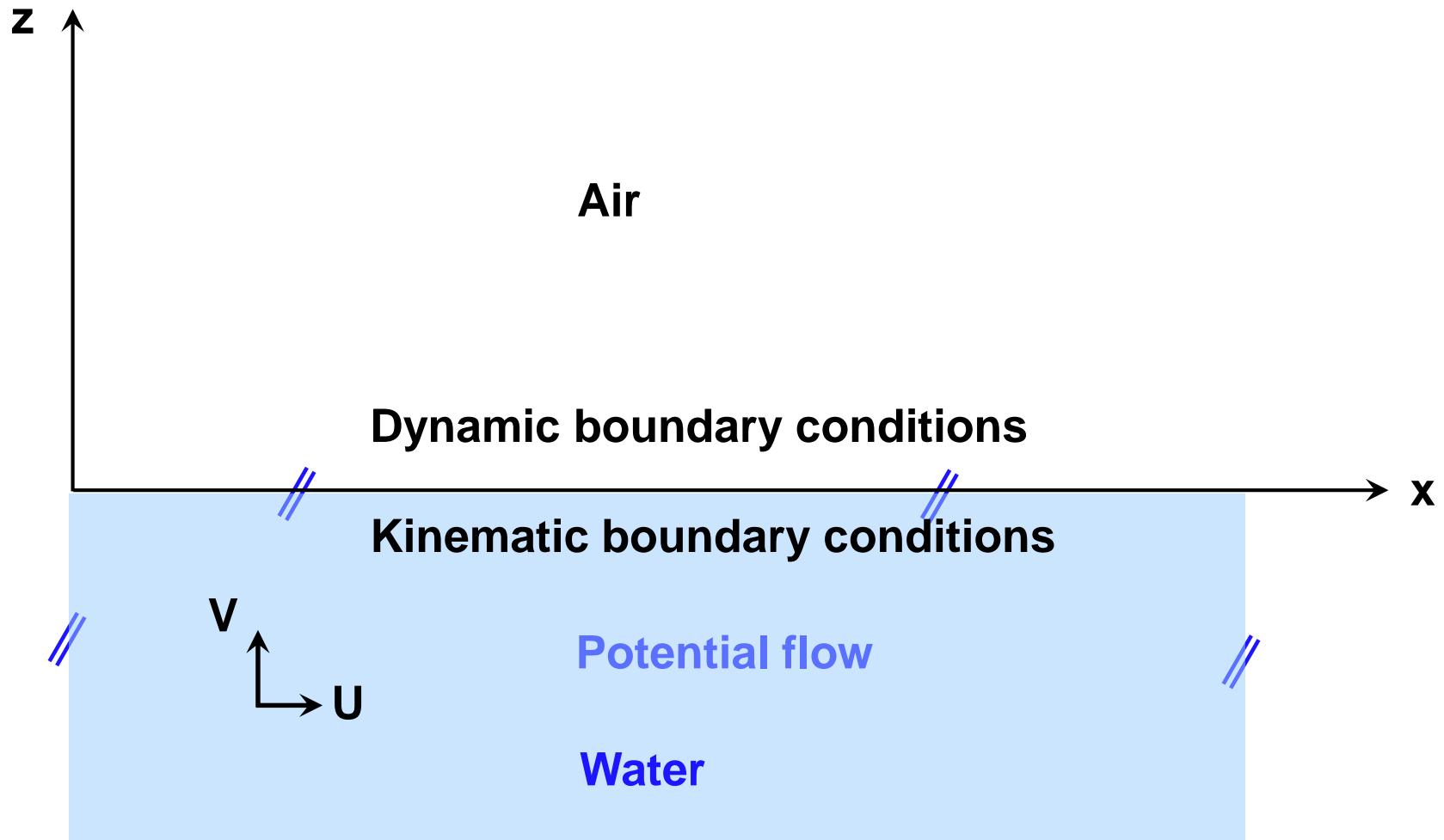


Hydrodynamics 13

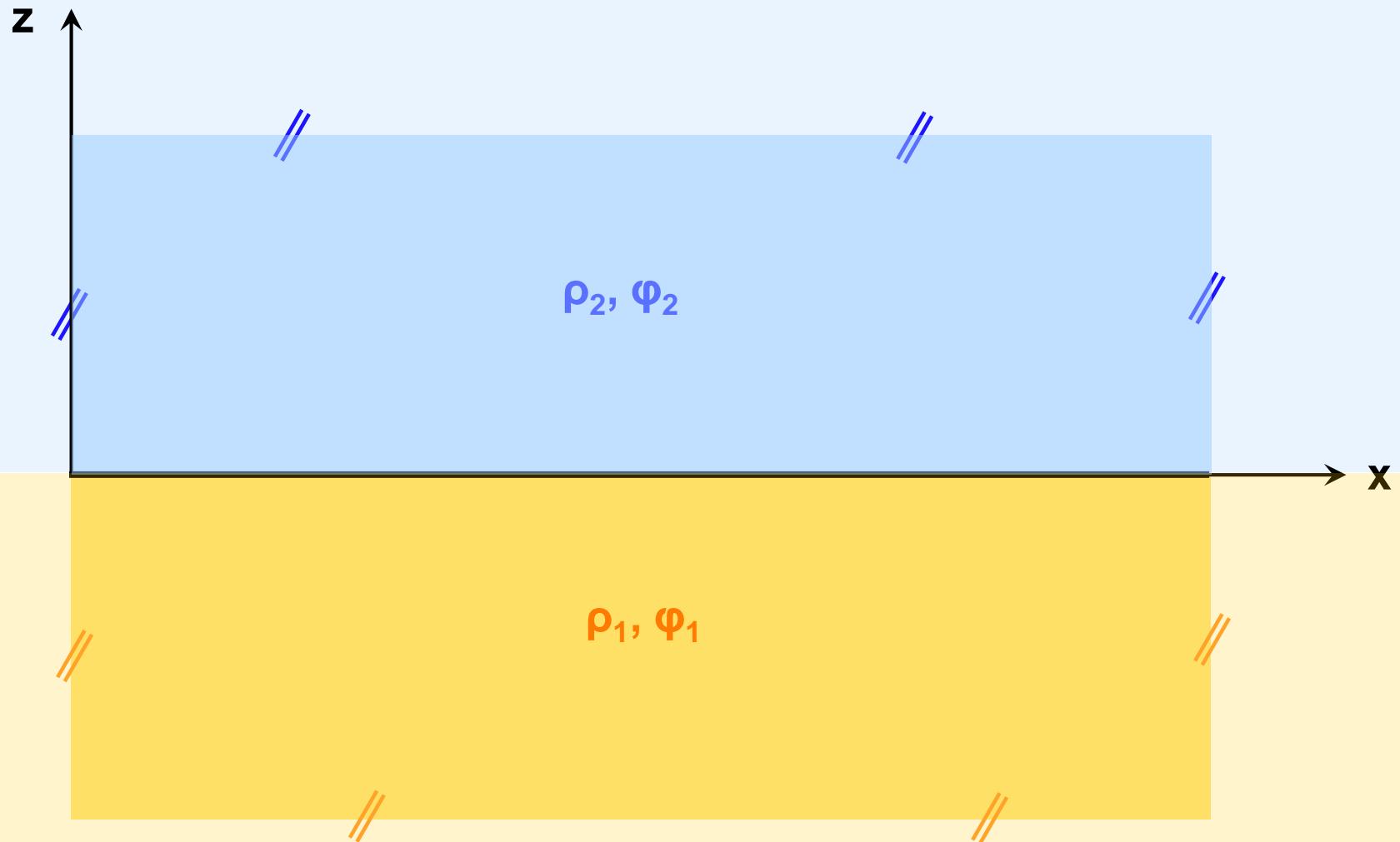
Waves



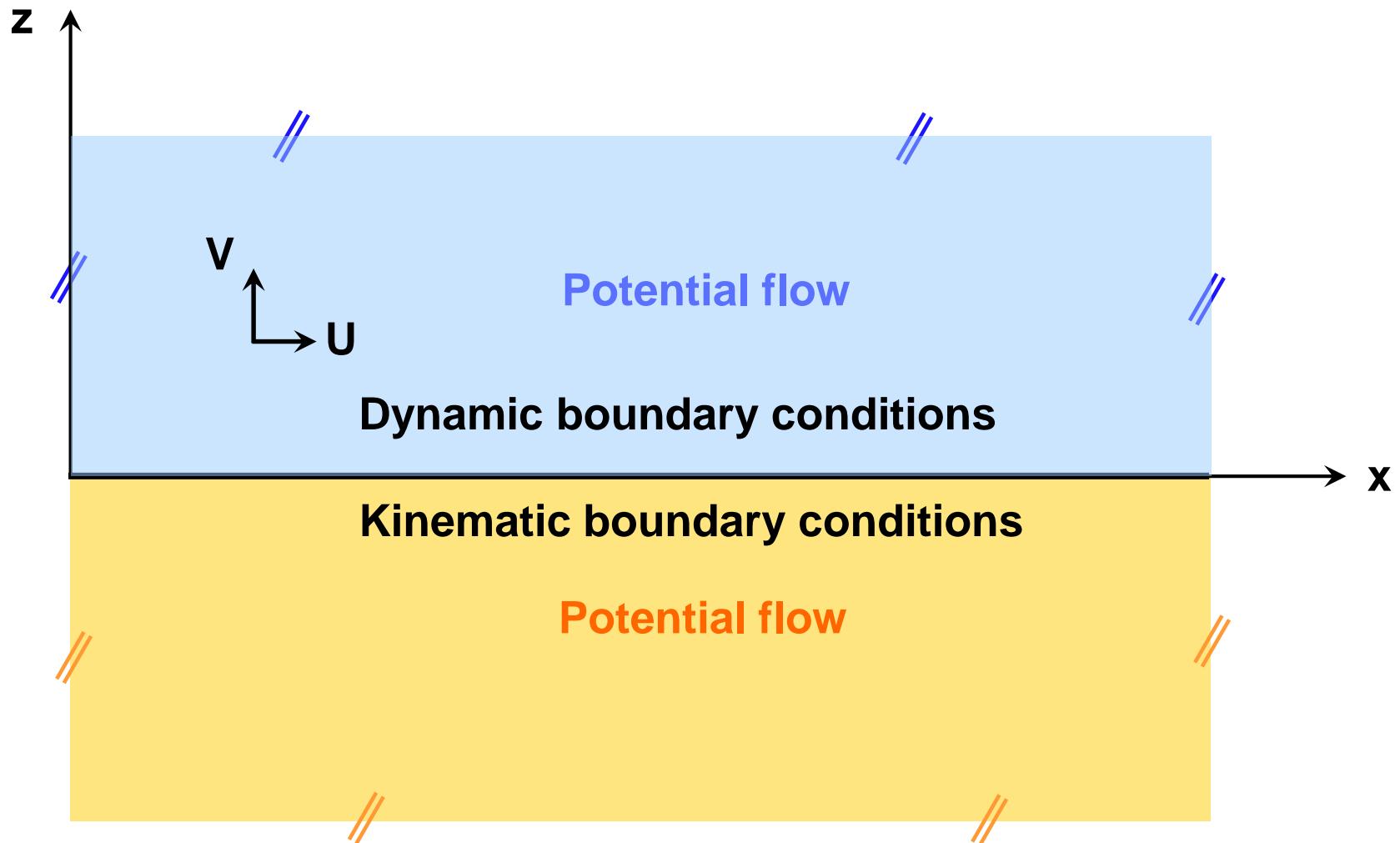
Waves



General case: two fluids



General case: two fluids



Linear waves dispersion relation

1. Equations and boundary conditions
2. Base state
3. Linearized equations
4. Normal mode expansion
5. Dispersion relation
6. Analysis of the dispersion relation

1. Equations

$$\begin{aligned}\Delta\Phi_1 &= 0 \\ \Delta\Phi_2 &= 0\end{aligned}$$

Potential flow

$$\begin{aligned}U_1 &= \frac{\partial\Phi_1}{\partial x}, & V_1 &= \frac{\partial\Phi_1}{\partial z} \\ U_2 &= \frac{\partial\Phi_2}{\partial x}, & V_2 &= \frac{\partial\Phi_2}{\partial z}\end{aligned}$$

Velocity field

1. Boundary conditions

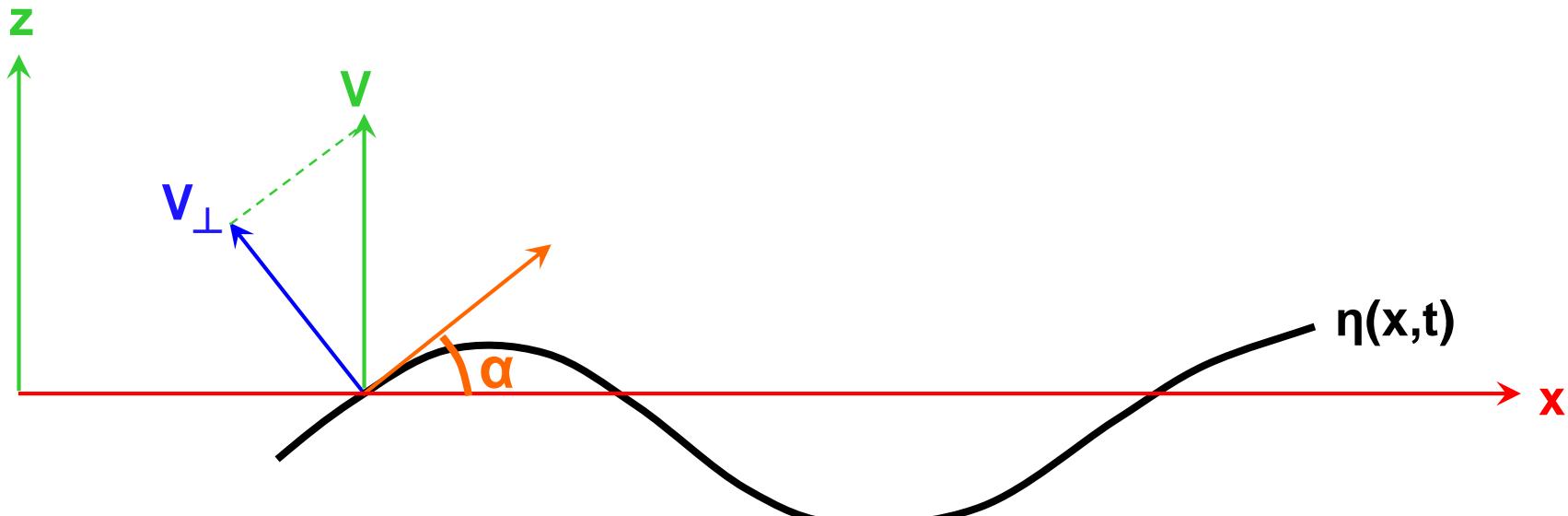
$\Phi_1 = 0$ at $z = -\infty$

$\Phi_2 = 0$ at $z = +\infty$

far-field

at $z = \eta$?

1. Kinematic boundary condition

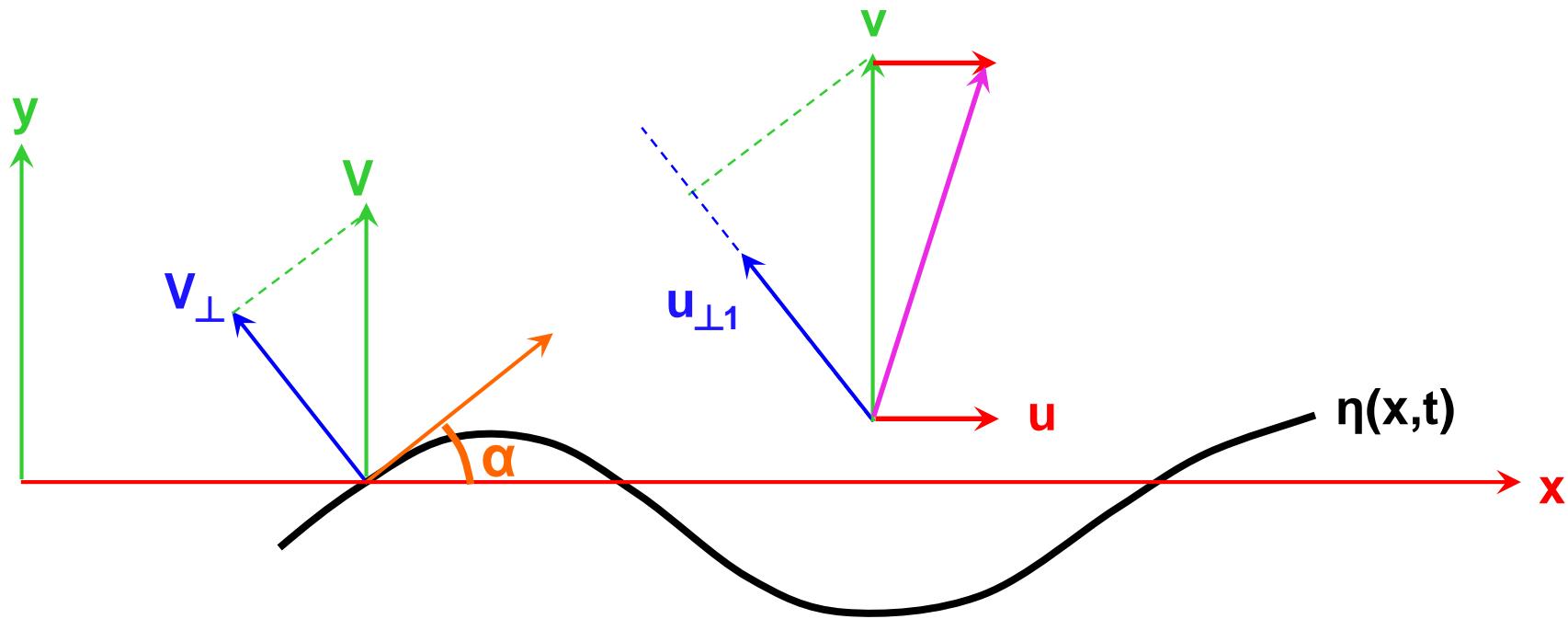


Kinematic condition : impermeability (no penetration)

No fluid particles going across the interface through the normal direction

$$v_{\perp} = \frac{\partial \eta}{\partial t} \cos(\alpha)$$

1. Kinematic boundary condition



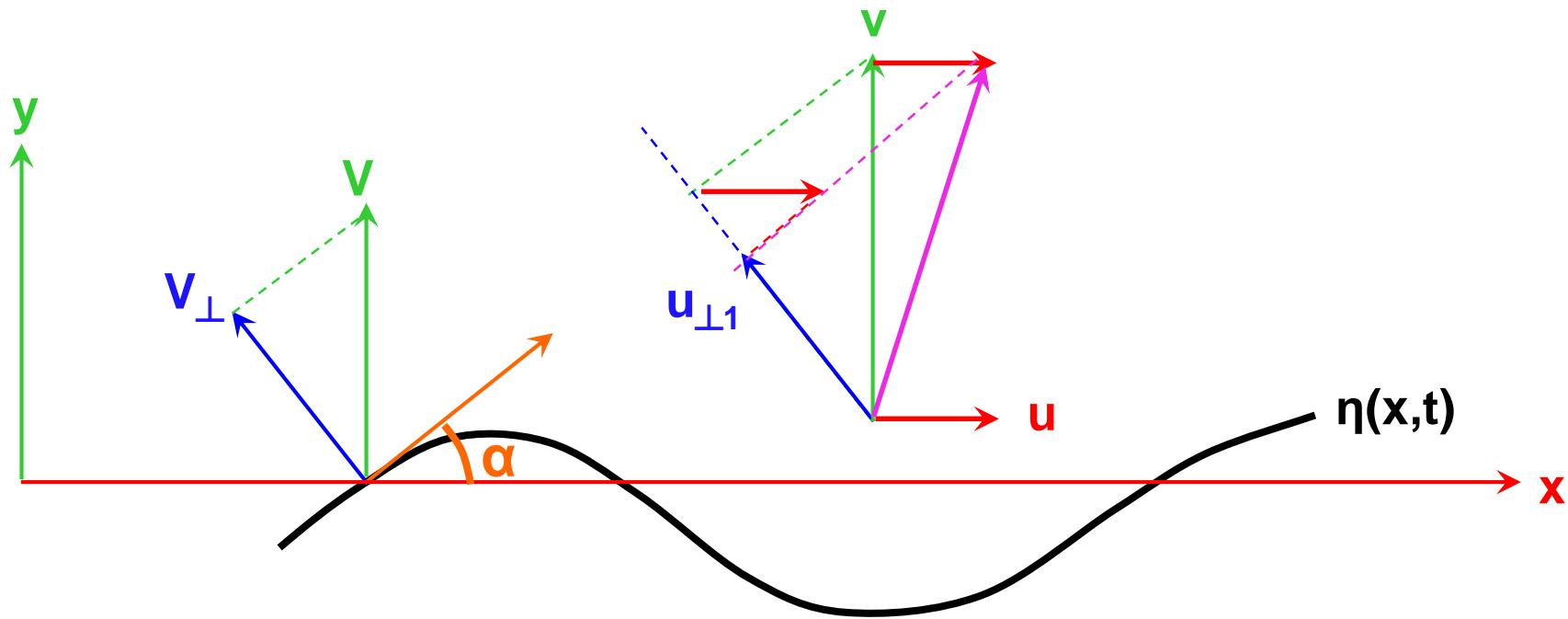
Kinematic condition : impermeability (no penetration)

No fluid particles going across the interface through the normal direction

$$v_{\perp} = \partial \eta / \partial t \cos(\alpha)$$

$$u_{\perp 1} = v_1 \cos(\alpha) +$$

1. Kinematic boundary condition



Kinematic condition : impermeability (no penetration)

No fluid particles going across the interface through the normal direction

$$v_{\perp} = \partial\eta/\partial t \cos(\alpha)$$

$$u_{\perp 1} = v_1 \cos(\alpha) - u_1 \sin(\alpha)$$

$$\left. \begin{array}{l} v_{\perp} = \partial\eta/\partial t \cos(\alpha) \\ u_{\perp 1} = v_1 \cos(\alpha) - u_1 \sin(\alpha) \end{array} \right\} \partial\eta/\partial t = v_1 - u_1 \tan(\alpha) \Rightarrow \boxed{\partial\eta/\partial t = v_1 - u_1 \partial\eta/\partial x}$$

1. Kinematic boundary conditions

$$\Phi_1 = 0 \text{ at } z = -\infty$$

$$\Phi_2 = 0 \text{ at } z = +\infty$$

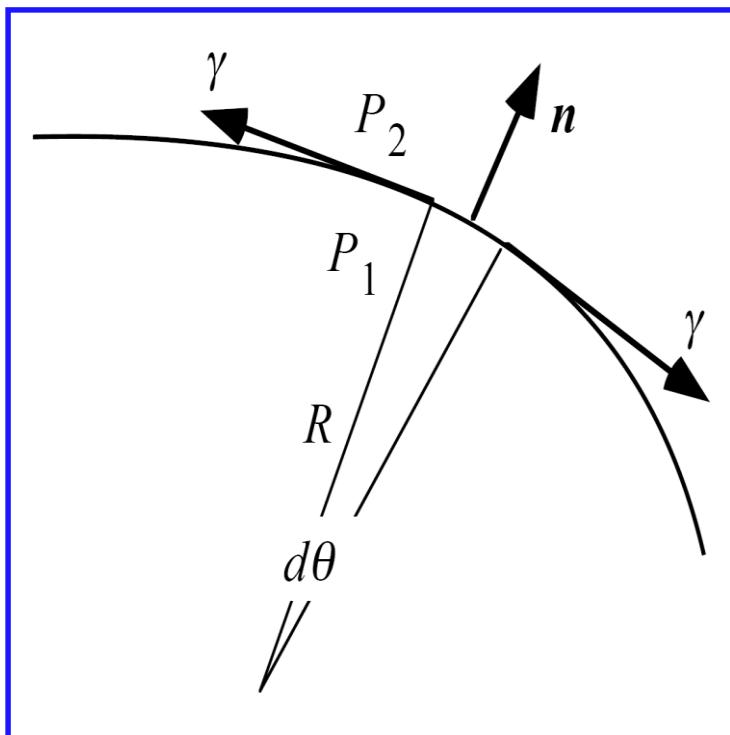
far-field

$$U_1 \frac{\partial \eta}{\partial x} - V_1 = - \frac{\partial \eta}{\partial t} \quad \text{at } z = \eta$$

$$U_2 \frac{\partial \eta}{\partial x} - V_2 = - \frac{\partial \eta}{\partial t}$$

1. Dynamic boundary conditions

$$P_1 - P_2 = -\gamma \frac{\frac{\partial^2 \eta}{\partial x^2}}{\left(1 + \frac{\partial \eta}{\partial x}\right)^{3/2}} \text{ at } z = \eta$$



$$\mathbf{n} = \frac{(-\partial_x \eta, 1)}{\sqrt{1 + \partial_x^2 \eta}}$$

$$\mathcal{C} = \nabla \cdot \mathbf{n}$$

1. More equations

$$\frac{\partial \Phi_1}{\partial t} + \frac{U_1^2 + V_1^2}{2} + \frac{P_1}{\rho_1} + gz = C_1(t) = 0$$

$$\frac{\partial \Phi_2}{\partial t} + \frac{U_2^2 + V_2^2}{2} + \frac{P_2}{\rho_2} + gz = C_2(t) = 0$$

2nd Bernoulli relations

2. Base state

$$\Phi_1 = 0,$$

$$\Phi_2 = 0$$

$$\eta = 0$$

$$P_1 = -\rho_1 g z$$

3. Perturb and linearize perturbation expansion

$$\begin{array}{lll} \Phi_1 & = 0 & + \epsilon \phi_1 \\ \Phi_2 & = 0 & + \epsilon \phi_2 \\ U_1 & = 0 & + \epsilon u_1 \\ V_1 & = 0 & + \epsilon v_1 \\ U_2 & = 0 & + \epsilon u_2 \\ V_2 & = 0 & + \epsilon v_2 \\ P_1 & = -\rho_1 g z & + \epsilon p_1 \\ P_2 & = -\rho_2 g z & + \epsilon p_2 \\ \eta & = 0 & + \epsilon \sigma \end{array} \quad \epsilon \ll 1$$

Variables **Base state** **Small perturbation**

3. Linearized equations

$$\begin{aligned}\Delta\phi_1 &= 0 \\ \Delta\phi_2 &= 0\end{aligned}$$

perturbed potential flow

$$\begin{aligned}u_1 &= \frac{\partial\phi_1}{\partial x}, & v_1 &= \frac{\partial\phi_1}{\partial z} \\ u_2 &= \frac{\partial\phi_2}{\partial x}, & v_2 &= \frac{\partial\phi_2}{\partial z}\end{aligned}$$

3. Perturbed kinematic boundary conditions

$$\phi_1 = 0 \text{ at } z = -\infty$$

$$\phi_2 = 0 \text{ at } z = +\infty$$

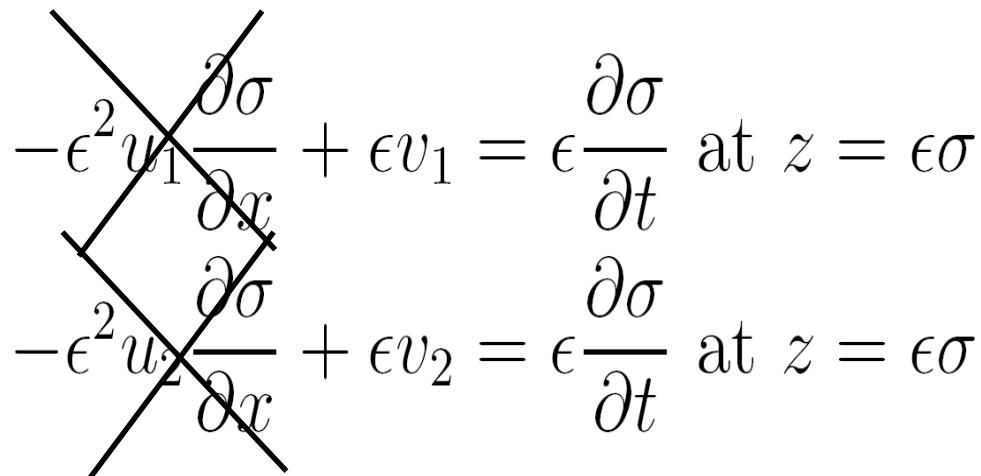
$$-\epsilon^2 u_1 \frac{\partial \sigma}{\partial x} + \epsilon v_1 = \epsilon \frac{\partial \sigma}{\partial t} \text{ at } z = \epsilon \sigma$$

$$-\epsilon^2 u_2 \frac{\partial \sigma}{\partial x} + \epsilon v_2 = \epsilon \frac{\partial \sigma}{\partial t} \text{ at } z = \epsilon \sigma$$

3. Perturbed kinematic boundary conditions

$$\phi_1 = 0 \text{ at } z = -\infty$$

$$\phi_2 = 0 \text{ at } z = +\infty$$


$$-\epsilon^2 u_1 \frac{\partial \sigma}{\partial x} + \epsilon v_1 = \epsilon \frac{\partial \sigma}{\partial t} \text{ at } z = \epsilon \sigma$$
$$-\epsilon^2 u_2 \frac{\partial \sigma}{\partial x} + \epsilon v_2 = \epsilon \frac{\partial \sigma}{\partial t} \text{ at } z = \epsilon \sigma$$

$$v_1 = \frac{\partial \sigma}{\partial t} \text{ at } z = \epsilon \sigma$$

$$v_2 = \frac{\partial \sigma}{\partial t} \text{ at } z = \epsilon \sigma$$

3. Flattened kinematic boundary conditions

$$\frac{\partial \phi_1}{\partial z} = \frac{\partial \sigma}{\partial t} \text{ at } z = \epsilon\sigma$$
$$\frac{\partial \phi_2}{\partial z} = \frac{\partial \sigma}{\partial t} \text{ at } z = \epsilon\sigma$$

Taylor expansion around 0: $\phi(\epsilon\sigma) = \phi(0) + (\epsilon\sigma) \frac{\partial \phi}{\partial z} \Big|_0$

$$\frac{\partial \phi_1}{\partial z} = \frac{\partial \sigma}{\partial t} \text{ at } z = 0$$
$$\frac{\partial \phi_2}{\partial z} = \frac{\partial \sigma}{\partial t} \text{ at } z = 0$$

⇒ transforms a b.c. at an unknown interface into a fixed place!

3. Perturbed dynamic boundary conditions

$$(P_1 + \epsilon p_1 - P_2 - \epsilon p_2)|_{\epsilon\sigma} = -\gamma\epsilon \frac{\partial^2\sigma}{\partial x^2} \left(1 - 3/2\epsilon^2 \left(\frac{\partial\sigma}{\partial x} \right)^2 \right)$$

Replace $P_1 = -gp_1z, \dots$

and linearize

$$\mathbf{g}(\rho_2 - \rho_1)\sigma + (p_1 - p_2)|_{\epsilon\sigma} = -\gamma \frac{\partial^2\sigma}{\partial x^2}$$

flatten

$$(\rho_2 - \rho_1)g\sigma + (p_1 - p_2)|_0 = -\gamma \frac{\partial^2\sigma}{\partial x^2}$$

3. Perturbed and linearized Bernoulli

Perturbed 2nd Bernoulli relations

$$\epsilon \frac{\partial \phi_1}{\partial t} + \epsilon^2 \frac{u_1^2 + v_1^2}{2} + \epsilon \frac{p_1}{\rho_1} = 0$$
$$\epsilon \frac{\partial \phi_2}{\partial t} + \epsilon^2 \frac{u_2^2 + v_2^2}{2} + \epsilon \frac{p_2}{\rho_2} = 0$$

Linearized 2nd Bernoulli relations

$$\frac{\partial \phi_1}{\partial t} + \frac{p_1}{\rho_1} = 0$$
$$\frac{\partial \phi_2}{\partial t} + \frac{p_2}{\rho_2} = 0$$

4. Normal mode expansion

Fourier transform in x and t

$$\phi_1 = f_1(z) \exp(i(kx - \omega t)),$$

$$\phi_2 = f_2(z) \exp(i(kx - \omega t)),$$

$$\sigma = C \exp(i(kx - \omega t)),$$

k is the wavenumber and ω the frequency (in rad/s)

$$\lambda = 2\pi/k$$

$$T = 2\pi/\omega$$

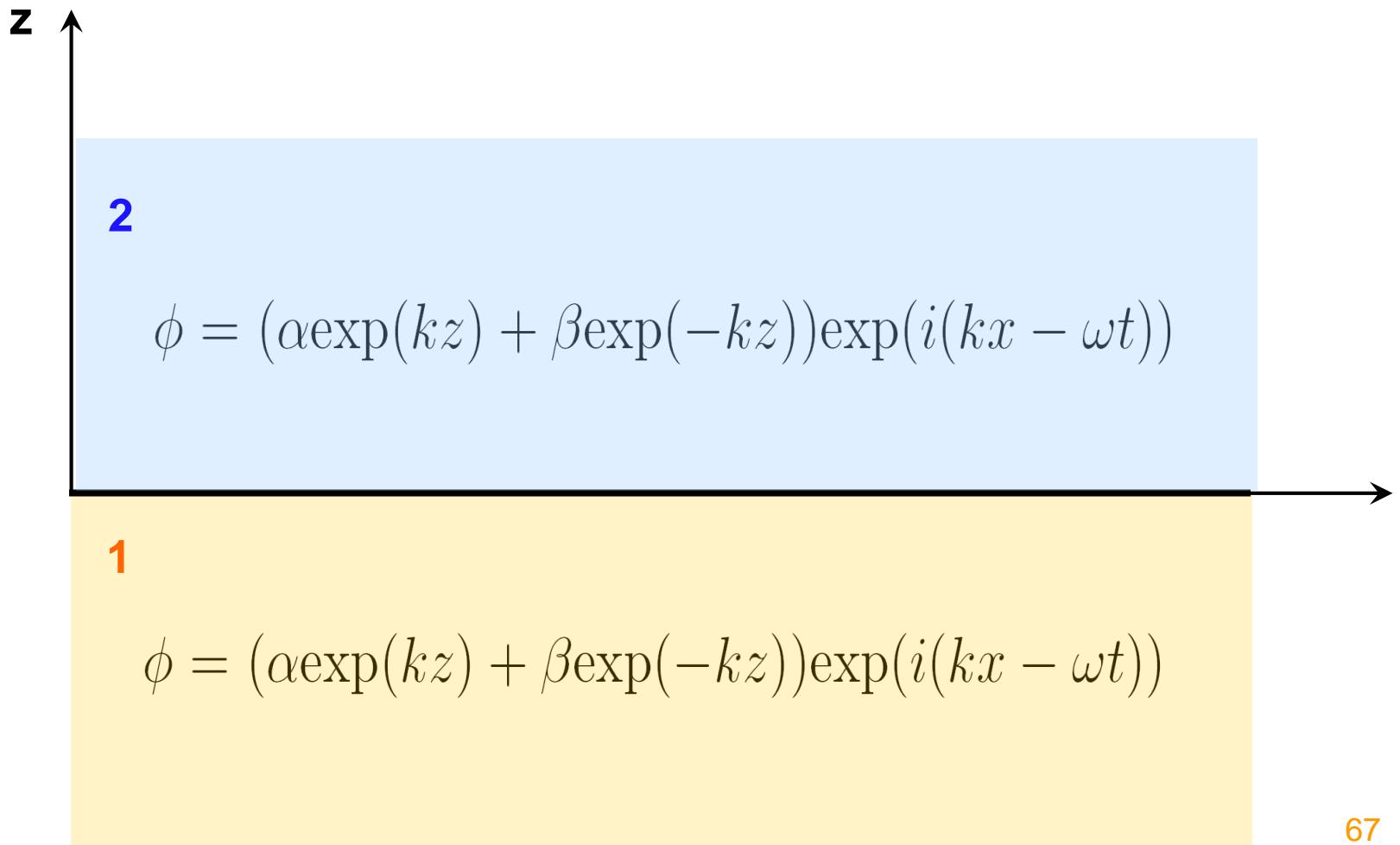
$$f = \omega/(2\pi)$$

4. Normal mode expansion

Solution to Laplace equation:

4. Normal mode expansion

Solution to Laplace equation:



4. Normal mode expansion

Solution to Laplace equation:

$$\begin{aligned}\phi_1 &= A \exp(kz) \exp(i(kx - \omega t)), \\ \phi_2 &= B \exp(-kz) \exp(i(kx - \omega t)), \\ \sigma &= C \exp(i(kx - \omega t)).\end{aligned}$$

4. Normal mode expansion

Replace in boundary conditions

$$\begin{aligned} g(\rho_2 - \rho_1)C + i\omega\rho_1A - i\omega\rho_2B &= \gamma k^2 C \\ kA &= -i\omega C \\ -kB &= -i\omega C \end{aligned}$$

This is an eigenvalue problem $i\omega X = MX$!

$$kg(\rho_2 - \rho_1)C + \omega^2\rho_1C + \omega^2\rho_2C = \gamma k^3 C.$$

5. Dispersion relation

$$\omega^2 = \frac{-kg(\rho_2 - \rho_1) + \gamma k^3}{\rho_1 + \rho_2}$$

- **Unstable if there exists one ω , $\text{Im}(\omega) > 0$**

$$\rho_2 > \rho_1$$

- **Neutral if for all ω , $\text{Im}(\omega) = 0$:**

$$\rho_1 > \rho_2$$

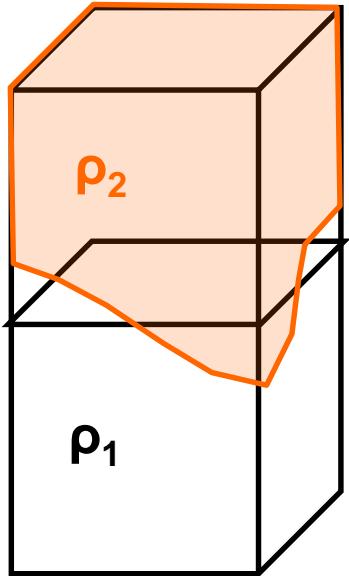
- **Stable (or damped) if for all ω , $\text{Im}(\omega) < 0$:**

The flow considered is not damped, we have neglected dissipation by neglecting viscosity

Instability analysis:

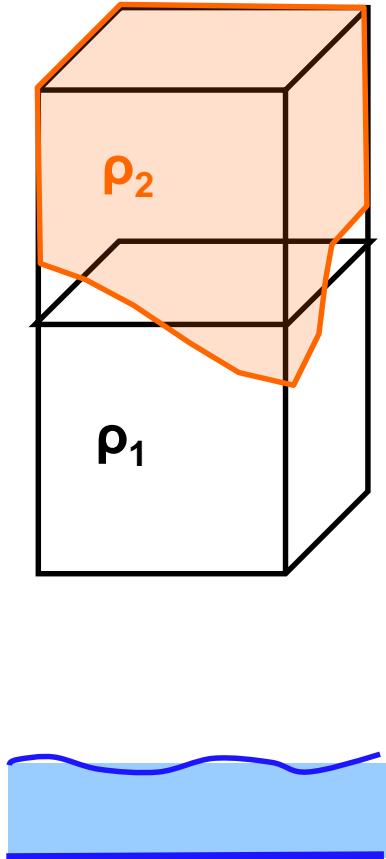
1. Equations and boundary conditions
2. Base state
3. Linearized equations
4. Normal mode expansion
5. Dispersion relation
6. Analysis of the dispersion relation

Dispersion relation



$$\omega^2 = \frac{-kg(\rho_2 - \rho_1) + \gamma k^3}{\rho_1 + \rho_2}$$

Dispersion relation



$$\omega^2 = \frac{-kg(\rho_2 - \rho_1) + \gamma k^3}{\rho_1 + \rho_2}$$

$$\omega^2 = \tanh(kH) \left(\frac{\gamma k^3}{\rho} + gk \right)$$

Dispersion relation

$$\omega^2 = \tanh(kH) \left(\frac{\gamma k^3}{\rho} + gk \right)$$

Capillary wavenumber: $k_c = \sqrt{\rho g / \gamma}$

Length scale: $\tilde{k} = k/k_c$

Time scale $\tilde{\omega} = \omega / \sqrt{gk_c}$

One single non-dimensional parameter $\tilde{H} = Hk_c$

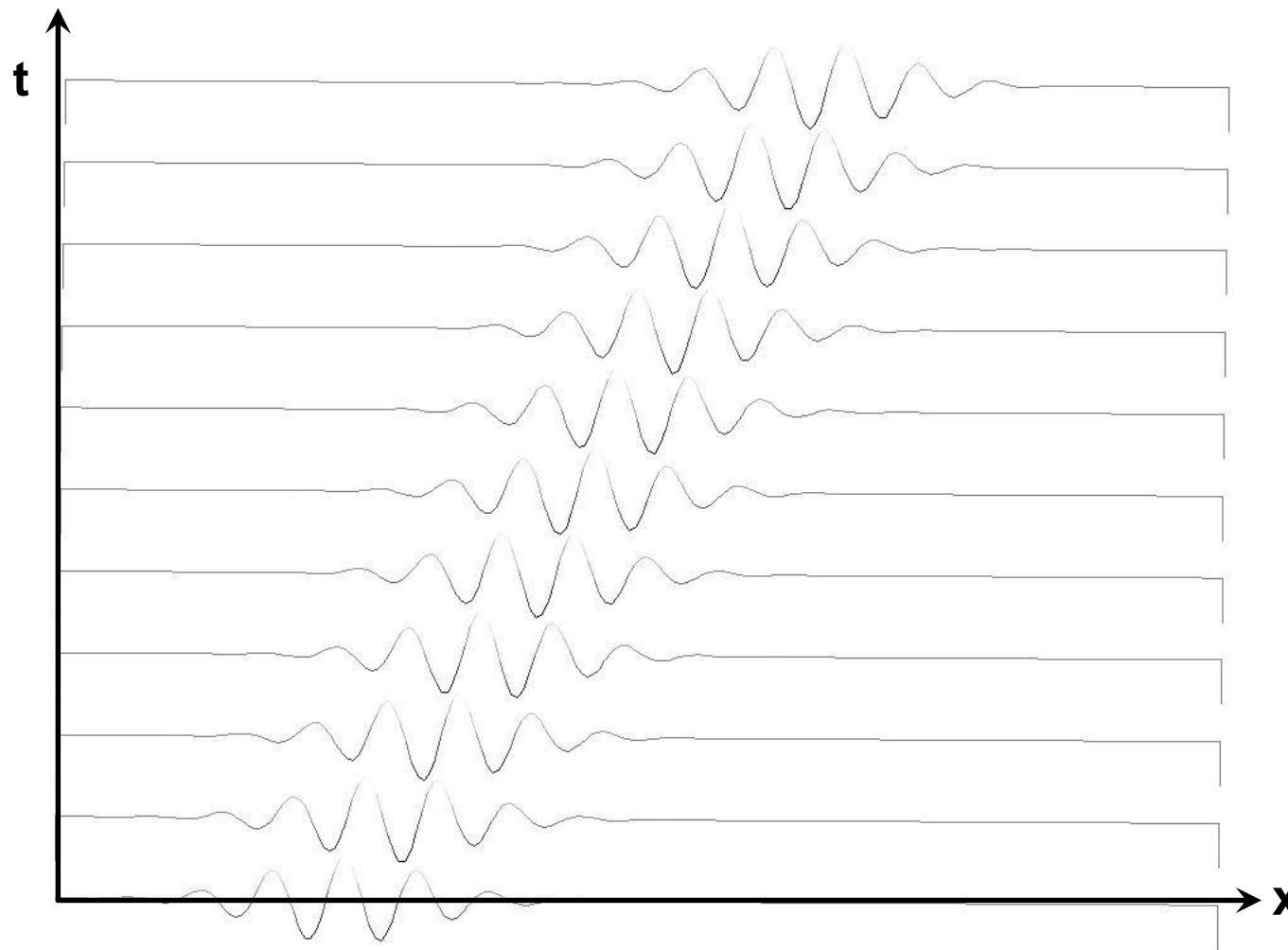
$$\tilde{\omega}^2 = \tanh(\tilde{k}\tilde{H}) \left(\tilde{k}^3 + \tilde{k} \right)$$

Dispersion relation

$$\tilde{\omega}^2 = \tanh(\tilde{k}\tilde{H}) \left(\tilde{k}^3 + \tilde{k} \right)$$

	gravity $\tilde{k} \ll 1$	capillary $\tilde{k} \gg 1$
shallow water $\tilde{k} \ll 1/\tilde{H}$	$\pm \tilde{k}$	$\pm \tilde{k}^2 \sqrt{\tilde{H}}$
Deep water $\tilde{k} \gg 1/\tilde{H}$	$\pm \sqrt{\tilde{k}}$	$\pm \tilde{k} \sqrt{\tilde{k}}$

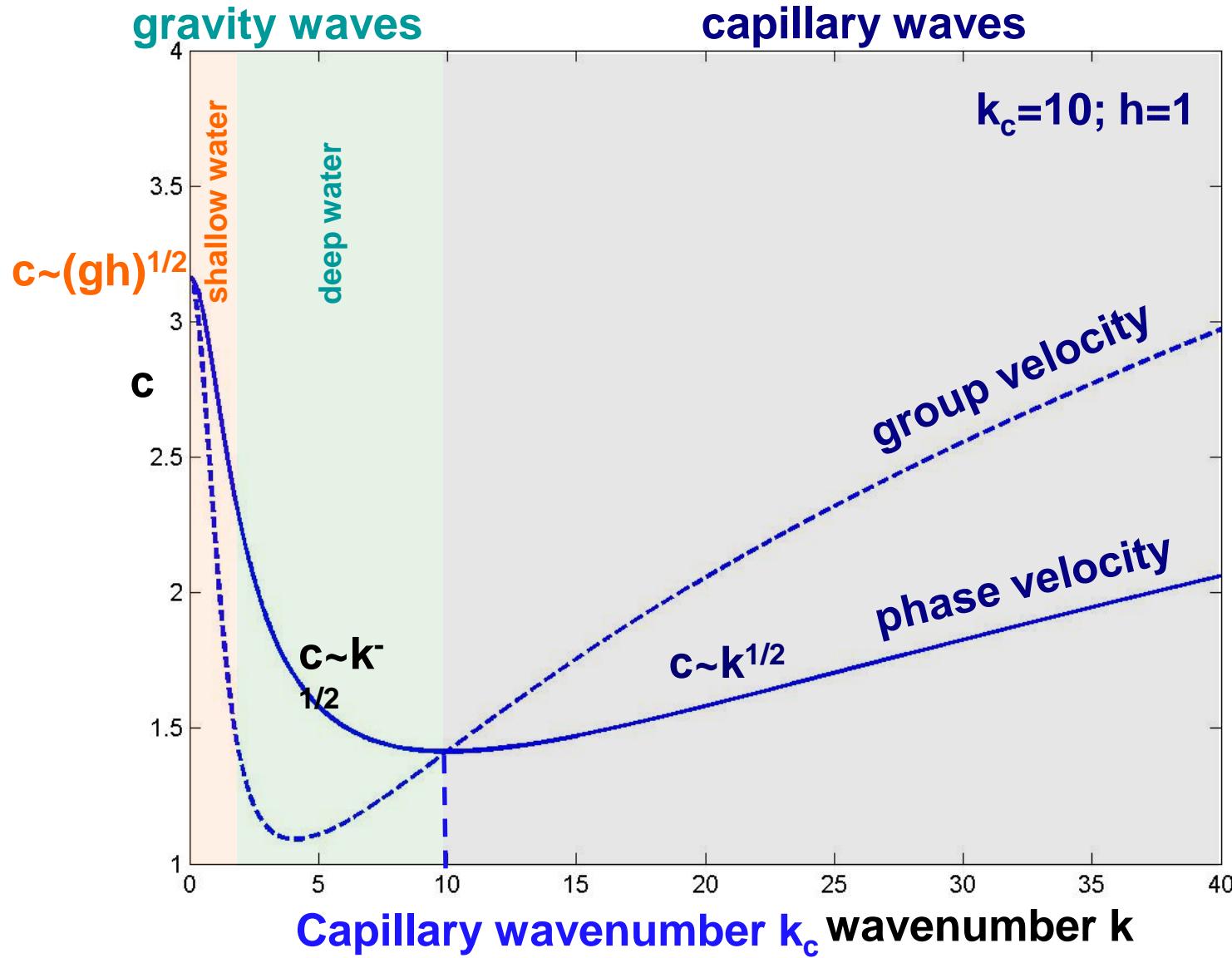
Difference between group velocity v and phase velocity c



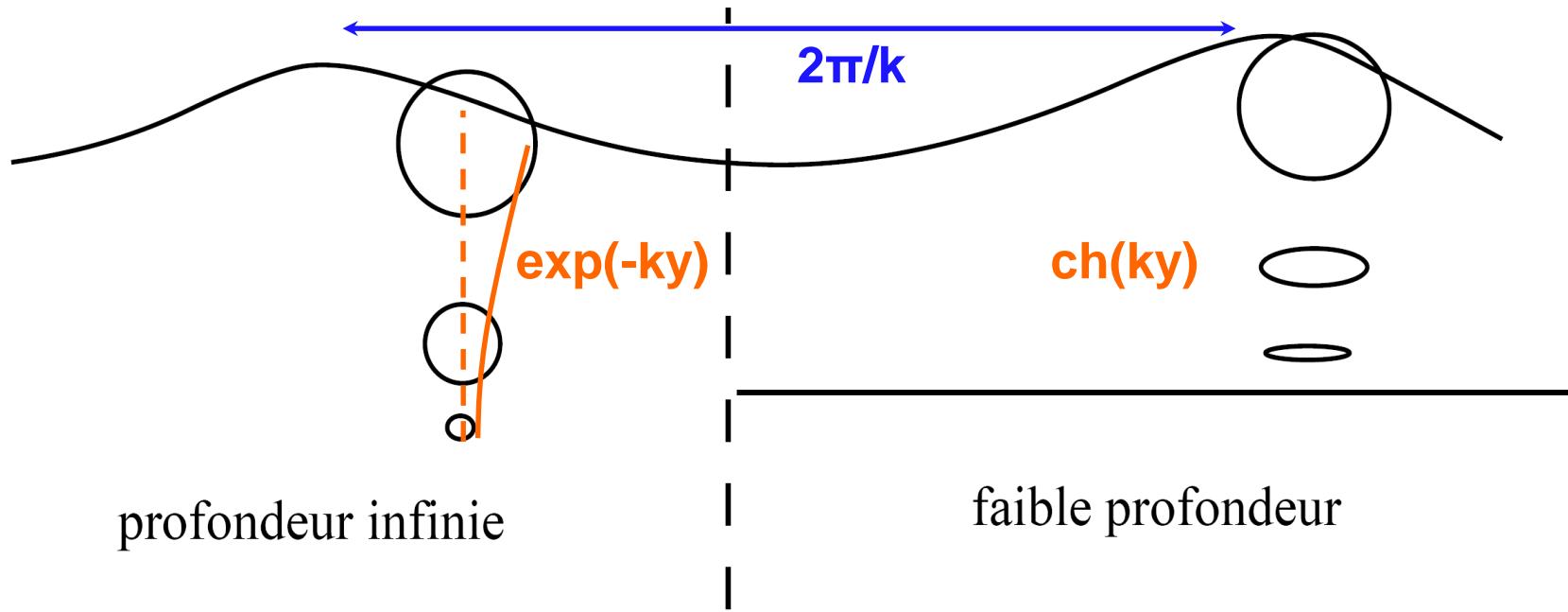
Dispersion relation

	gravity $\tilde{k} \ll 1$	capillary $\tilde{k} \gg 1$
shallow water $\tilde{k} \ll 1/\tilde{H}$	$\omega_{shallow/gravity} \sim \pm k \sqrt{gH}$ $c_{shallow/gravity} \sim \pm \sqrt{gH}$ $v_{shallow/gravity} \sim \pm \sqrt{gH}$	$\omega_{shallow/capillary} \sim \pm k^2 \sqrt{\gamma H / \rho}$ $c_{shallow/capillary} \sim \pm k \sqrt{\gamma H / \rho}$ $v_{shallow/capillary} \sim \pm 2k \sqrt{\gamma H / \rho}$
Deep water $\tilde{k} \gg 1/\tilde{H}$	$\omega_{deep/gravity} \sim \pm \sqrt{gk}$ $c_{deep/gravity} \sim \pm \sqrt{\frac{g}{k}}$ $v_{deep/gravity} \sim \pm \frac{1}{2} \sqrt{\frac{g}{k}}$	$\omega_{deep/capillary} \sim \pm k^{3/2} \sqrt{\gamma / \rho}$ $c_{deep/capillary} \sim \pm k^{1/2} \sqrt{\gamma / \rho}$ $v_{deep/capillary} \sim \pm 3/2 k^{1/2} \sqrt{\gamma / \rho}$

Dispersion relation

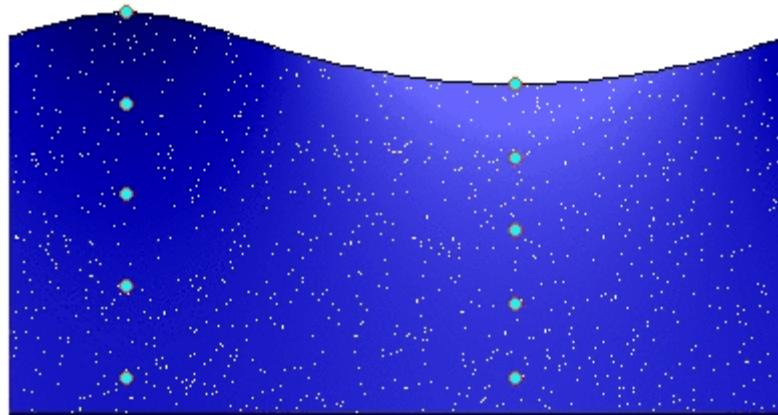


Trajectories below the waves



Stokes drift!

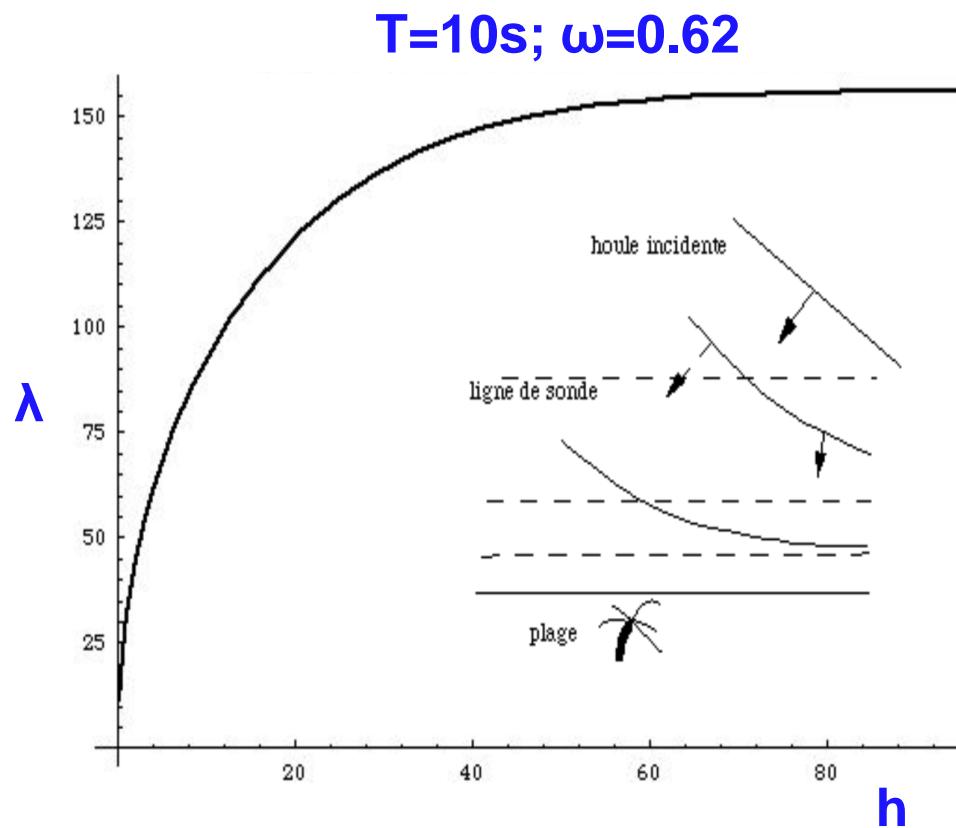
wave phase : $t / T = 0.000$



Why are the waves parallel to the shore?

$$c \sim (gh)^{1/2}$$

$$\lambda \sim T(gh)^{1/2}$$



Refraction and diffraction of waves

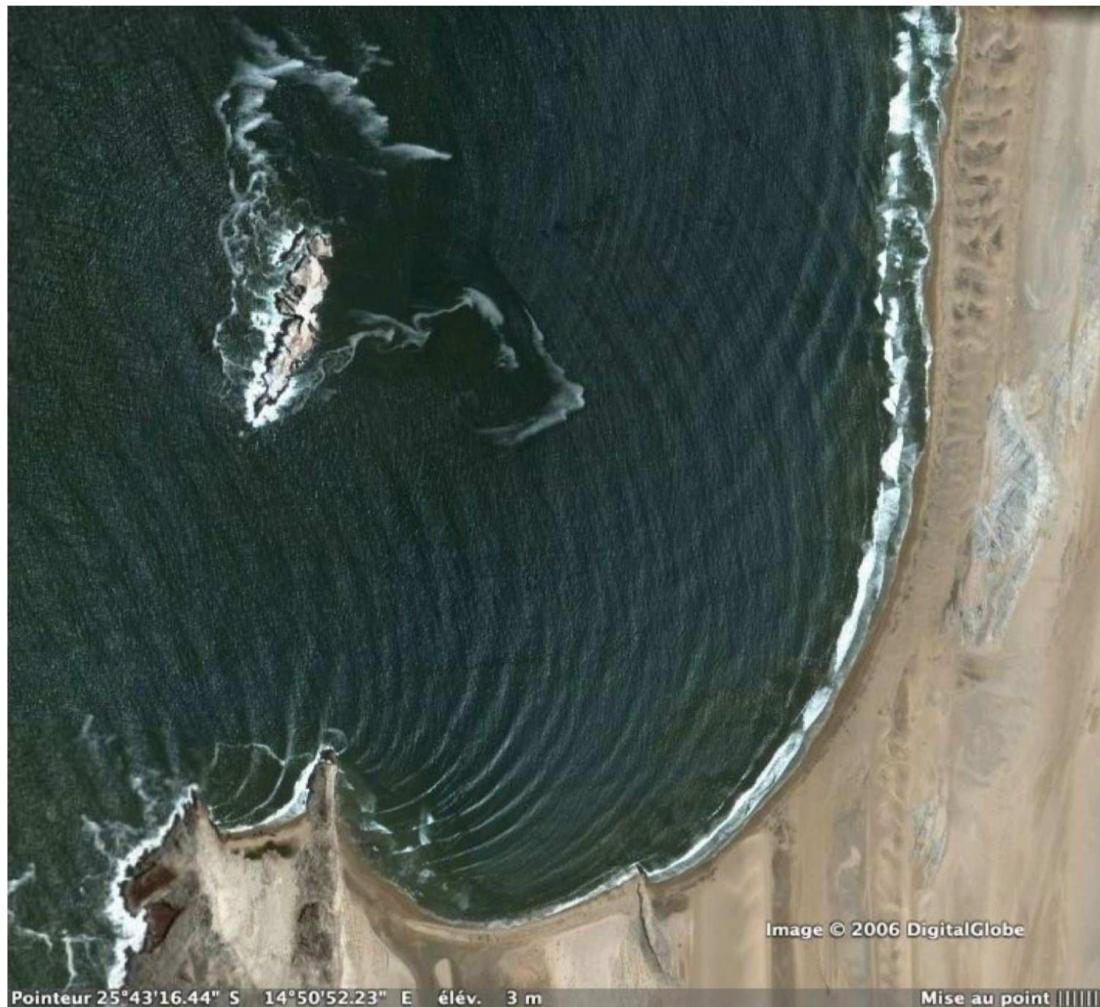


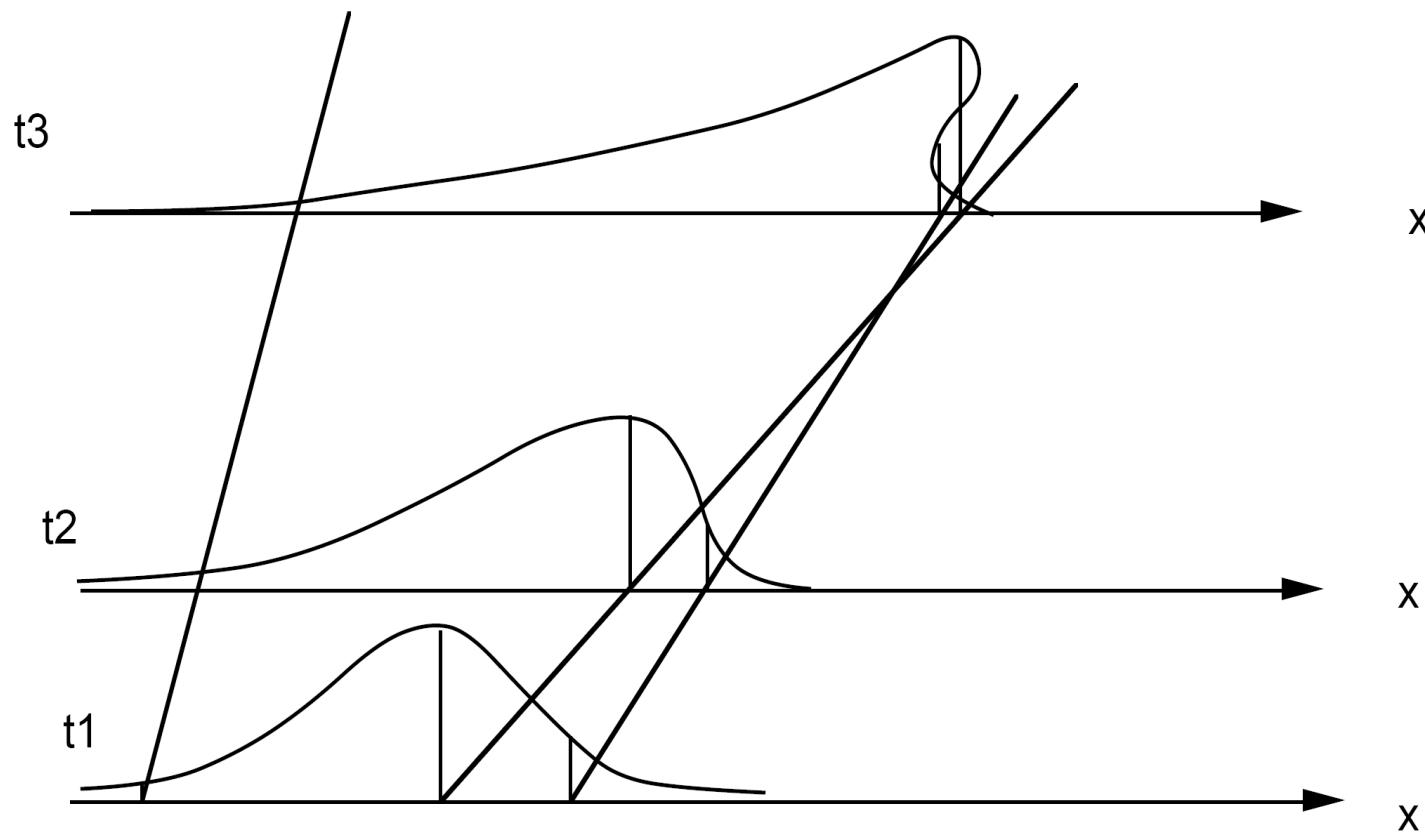
Image © 2006 DigitalGlobe

Pointeur 25°43'16.44" S 14°50'52.23" E élév. 3 m

Mise au point |||||||

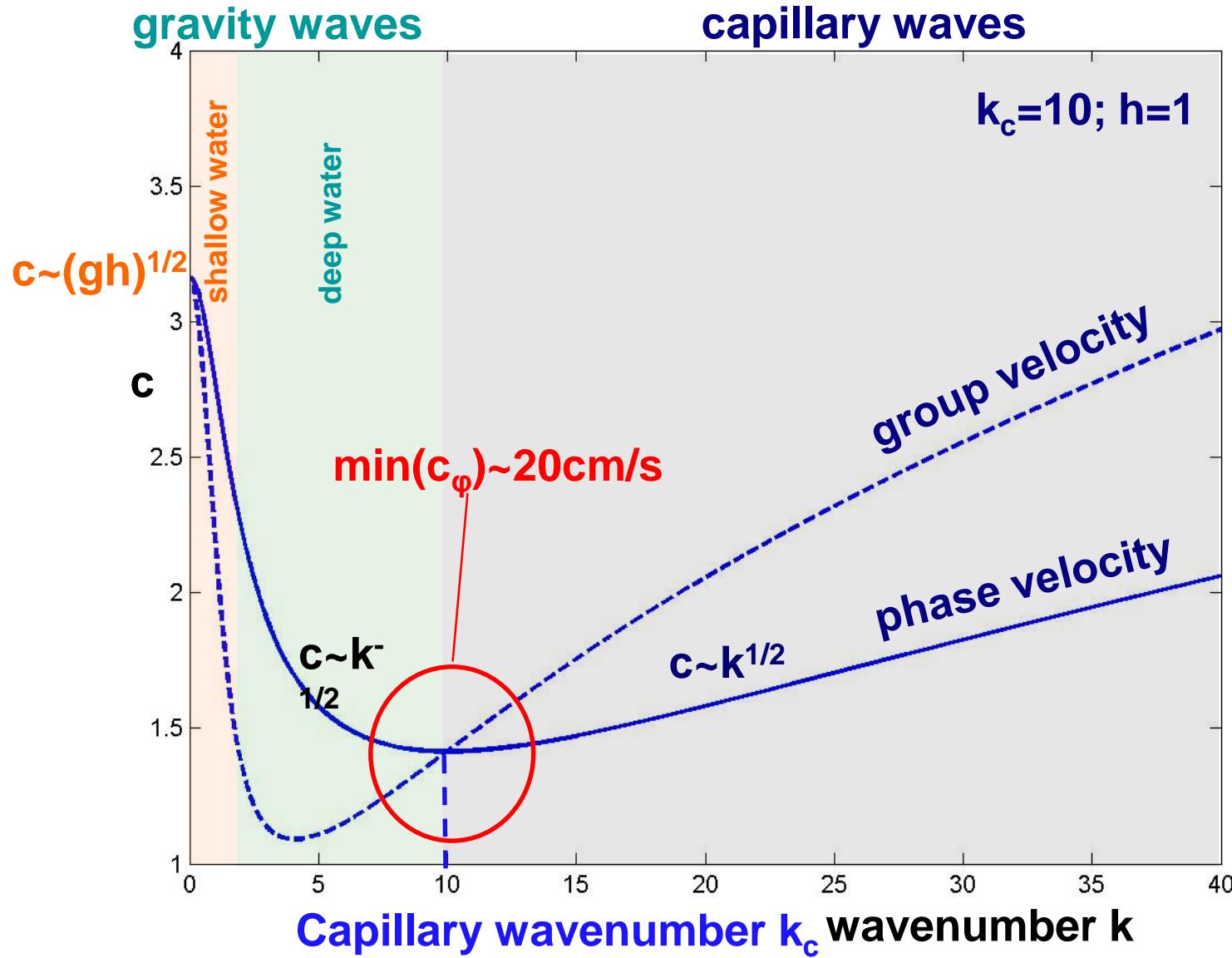
Satellite view Namibian coast

Nonlinear waves, wavebreaking



The celerity increases with the depth

Dispersion relation



Conditions for wave pattern formation?



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$$V_{\text{duck}} \leq C_{\min} \quad ?$$