

Hydrodynamics

potential flow

Outline

1. Intro+kinematics
2. Dynamics
3. Dimensional Analysis
4. Low Reynolds number flow/ Stokes drag
5. Stokes drag-sedimentation-reversibility
6. Hele Shaw-Lubrication--Pipe Flows

7. Unsteady flow

8. Boundary layer [1]

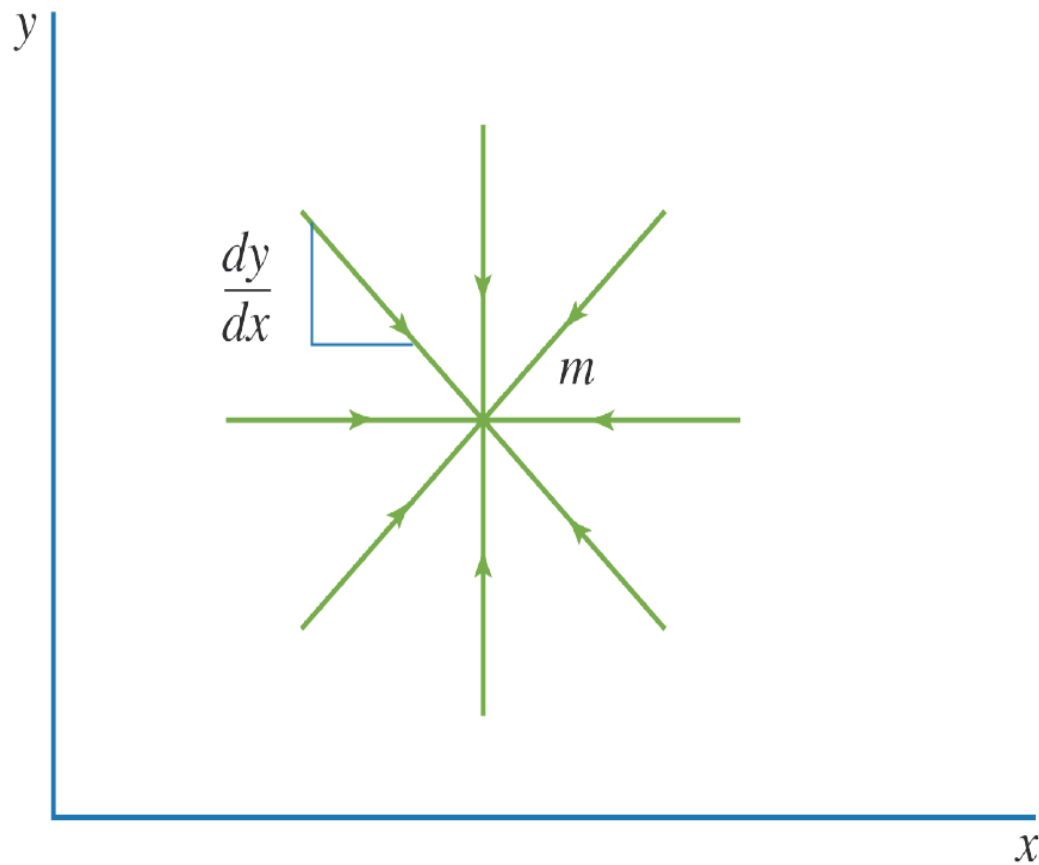
9. Boundary layer [2]

10. Vorticity-Bernoulli-Potential flow

11. Potential flow

Holomorphic functions

$$Z = P(x, y) + iQ(x, y)$$



Holomorphic functions

$$\frac{dZ}{dz} = \frac{\frac{\partial P}{\partial x} + i \frac{\partial Q}{\partial x} + \left(\frac{\partial P}{\partial y} + i \frac{\partial Q}{\partial y} \right) \frac{dy}{dx}}{1 + i \frac{dy}{dx}}$$

$$f'(z) = \lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

$$\frac{\partial P}{\partial x} + i \frac{\partial Q}{\partial x} = -i \left(\frac{\partial P}{\partial y} + i \frac{\partial Q}{\partial y} \right)$$

Holomorphic functions

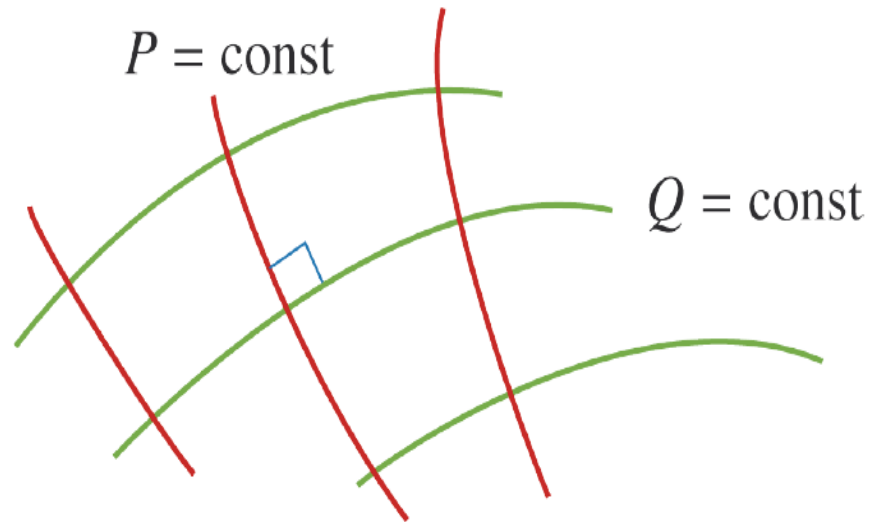
Cauchy-Riemann conditions

$$\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y} \quad , \quad \frac{\partial P}{\partial y} = -\frac{\partial Q}{\partial x}$$

Conjugate Holomorphic functions

$$\Delta P = 0 \quad , \quad \Delta Q = 0$$

Orthogonal lattice



2D potential flow

$$z = x + iy$$

$$f(z) = \varphi(x, y) + i\psi(x, y)$$

$$w(z) = \frac{df}{dz} = u(x, y) - iv(x, y)$$

Complex Potential

$$f(z) = \phi(x, y) + i\psi(x, y)$$

$$\frac{\partial \phi}{\partial y} = - \frac{\partial \psi}{\partial x}$$

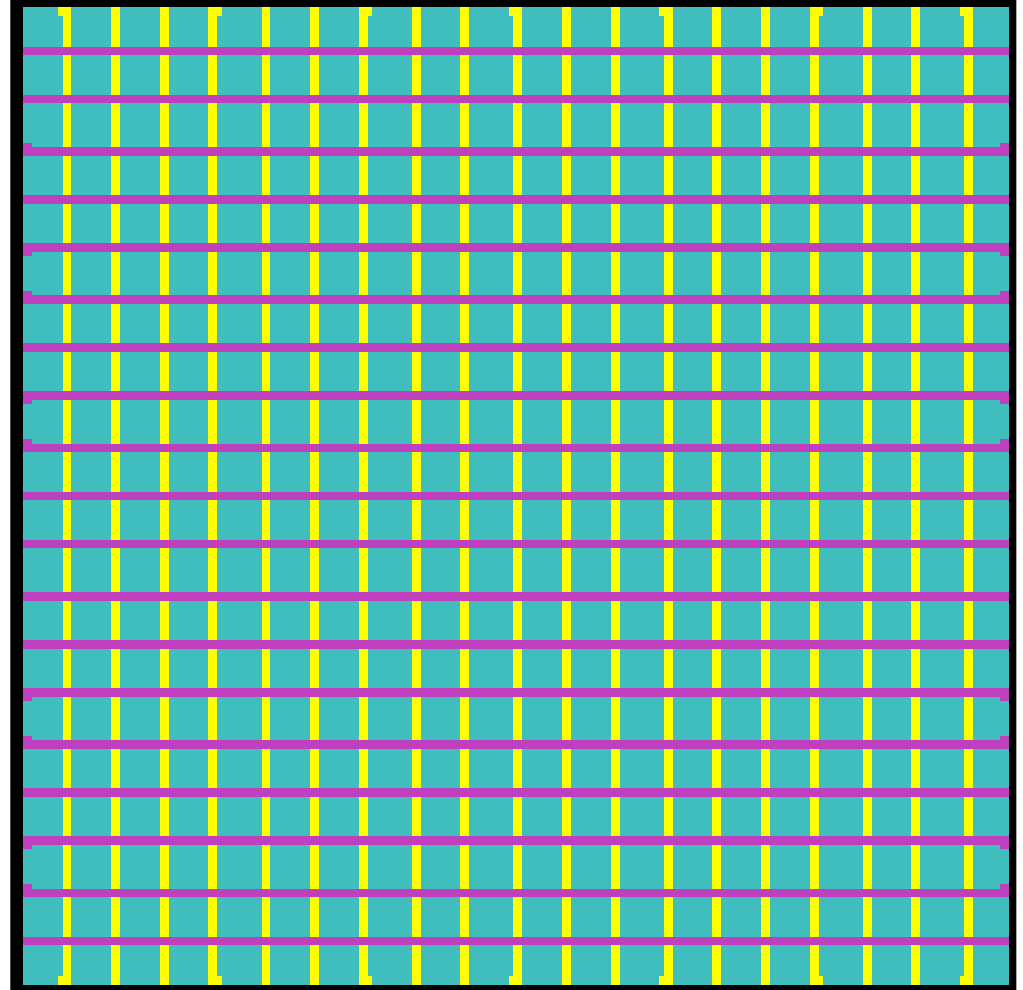
$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}$$

Complex velocity

$$\frac{df(z)}{dz} = w(z) = u - iv$$

Uniform flow

$$f(z) = z$$



Equipotentielle

$$\phi(x, y) = x = \text{cte}$$

Ligne de courant

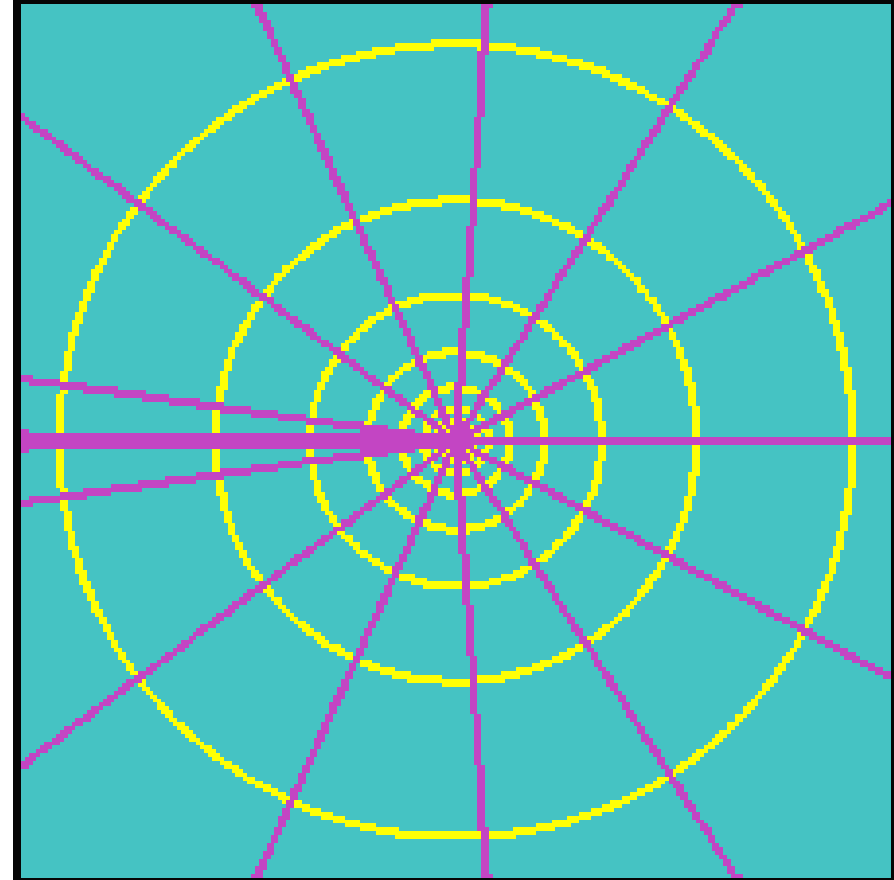
$$\psi(x, y) = y = \text{cte}$$

SOURCE $D>0$ or SINK $D<0$

$$f(z) = \frac{D}{2\pi} \text{Ln}(z)$$

$$\phi(x, y) = \frac{D}{2\pi} \text{Ln} r = \text{cte}$$

$$\psi(x, y) = \frac{D\theta}{2\pi} = \text{cte}$$

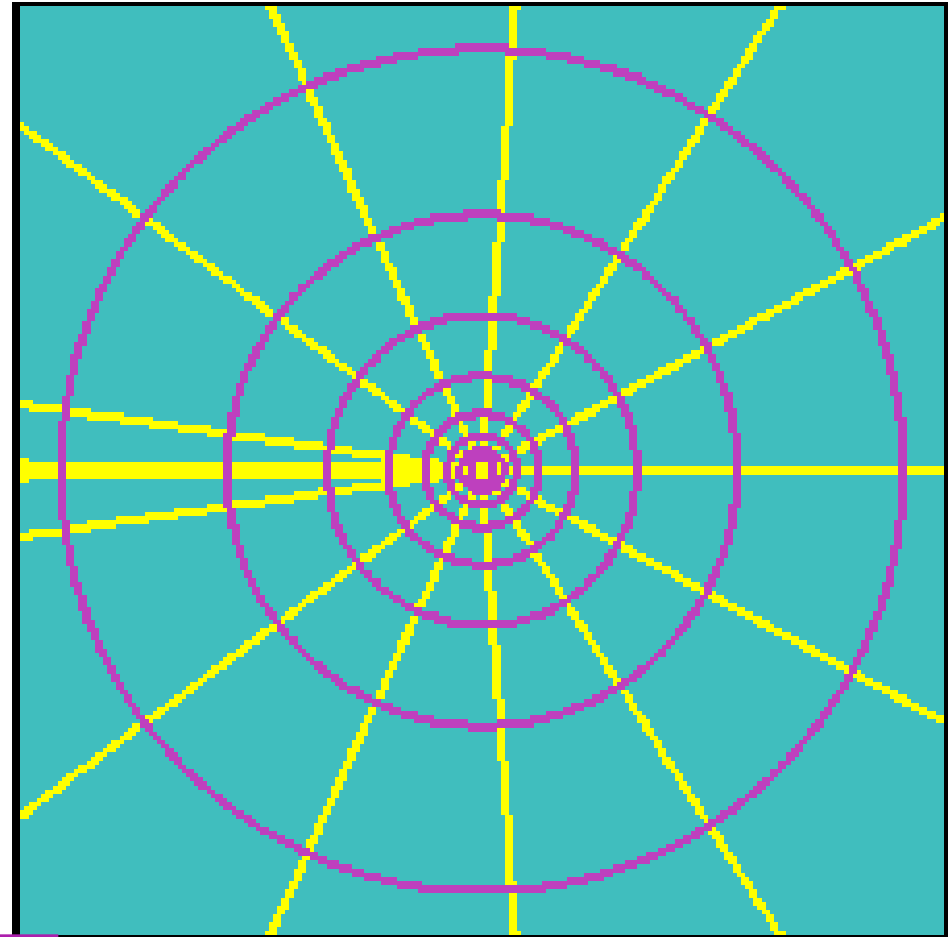


VORTEX

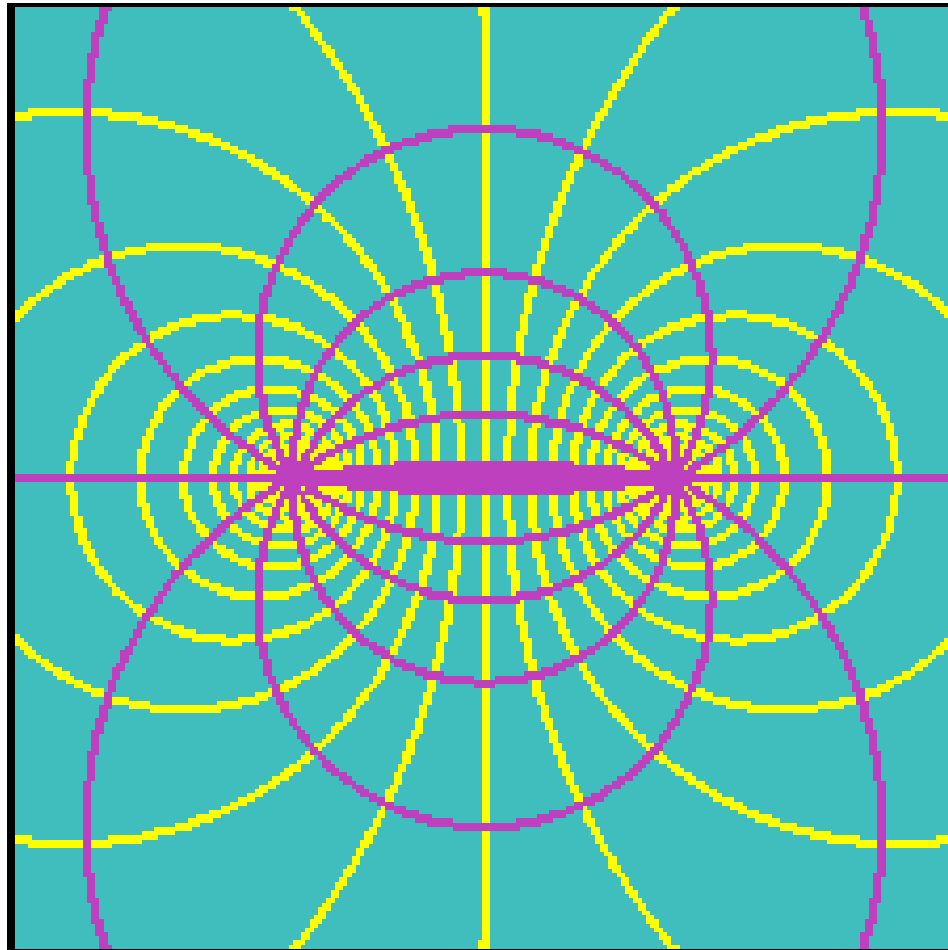
$$f(z) = -\frac{i\Gamma}{2\pi} \text{Ln}(z)$$

$$\phi(x, y) = \frac{\Gamma\theta}{2\pi} = \text{cte}$$

$$\psi(x, y) = \frac{-\Gamma}{2\pi} \text{Ln}r = \text{cte}$$



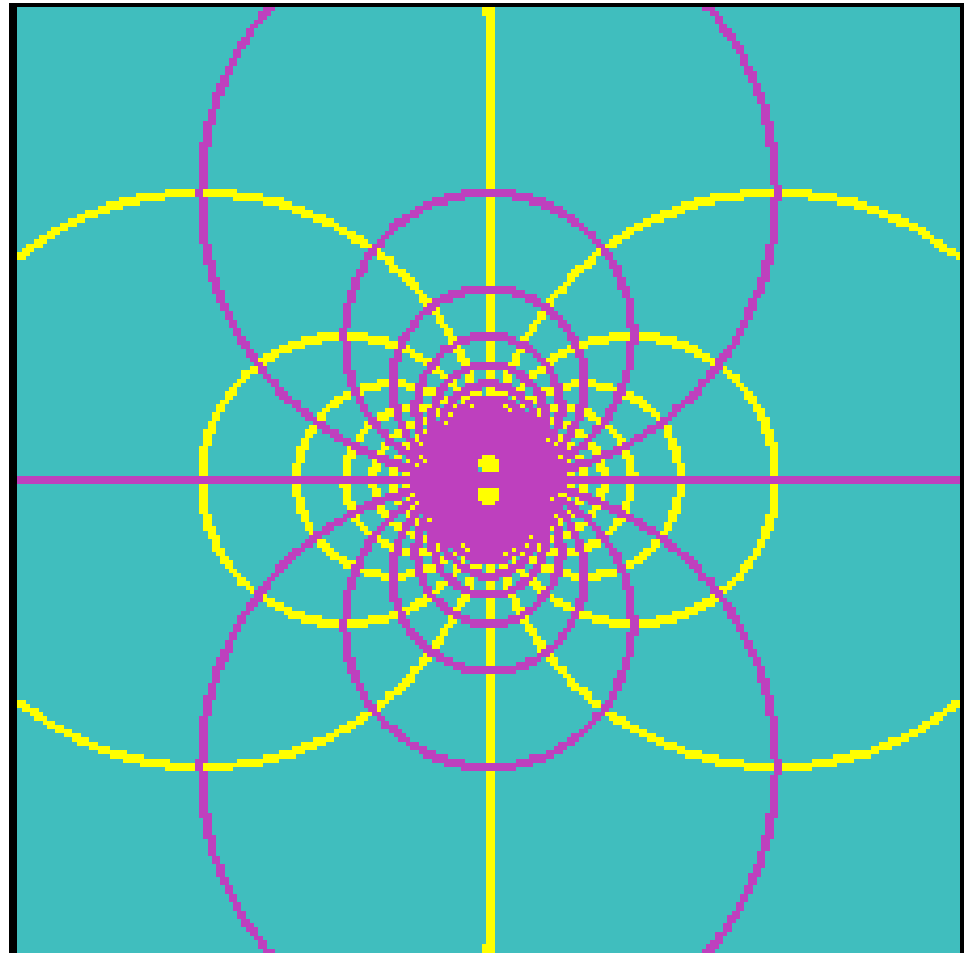
SUPERPOSITION, SOURCE and SINK



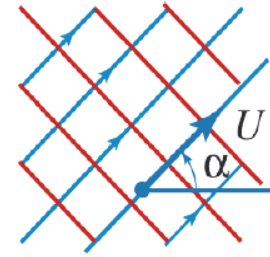
$$f(z) = \frac{D}{2\pi} \operatorname{Ln}(z - a) - \frac{D}{2\pi} \operatorname{Ln}(z + a)$$

DOUBLET

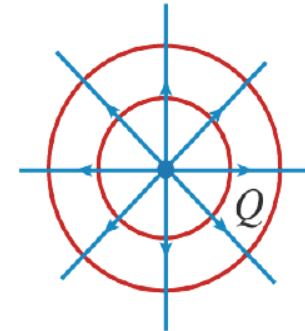
$$f(z) = \frac{k}{z}$$



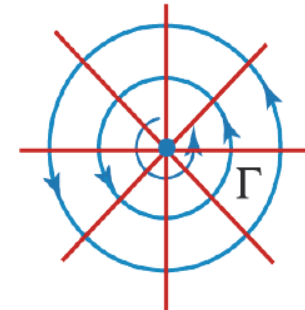
$$f(z) = Ue^{-i\alpha}z$$



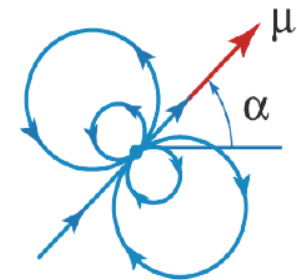
$$f(z) = \frac{Q}{2\pi} \log(z - z_0)$$



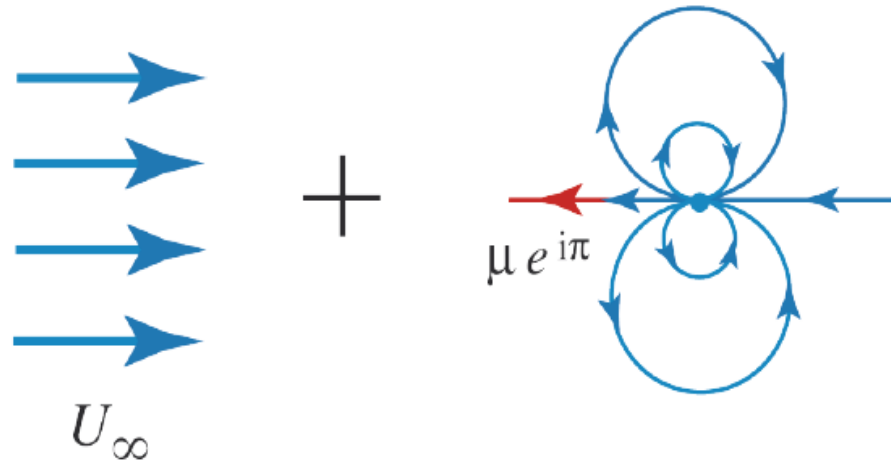
$$f(z) = -\frac{i\Gamma}{2\pi} \log(z - z_0)$$



$$f(z) = -\frac{\mu e^{i\alpha}}{z - z_0}$$



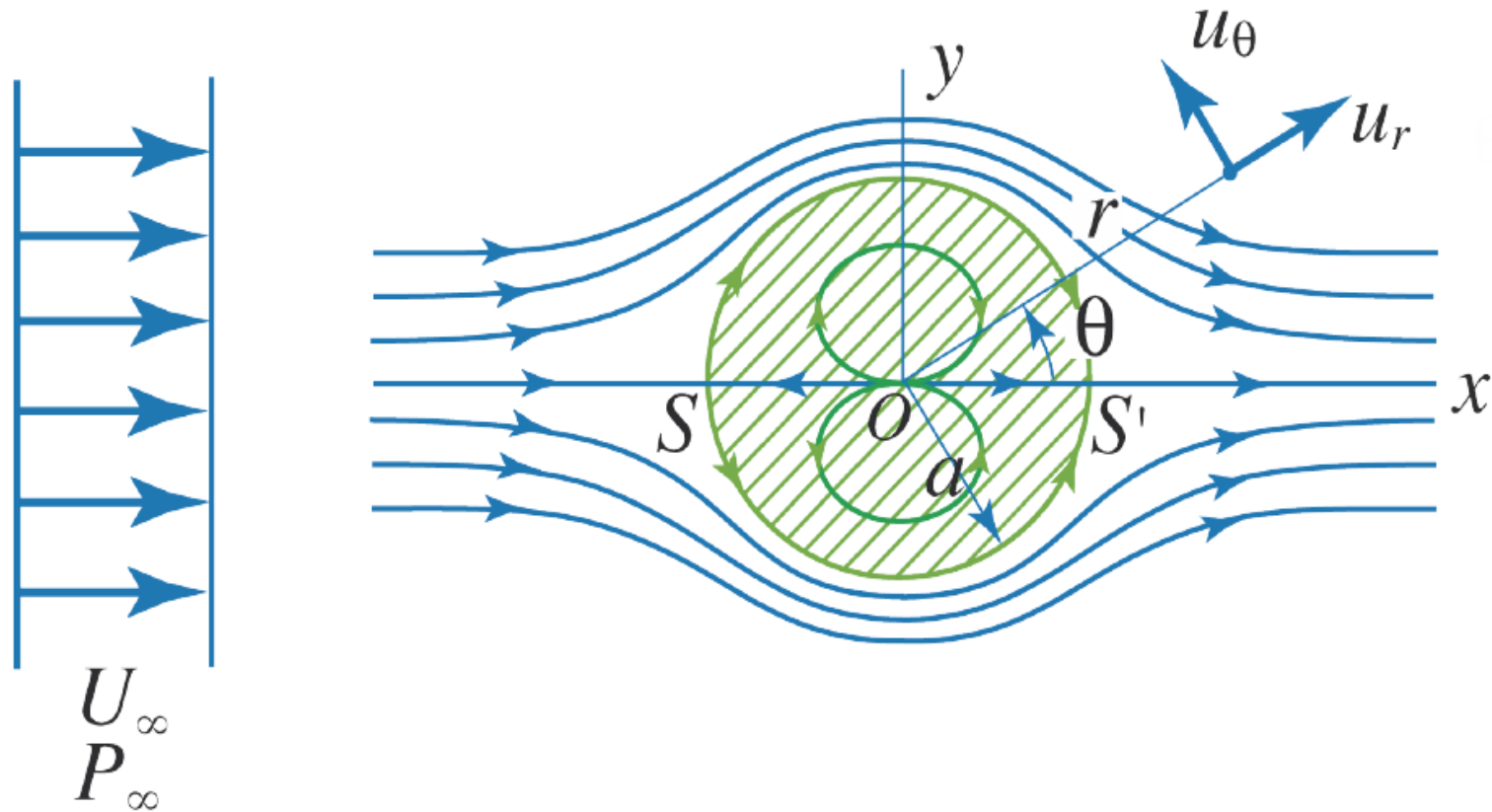
Acyclic flow around cylinder



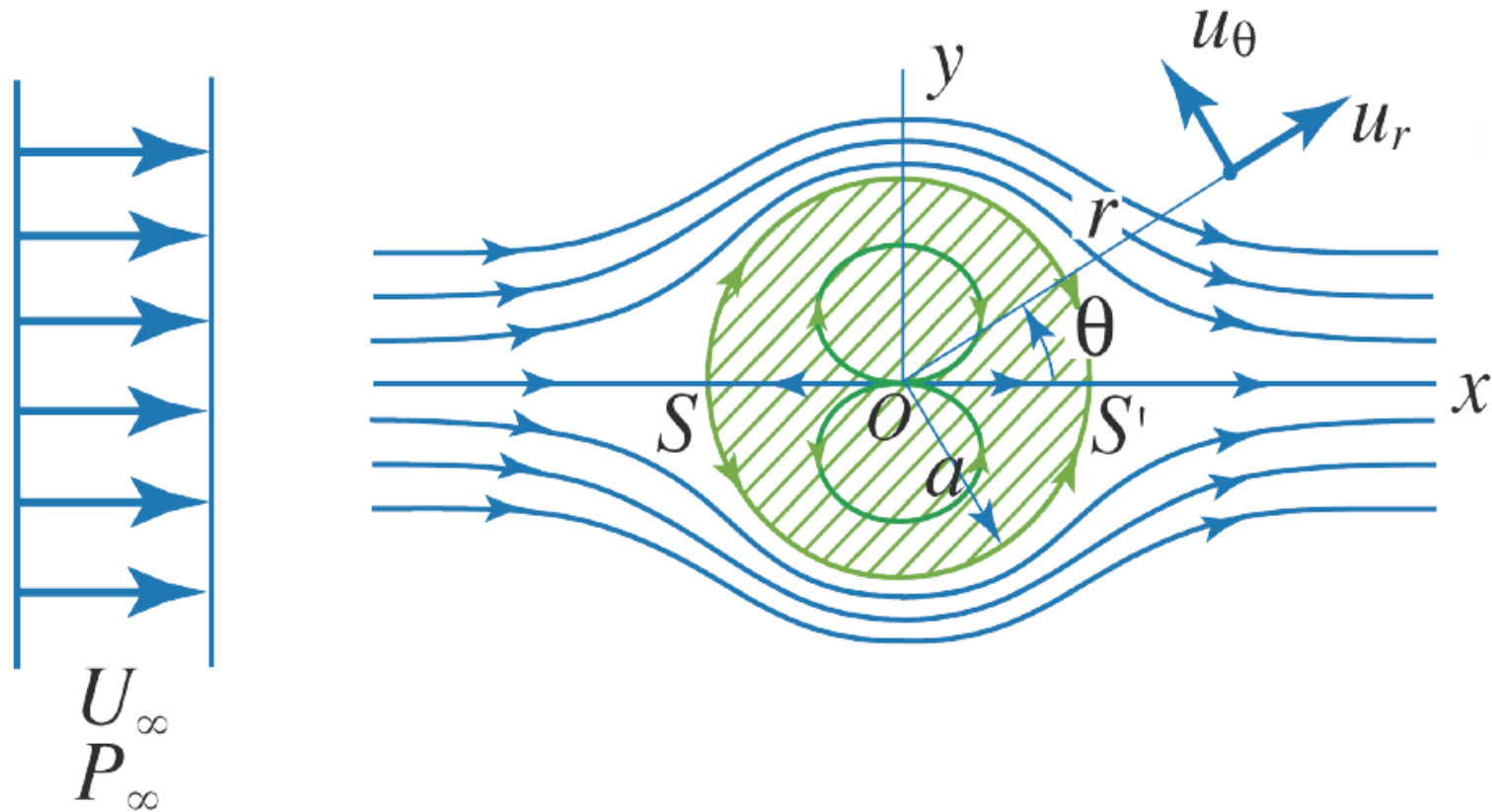
$$f(z) = U_\infty z + \frac{\mu}{z}$$

$$f(z) = U_\infty \left(z + \frac{a^2}{z} \right)$$

Acyclic flow around cylinder



Acyclic flow around cylinder



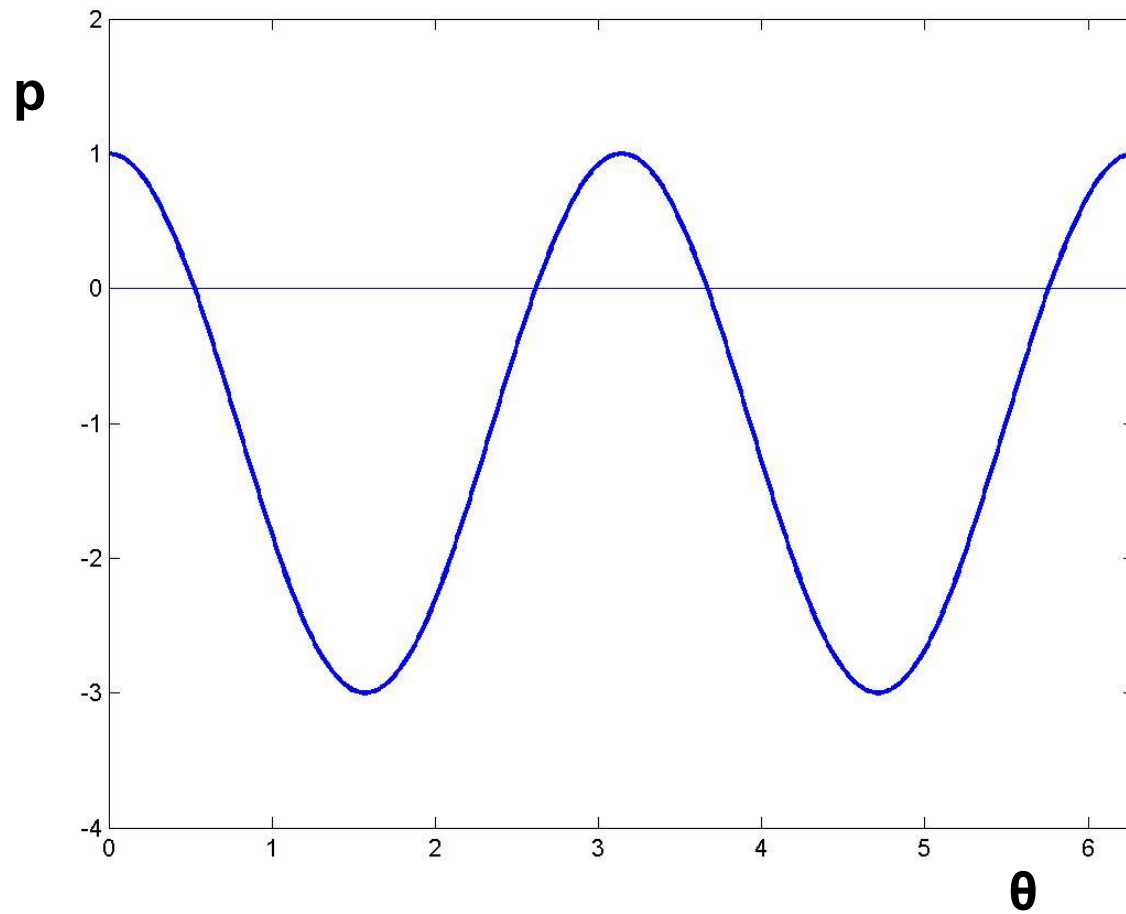
$$U_r(r, \theta) = U_\infty \left(1 - \frac{a^2}{r^2} \right) \cos\theta \quad , \quad U_\theta(r, \theta) = -U_\infty \left(1 + \frac{a^2}{r^2} \right) \sin\theta$$

Bernoulli

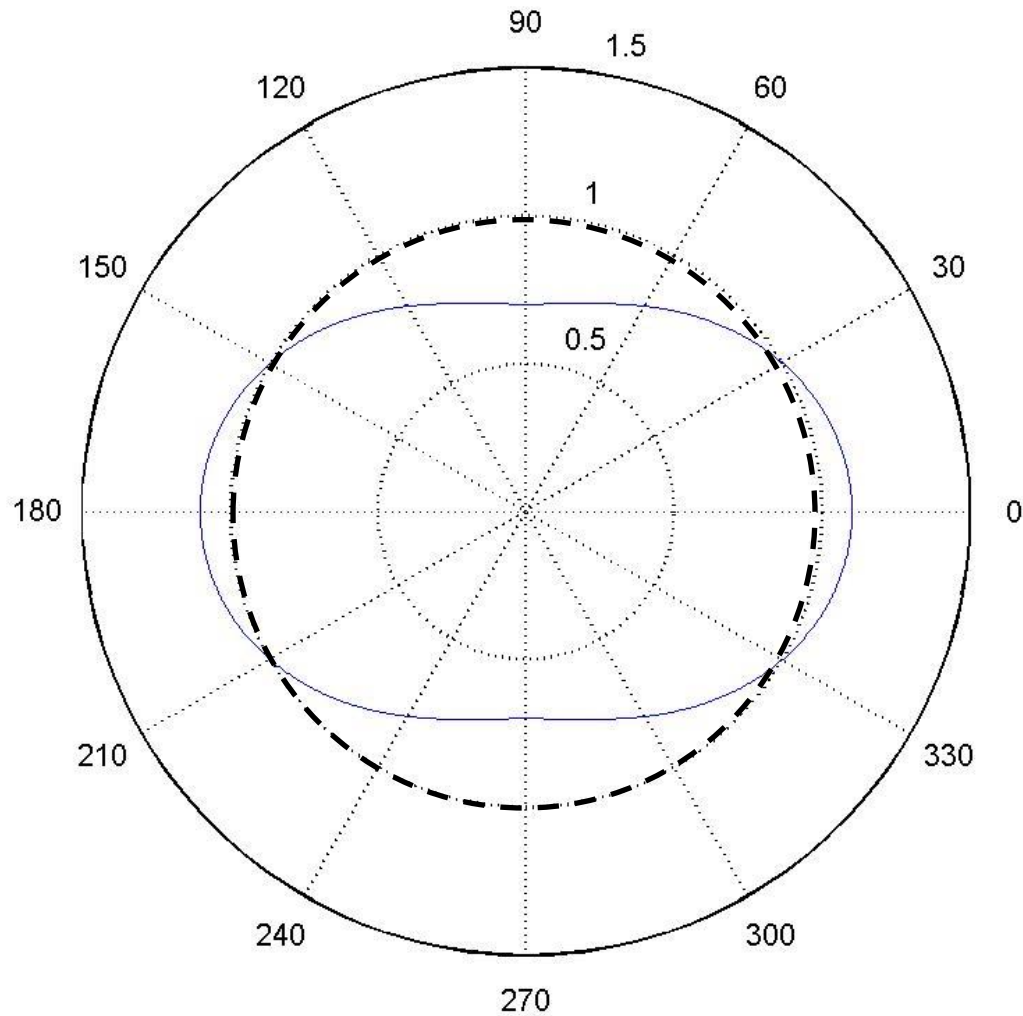
$$U_r(r, \theta) = U_\infty \left(1 - \frac{a^2}{r^2} \right) \cos\theta \quad , \quad U_\theta(r, \theta) = - U_\infty \left(1 + \frac{a^2}{r^2} \right) \sin\theta$$

$$p(a, \theta) = \frac{1}{2} \rho U_\infty^2 (1 - 4 \sin^2 \theta)$$

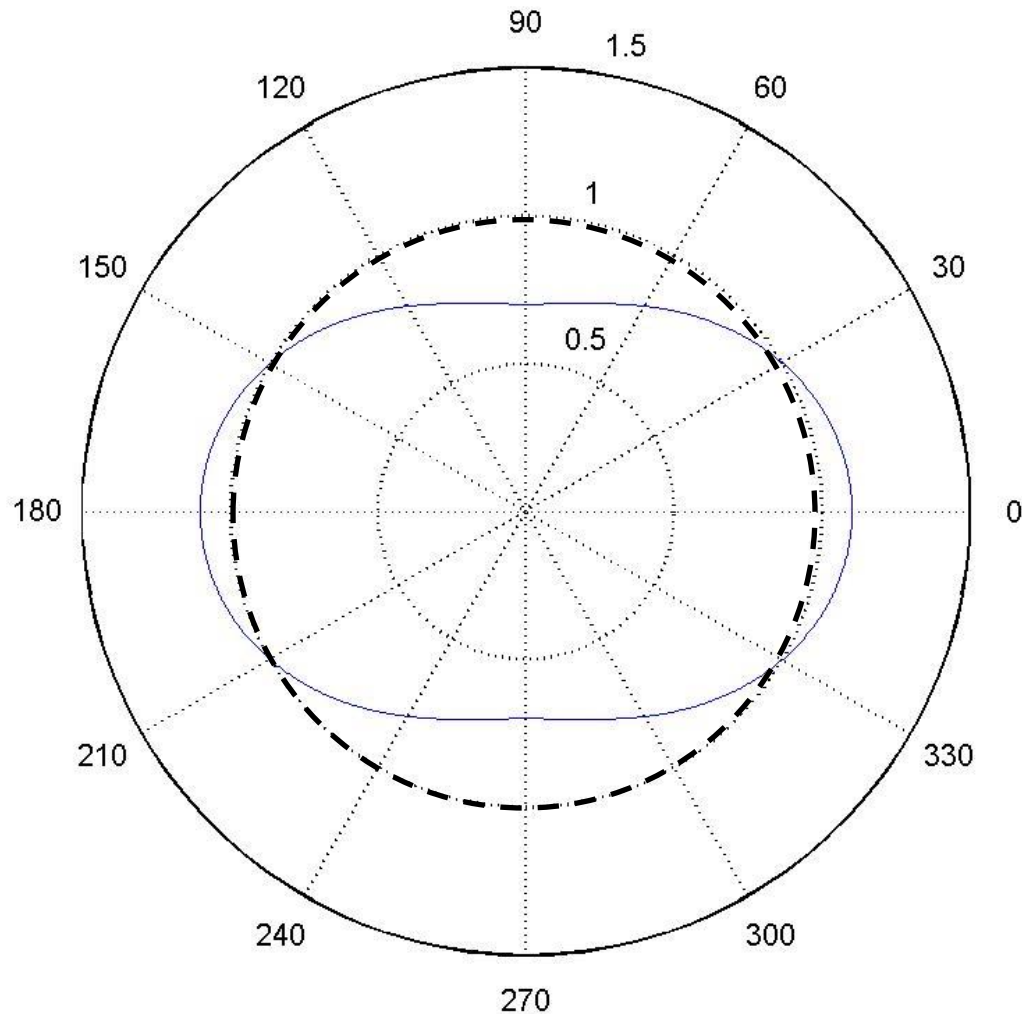
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$$p(a, \theta) = \frac{1}{2} \rho U_{\infty}^2 (1 - 4 \sin^2 \theta)$$



Traînée : $F_D=0$

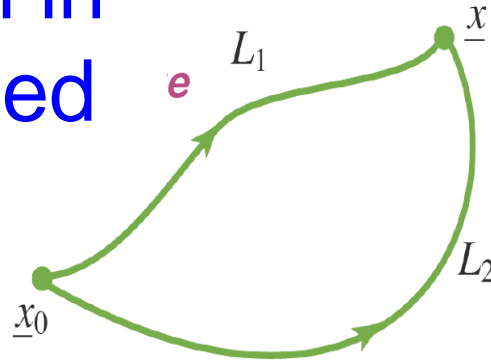
Portance: $F_L=0$

Achtung! Connectivity?

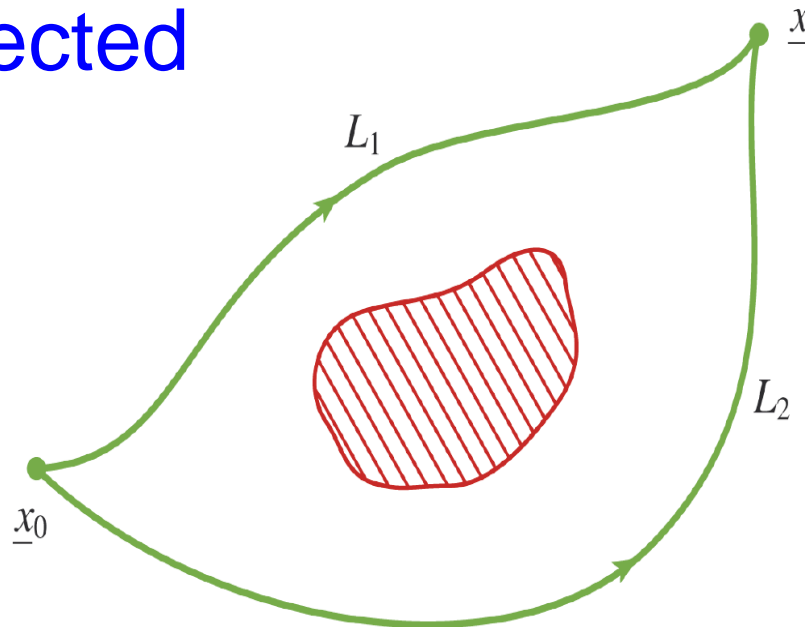
The solution to Laplace equation is unique only in simply connected domains



Unique solution in
simply connected
domains

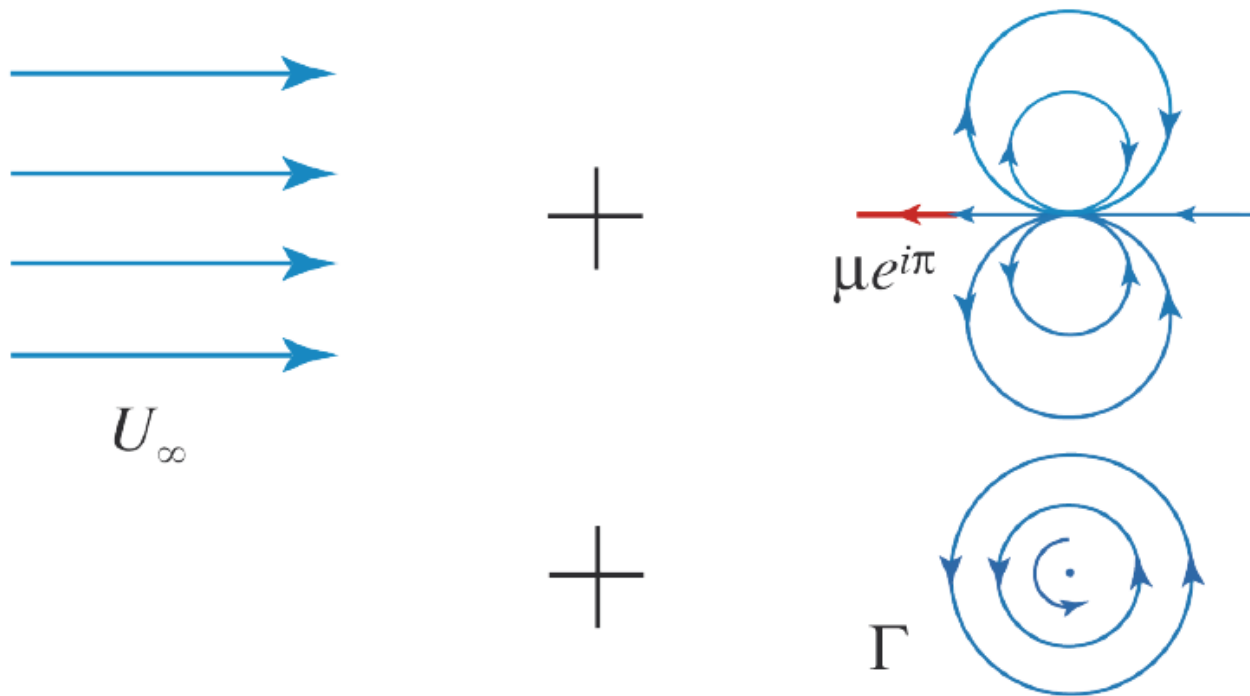


Multiple solutions in
multiply connected
domains



$$\varphi(\underline{x}, t) = \varphi(\underline{x}_0, t) + \int_{L_j} \underline{U}(\underline{x}, t) \cdot \underline{d\ell}$$

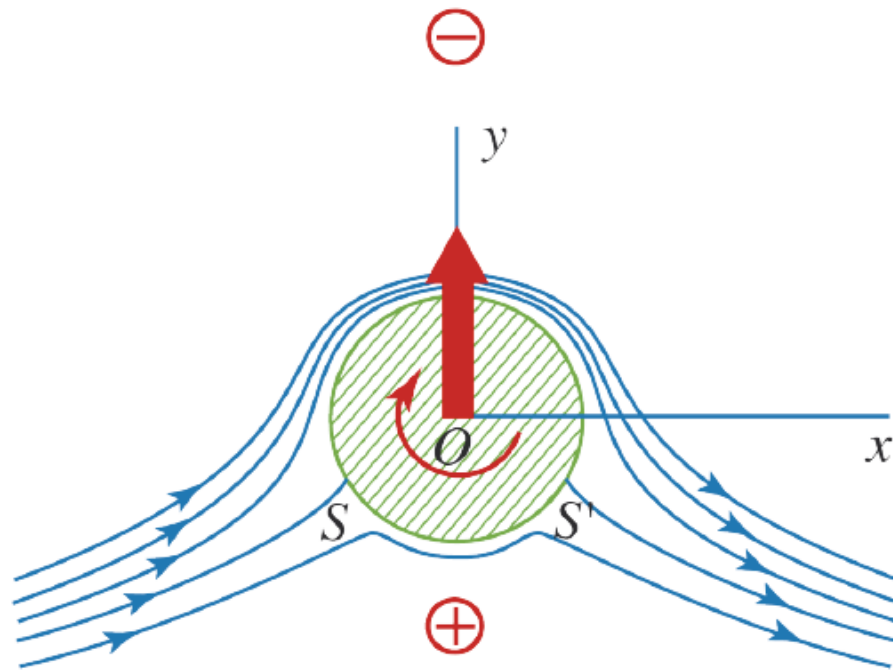
Cyclic flow around cylinder



$$f(z) = U_\infty z + \frac{\mu}{z} - \frac{i\Gamma}{2\pi} \log\left(\frac{z}{a}\right)$$

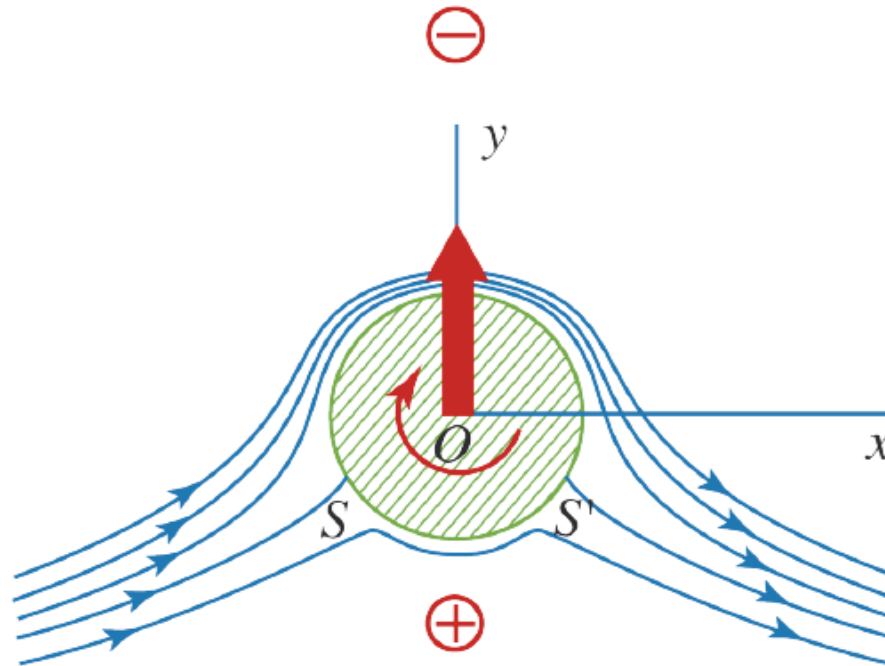
$$f(z) = U_\infty \left(z + \frac{a^2}{z} \right) - \frac{i\Gamma}{2\pi} \log\left(\frac{z}{a}\right)$$

Cyclic flow around cylinder



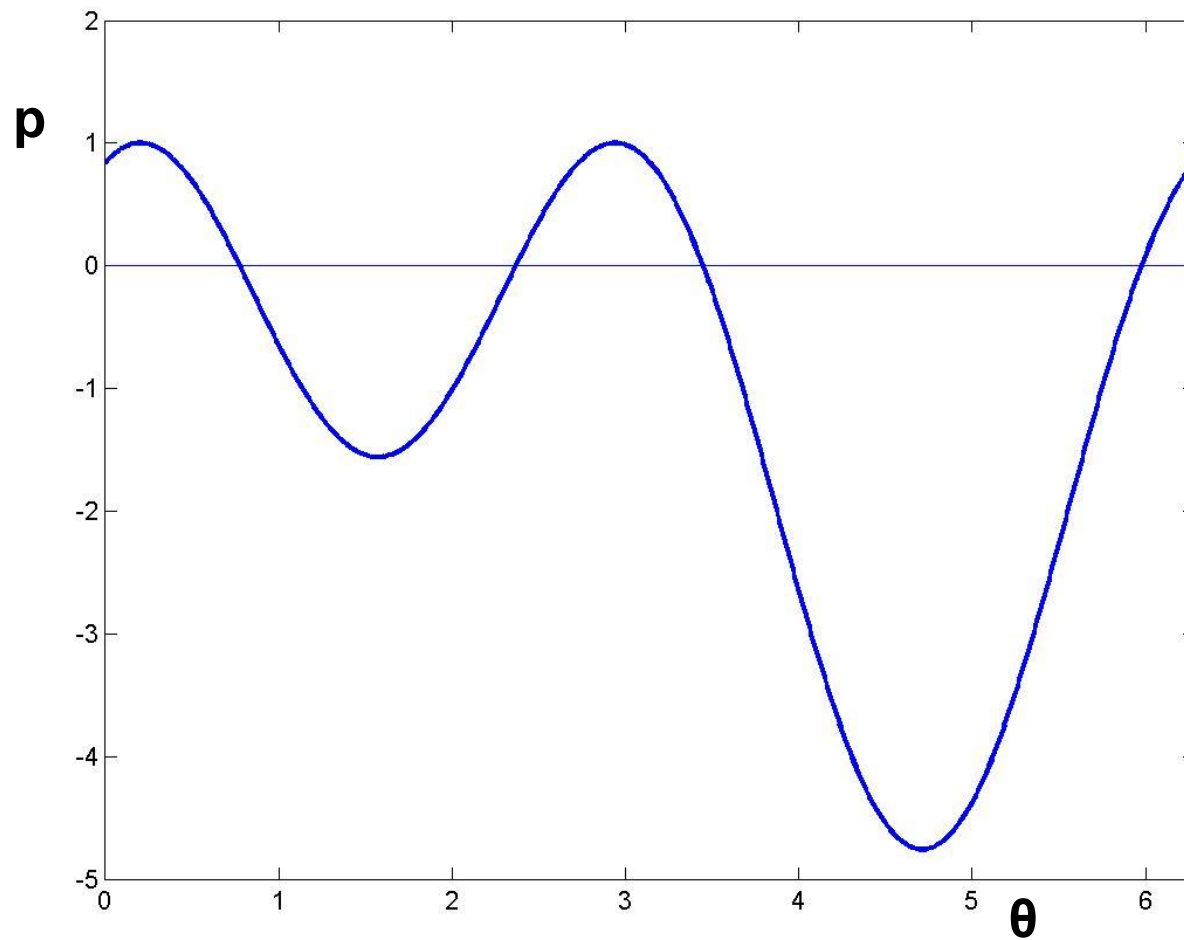
$$-1 < \frac{\Gamma}{4\pi U_{\infty} a} < 0$$

Magnus effect and lift

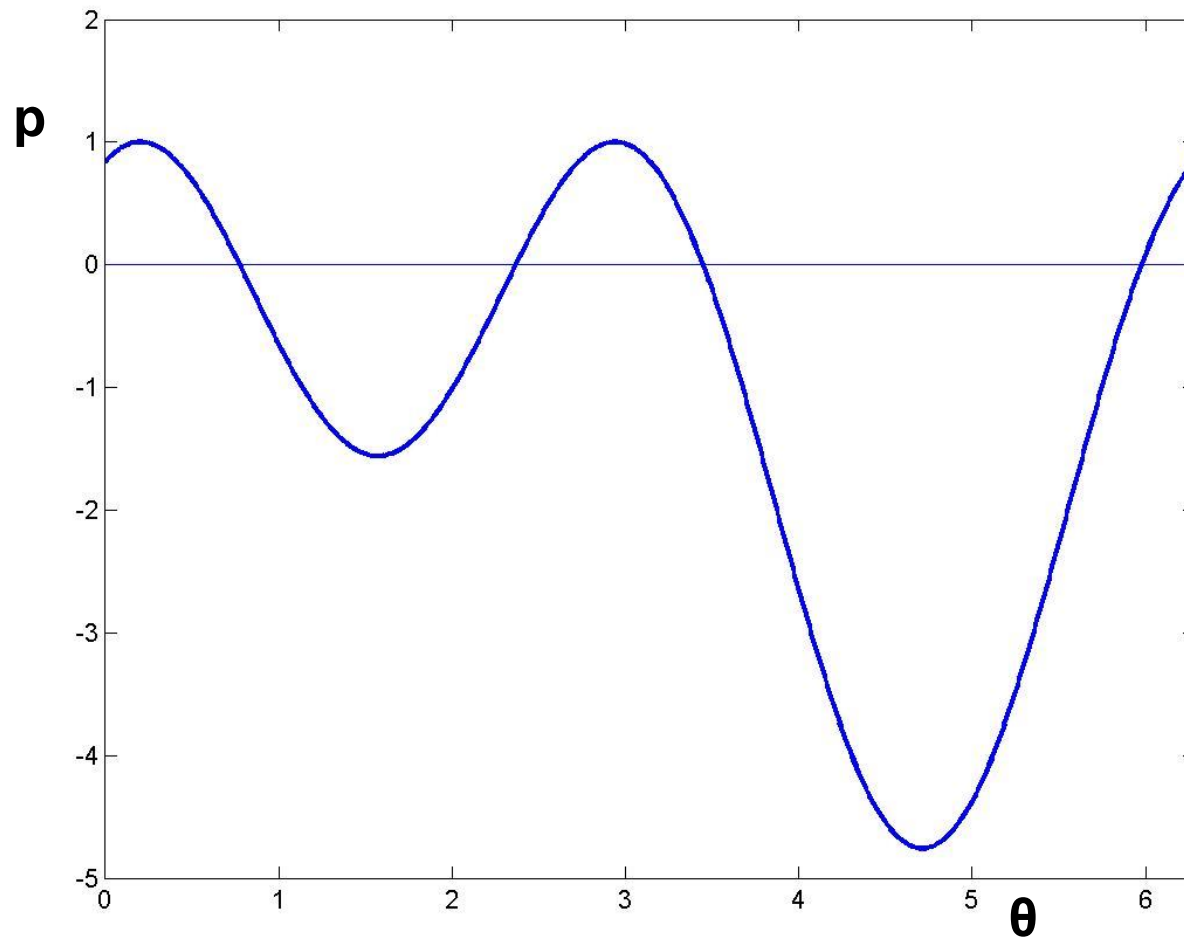


$$U_r(r, \theta) = U_\infty \left(1 - \frac{a^2}{r^2} \right) \cos \theta, \quad U_\theta(r, \theta) = -U_\infty \left(1 + \frac{a^2}{r^2} \right) \sin \theta + \frac{\Gamma}{2\pi r}$$

$$p(a, \theta) = \frac{1}{2} \rho U_{\infty}^2 \left(1 - 4 \left(\sin \theta - \frac{\Gamma}{4\pi U_{\infty} a} \right)^2 \right)$$



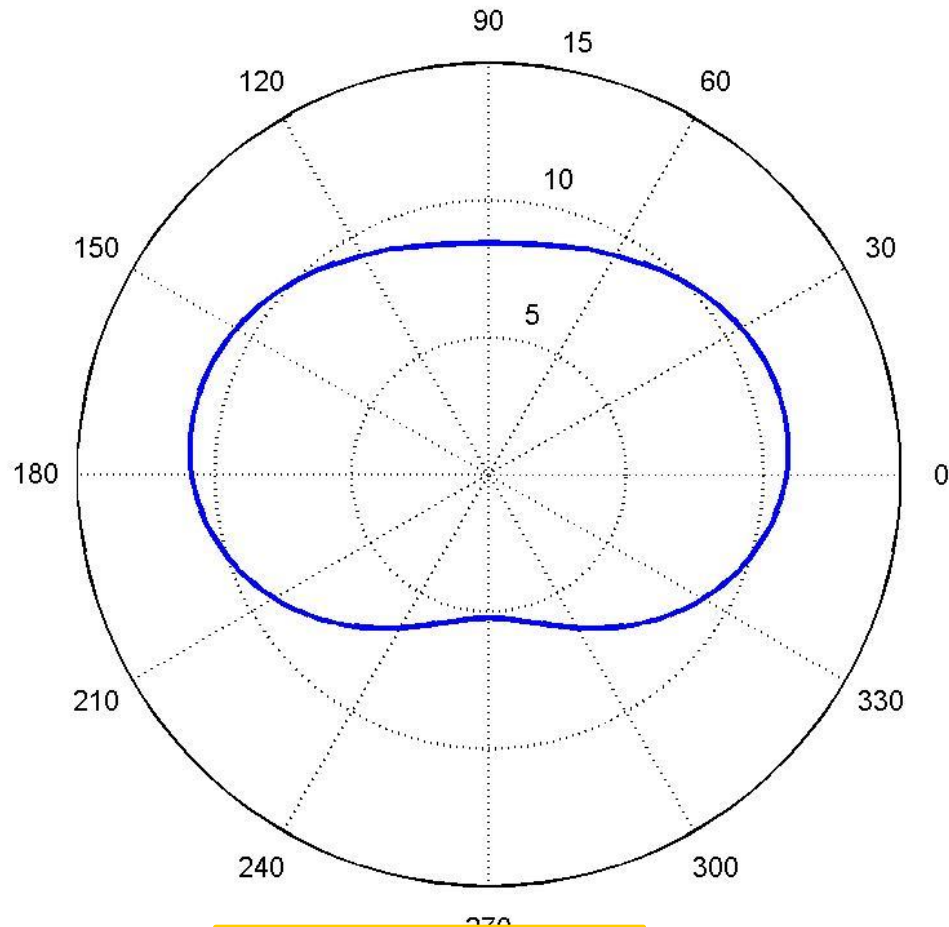
$$p(a, \theta) = \frac{1}{2} \rho U_{\infty}^2 \left(1 - 4 \left(\sin \theta - \frac{\Gamma}{4\pi U_{\infty} a} \right)^2 \right)$$



Drag : $F_D = 0$!

Lift: $F_L = 0$

Pressure distribution



- Drag : $F_D = 0$
- Lift: $F_L \neq 0$

Cyclic flow around cylinder

Drag coefficient

$$C_x = 0$$

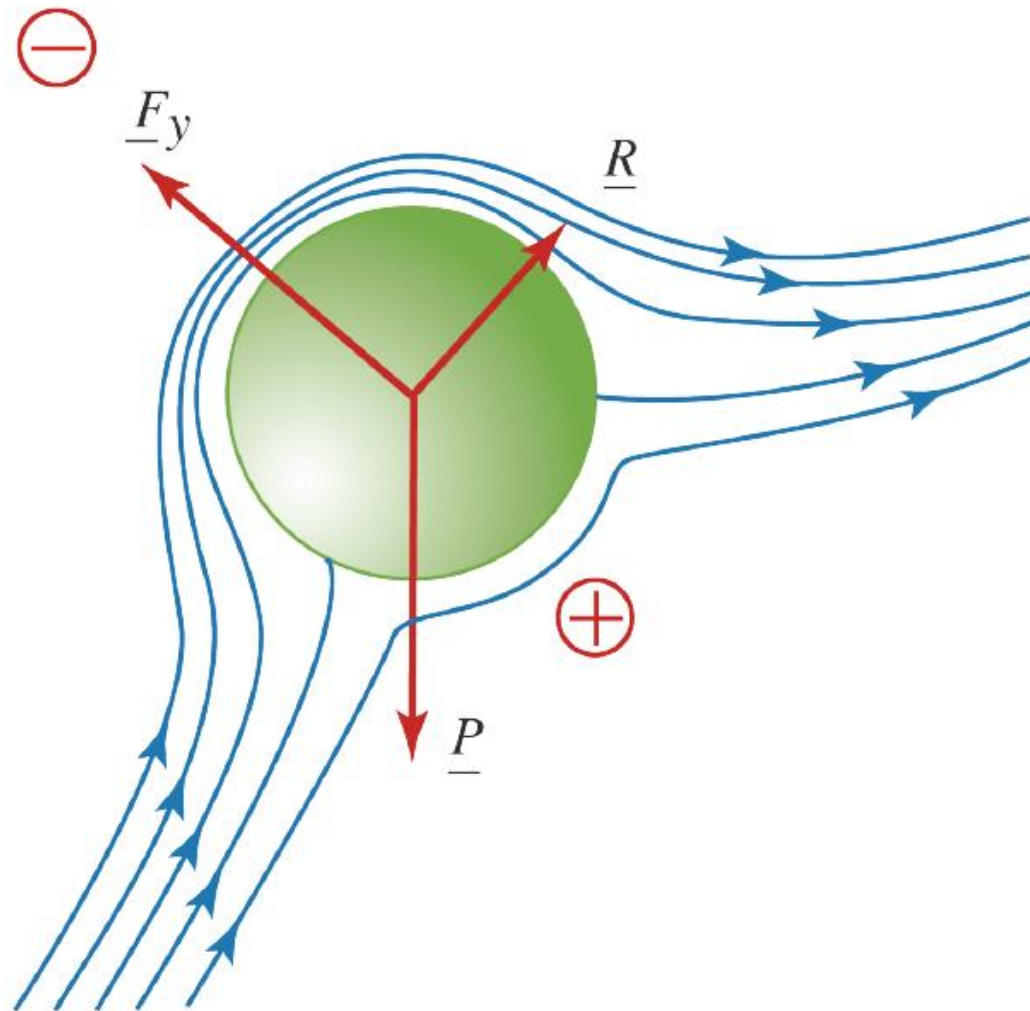
Lift

$$F_y = -\rho U_\infty \Gamma$$

Lift coefficient

$$C_y \neq 0$$

Floating ball



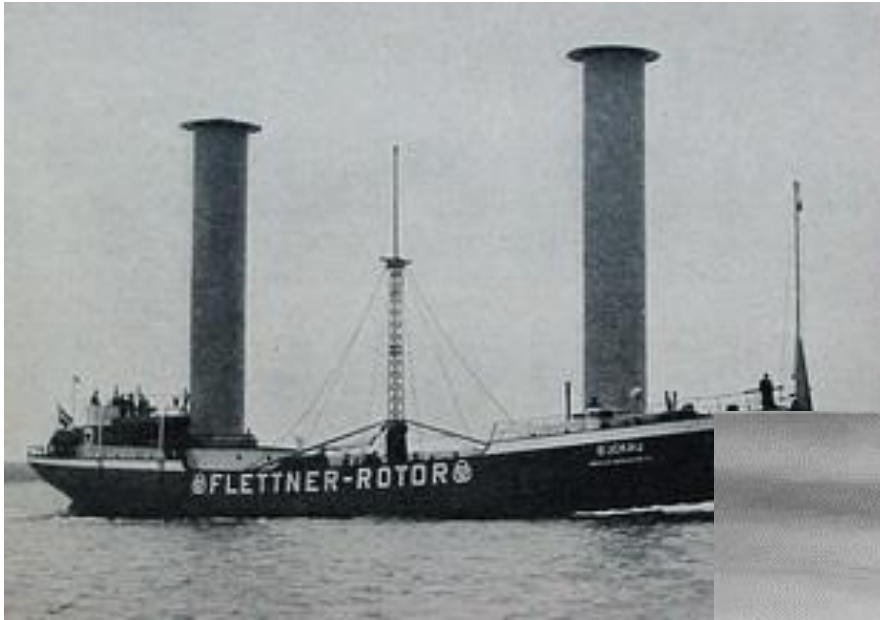
Tangente fluide



Garcia & Chomaz

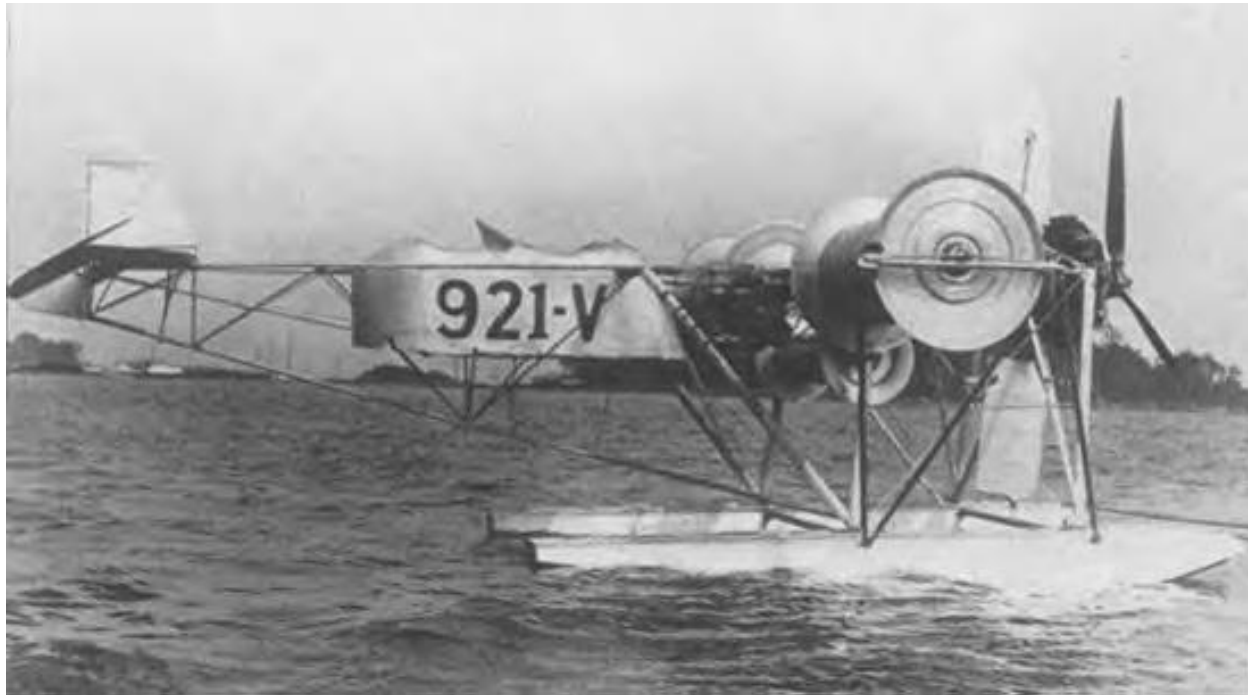


Le Baden Baden

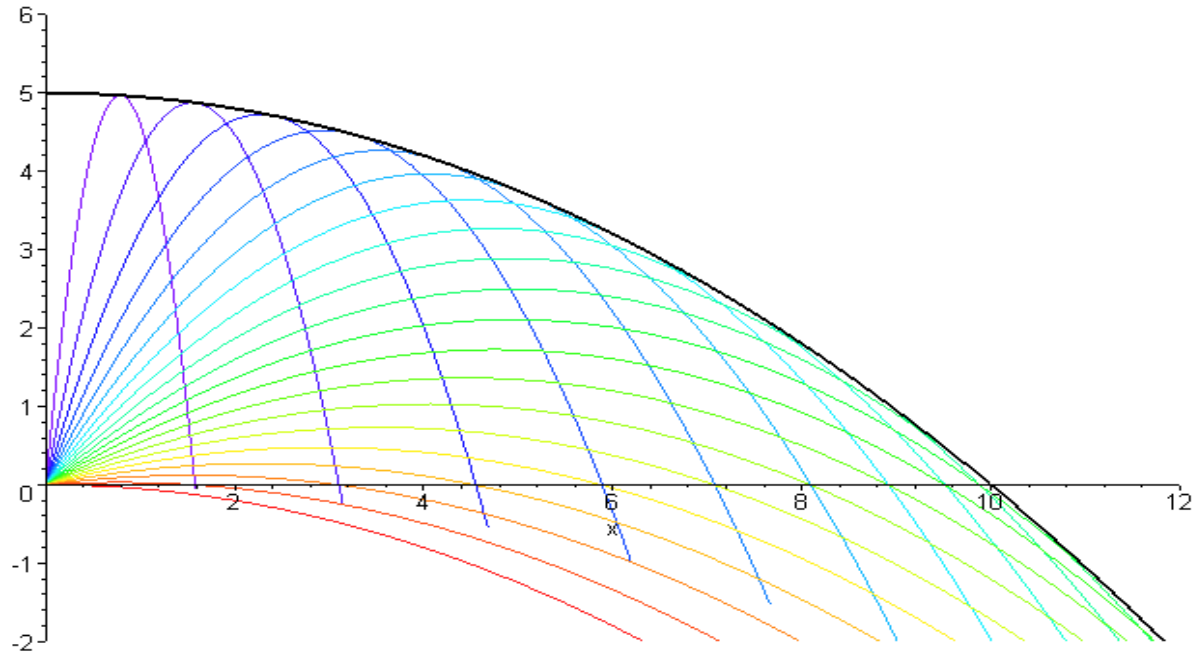


L'Alcyone

Rotating wings!



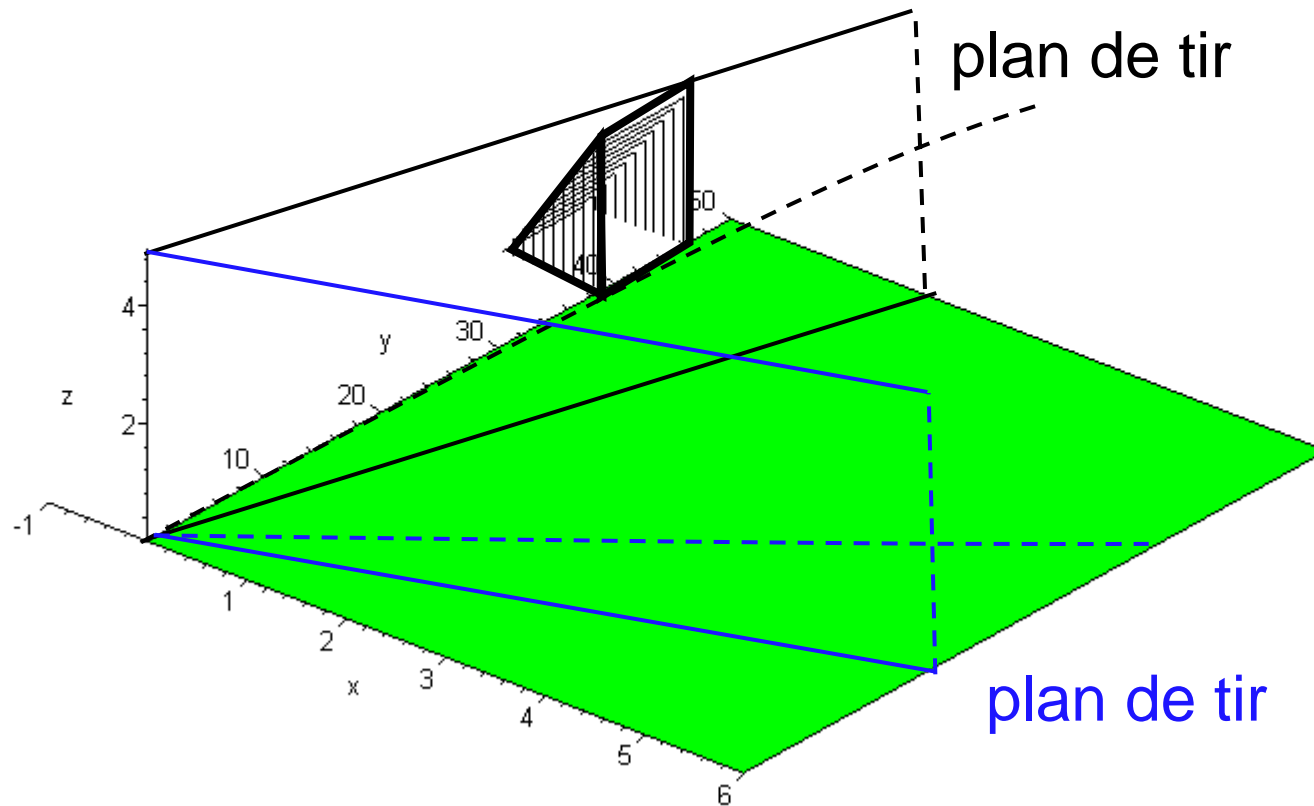
Parabolic trajectory



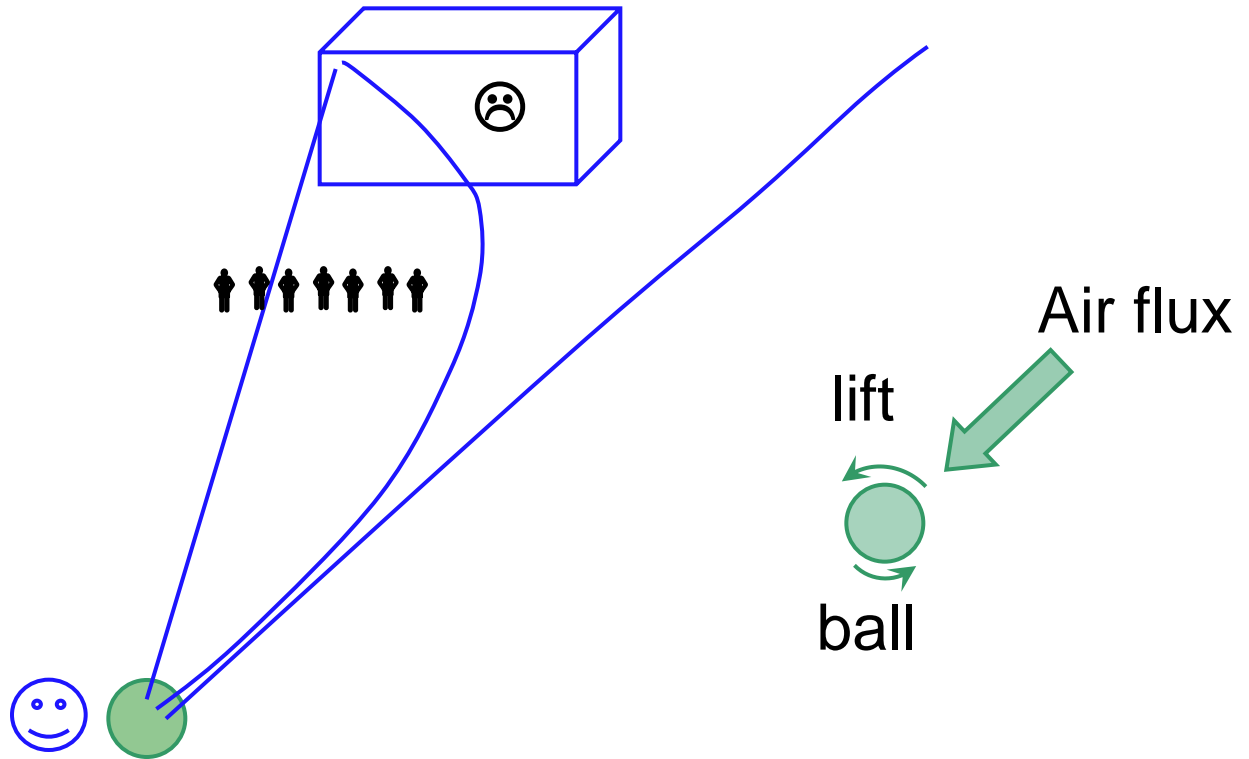
Trajectories with initial velocity 10m/s;
the shooting angle varies

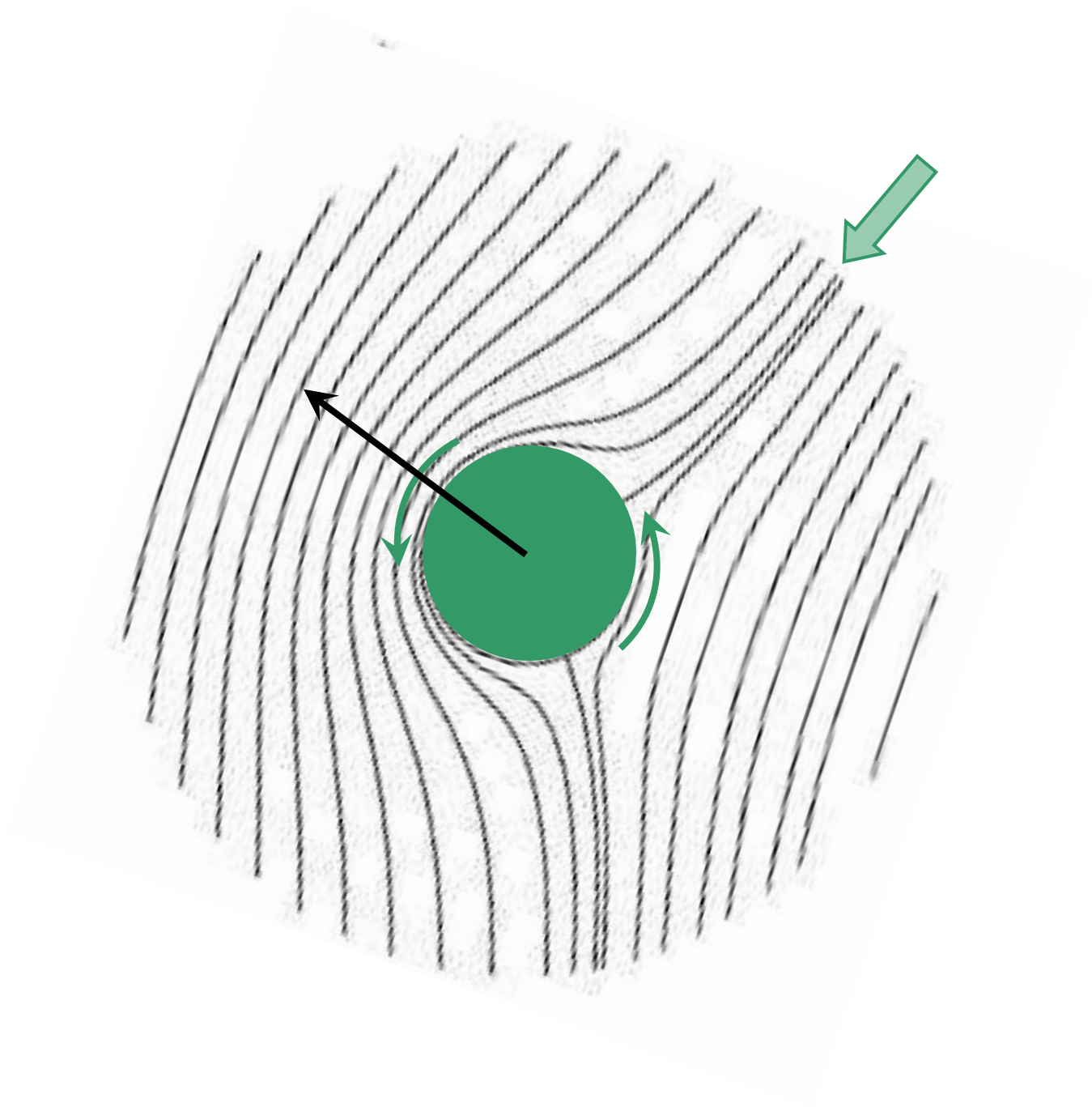
Reentrant cornerkick

Brushed kick with vertical rotation of the ball

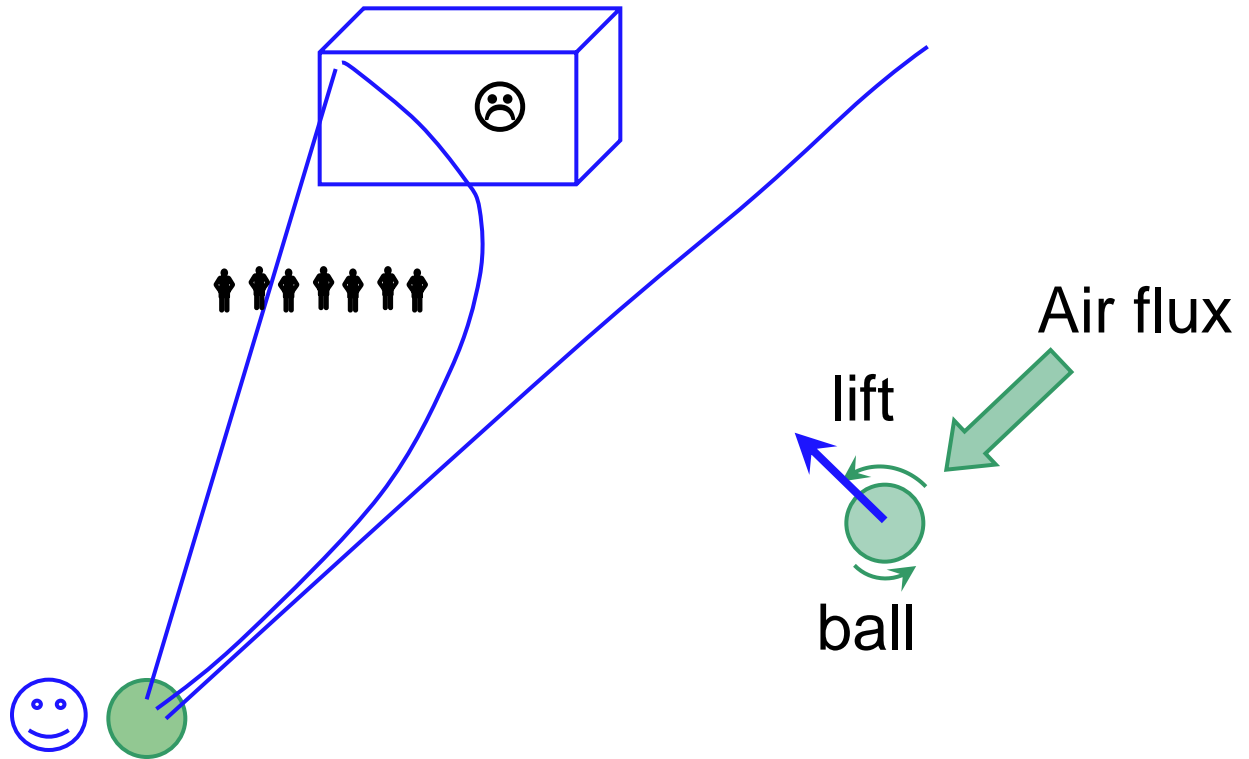


Free kick

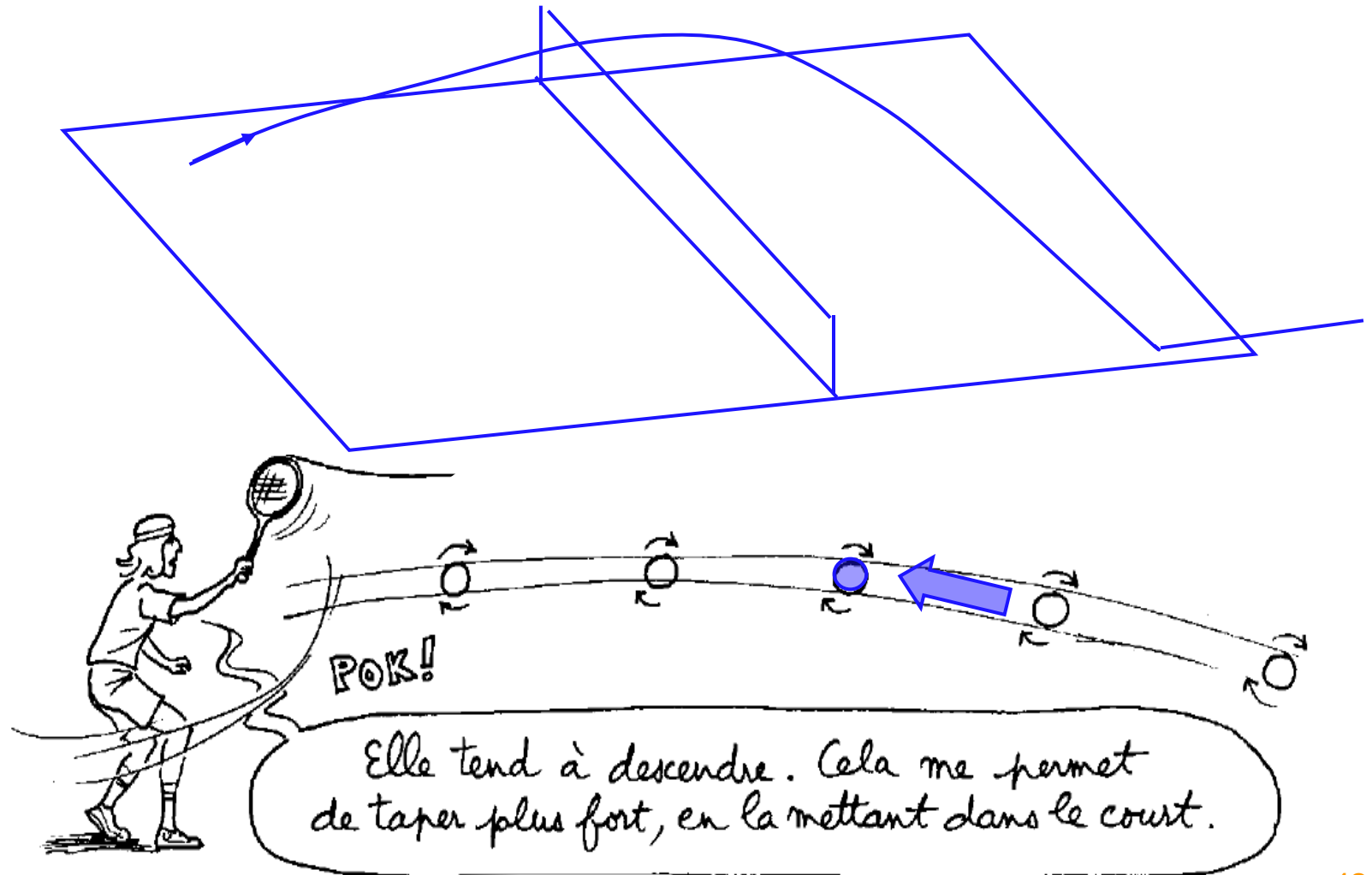


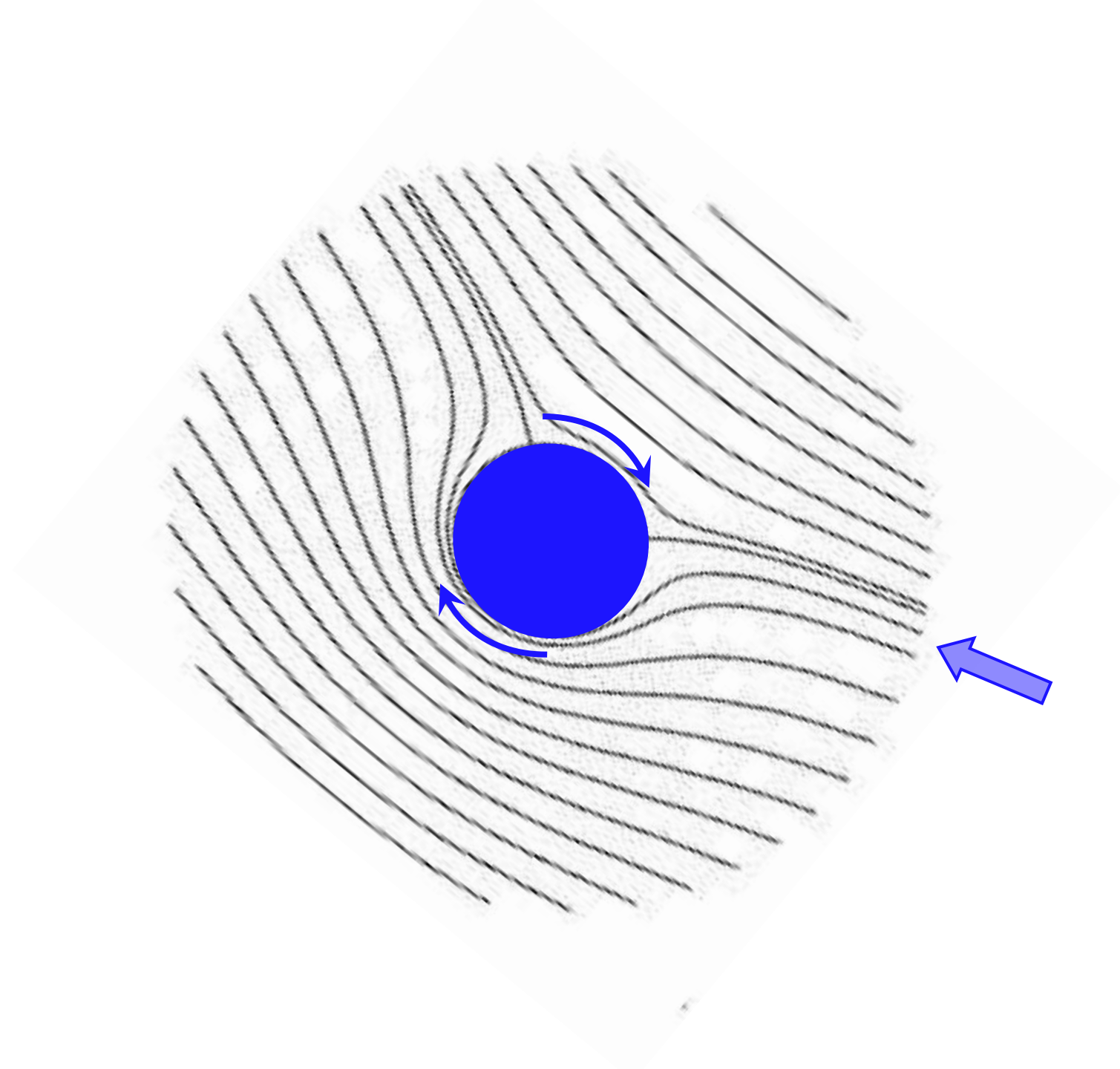


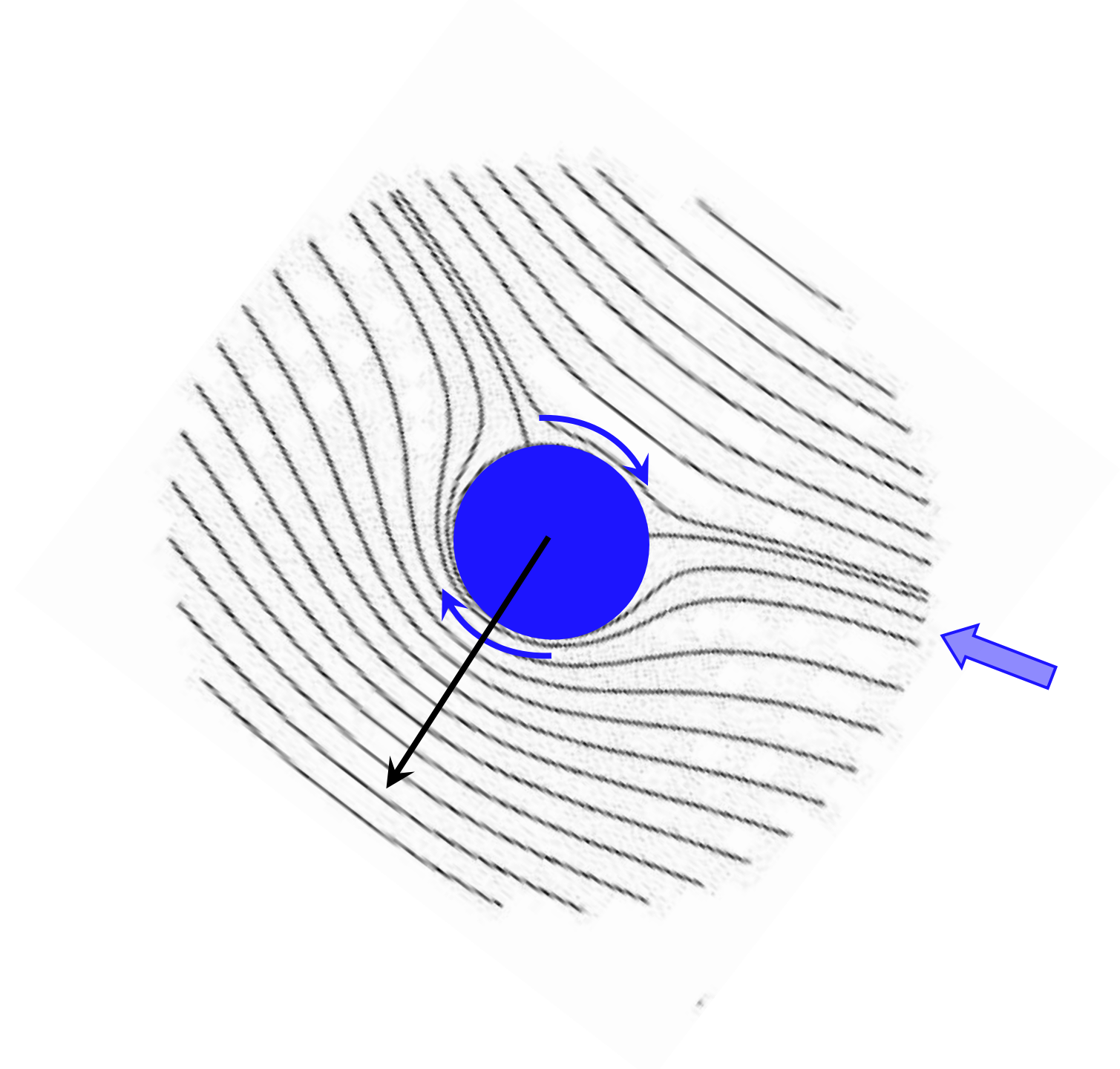
Free kick



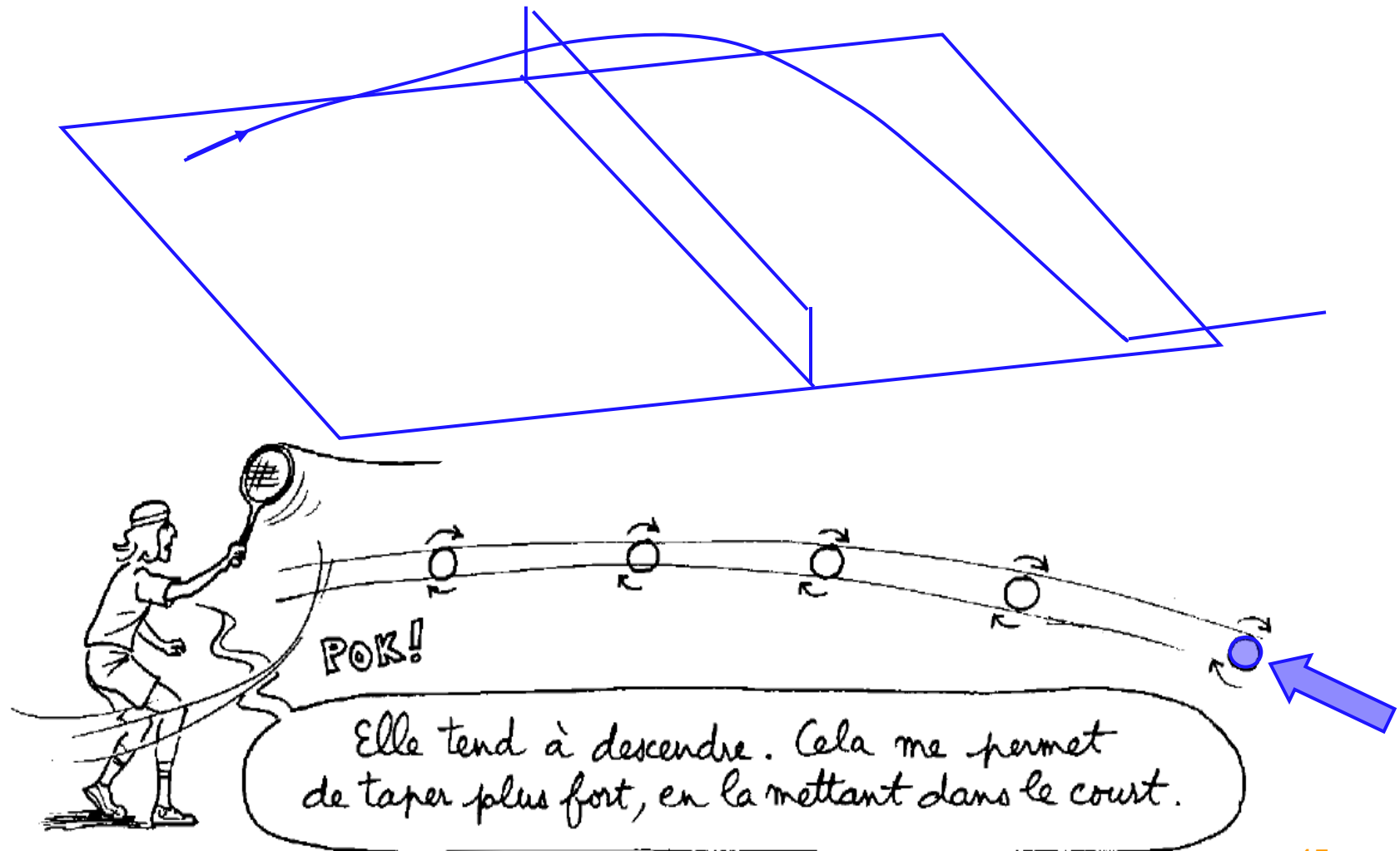
Lift



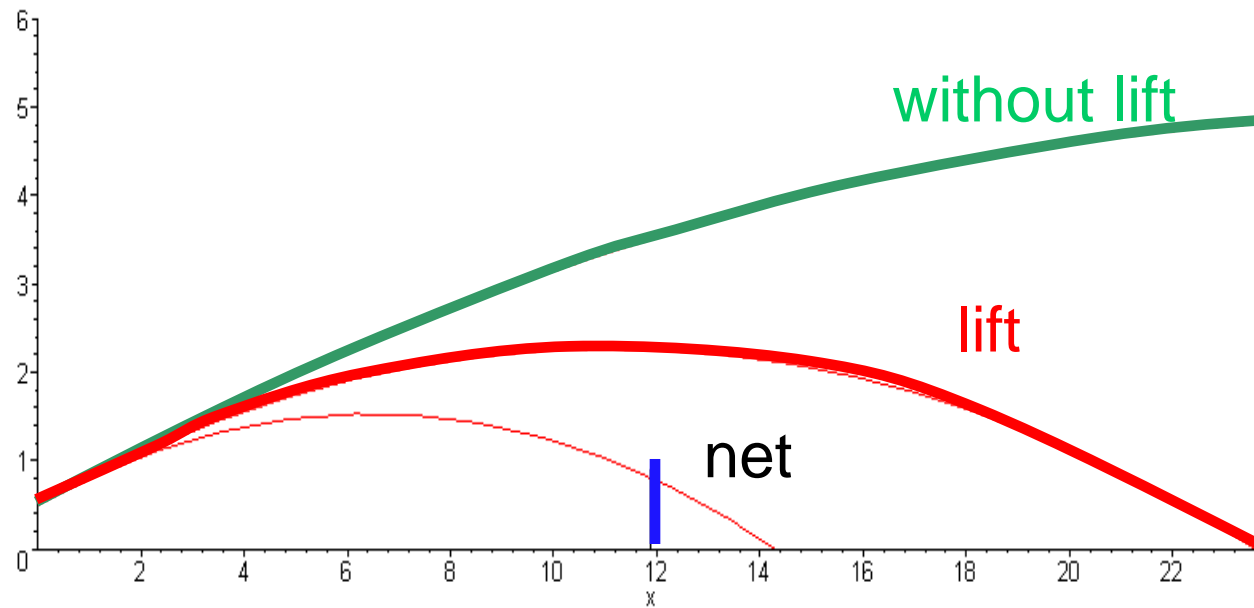




Lifted trajectory

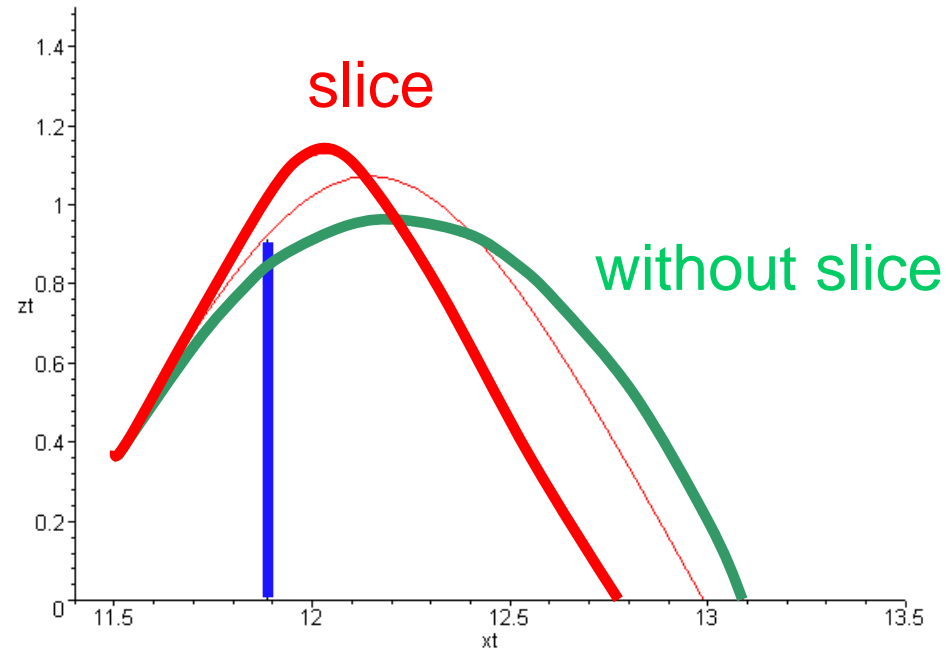


With lift



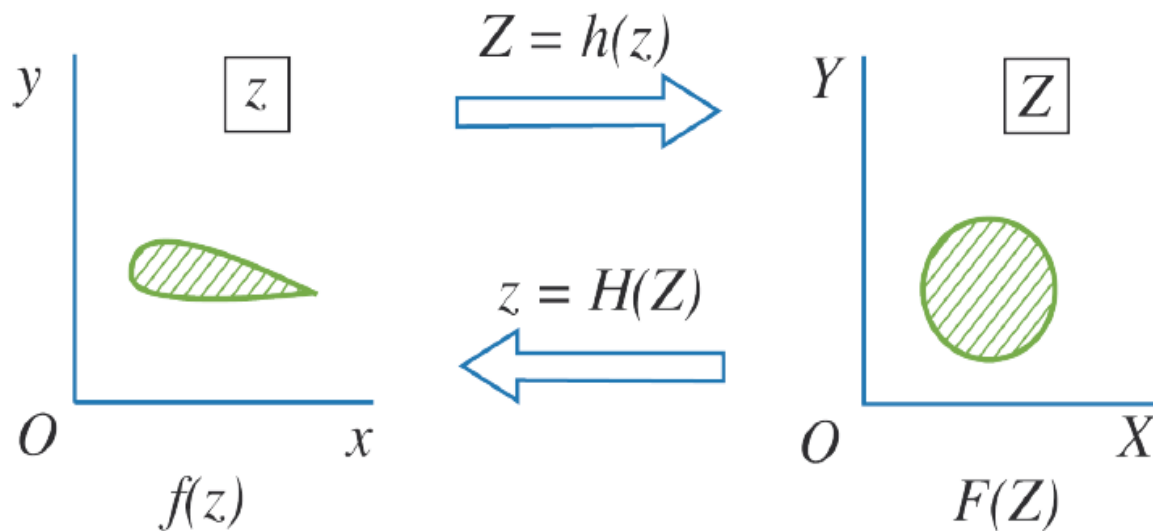
Lift at tennis tennis $V_0=30\text{m/s}$ angle 18°

Slice



Slice at tennis $V_0=5\text{m/s}$ angle 60°

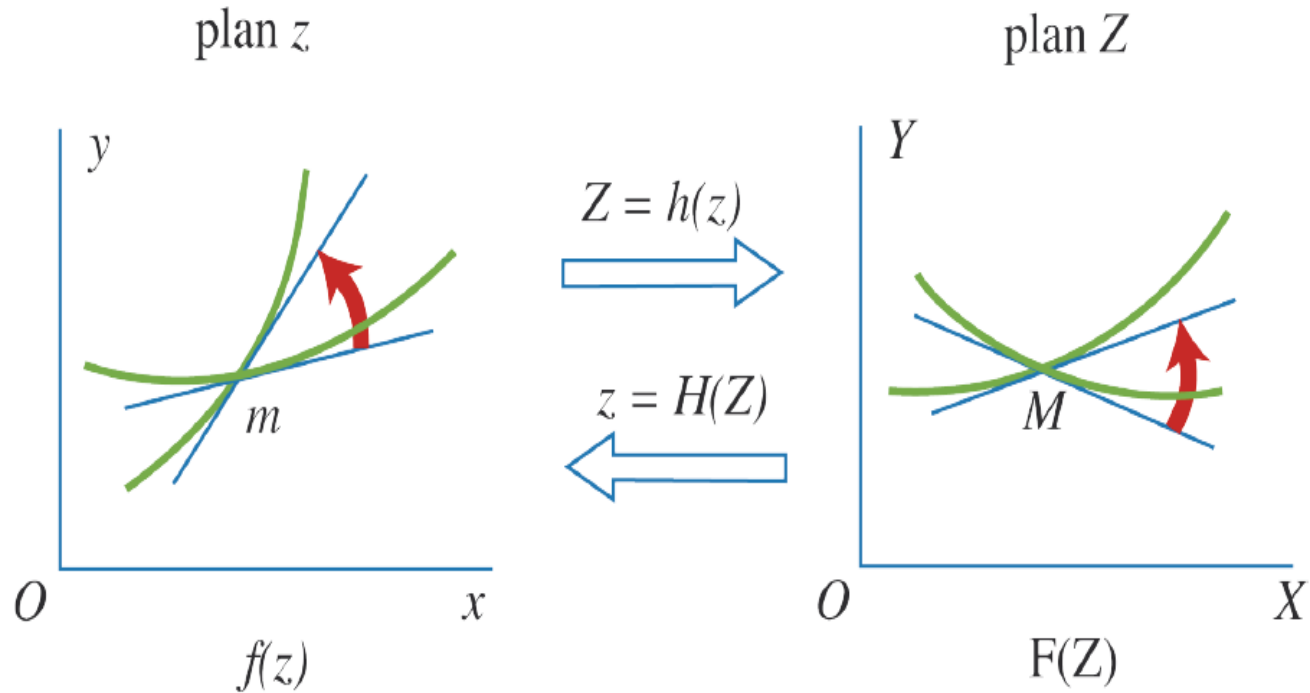
Conformal maps



$$f(z) = F[h(z)]$$

Conformal maps

Preservation of angles



Conformal maps

Preservation of angles

(except at critical points $h'(z)=0$)

⇒ The image of an holomorphic function is holomorphic

Link between z and Z planes

$$F(Z) = f[H(Z)]$$

Equality of complex potentials

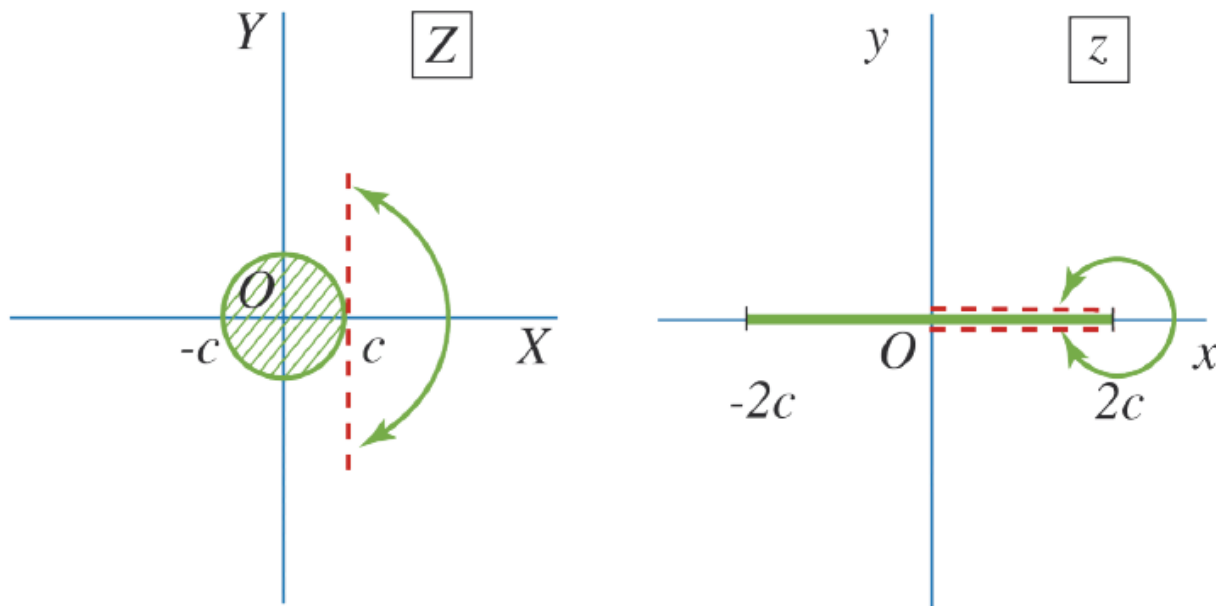
$$W(Z) = w[H(Z)]H'(Z)$$

Transformation of velocities

⇒ **Conservation of streamlines, flow rate, circulation but not stagnation points**

Joukowski's transform

$$z = Z + \frac{c^2}{Z}$$

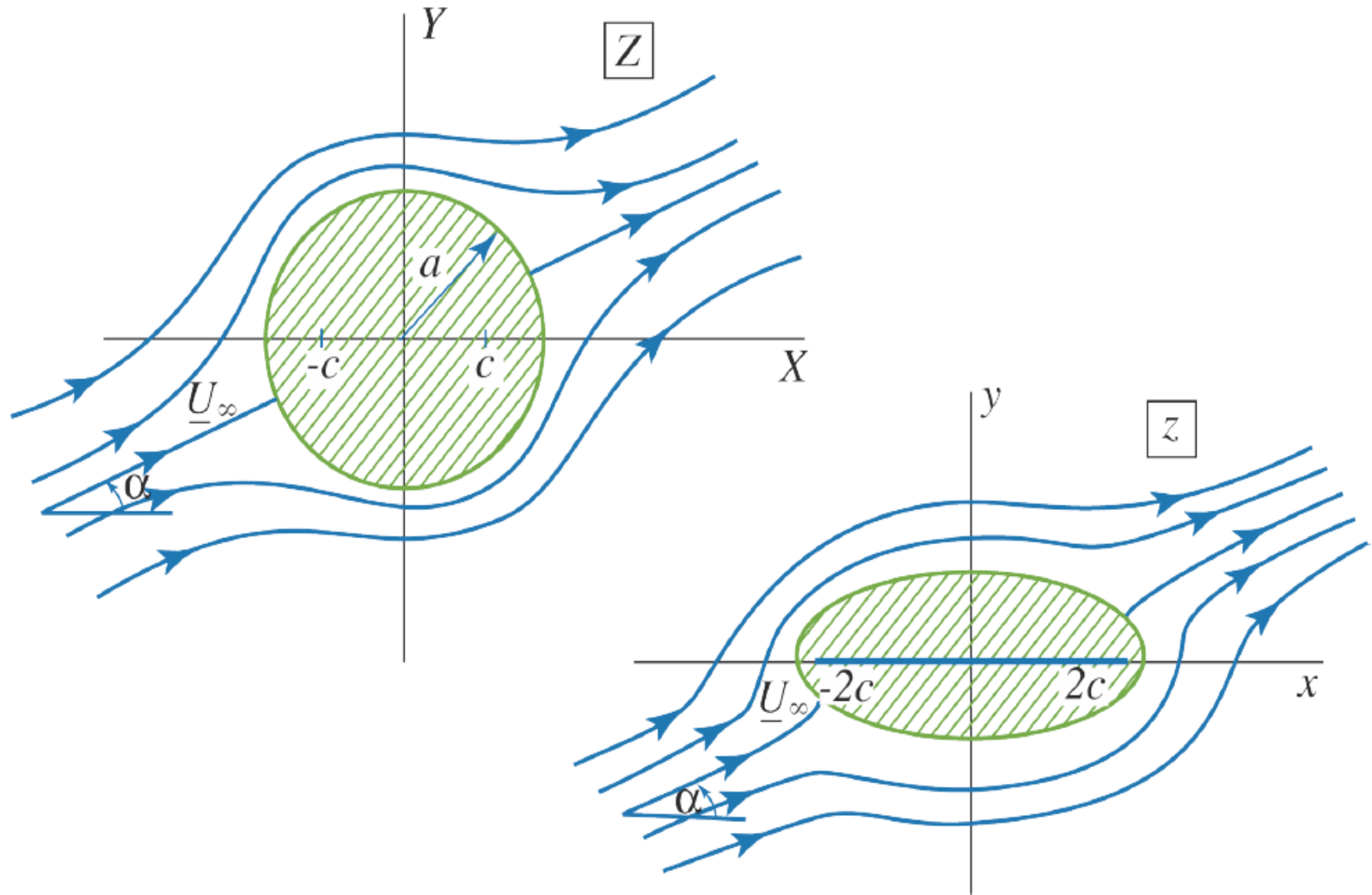


$$z - 2c \sim \frac{1}{c}(Z - c)^2$$

$$|Z - c| \rightarrow 0$$

Joukowski's transform

Flow around ellipse



Joukowski's transform

Flow around ellipse

$$F(Z) = U_{\infty} \left(Z e^{-i\alpha} + \frac{a^2}{Z} e^{i\alpha} \right)$$

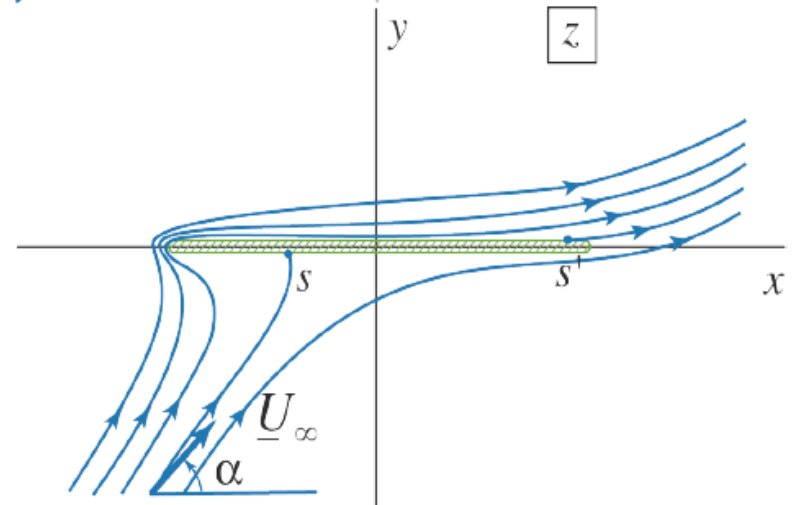
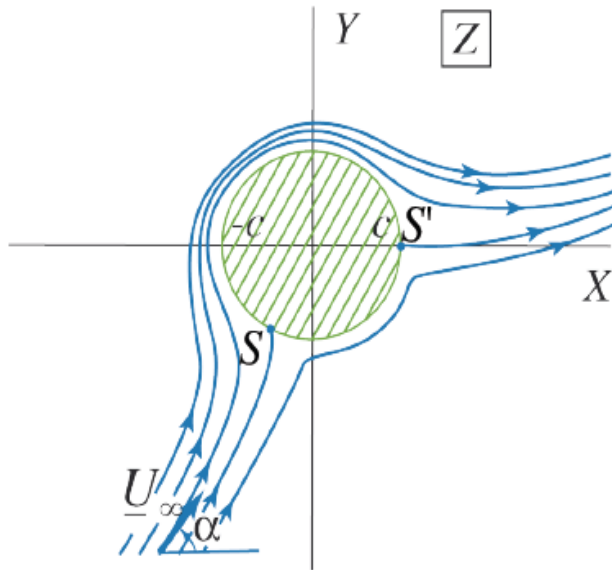
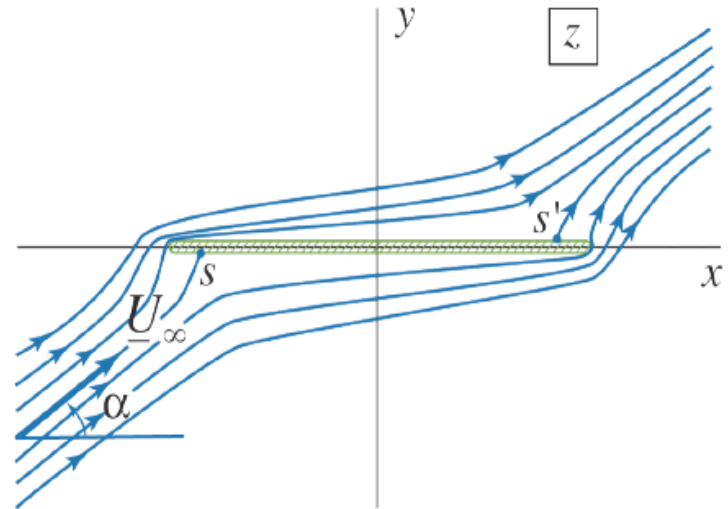
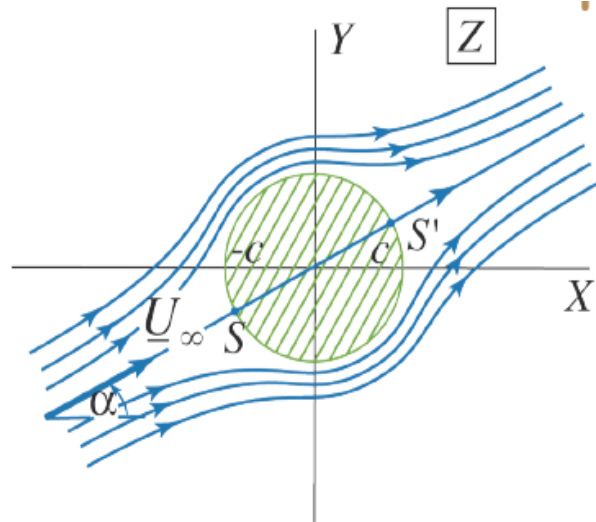
$$Z^2 - z Z + c^2 = 0$$

$$Z = h(z) = \frac{z}{2} + \sqrt{\left(\frac{z}{2}\right)^2 - c^2}$$

$$f(z) = U_{\infty} \left[z e^{-i\alpha} + \left(\frac{a^2}{c^2} e^{i\alpha} - e^{-i\alpha} \right) \left(\frac{z}{2} - \sqrt{\left(\frac{z}{2}\right)^2 - c^2} \right) \right]$$

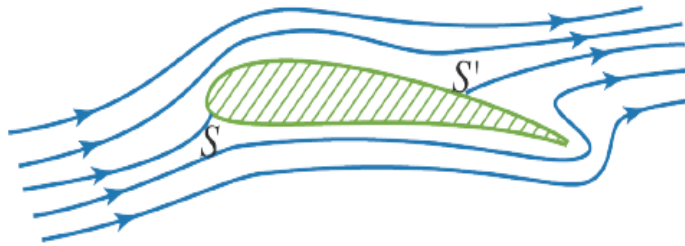
Joukowski's transform

Multiplicity of solutions



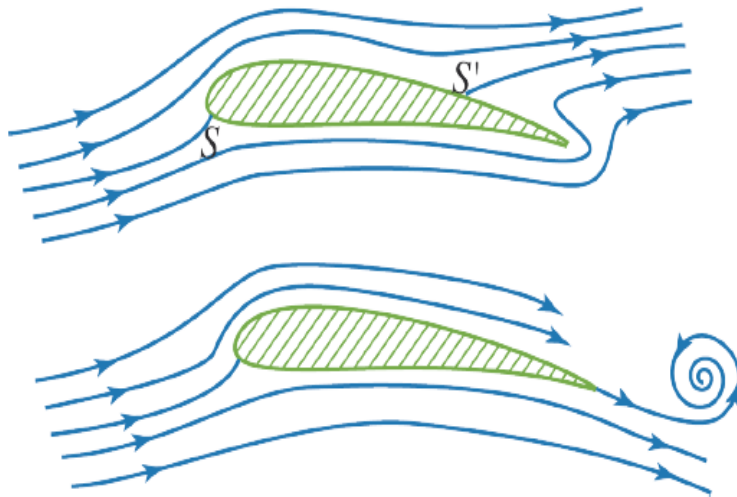
Flow around foil

Startup vortex and circulation selection



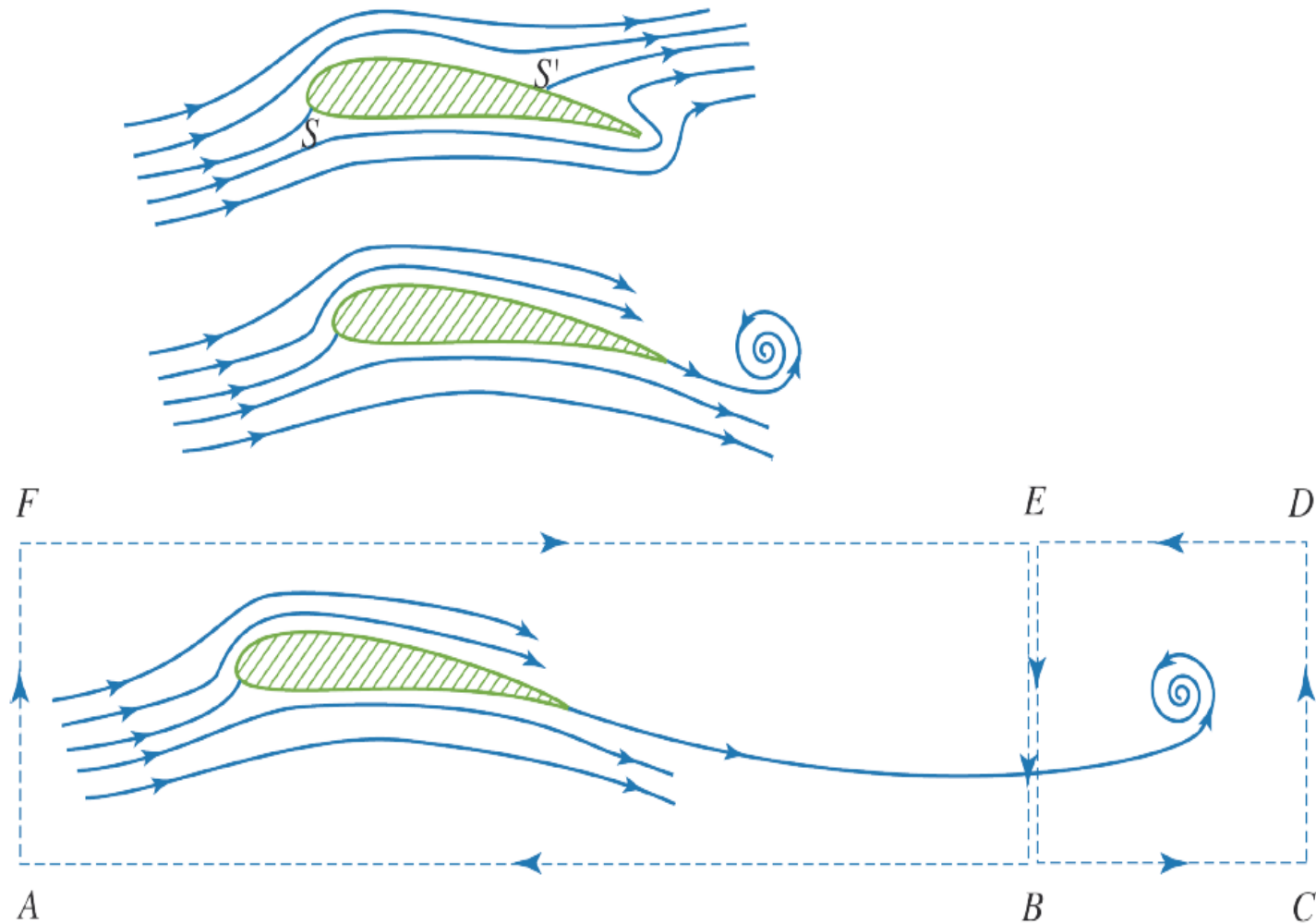
Flow around foil

Startup vortex and circulation selection

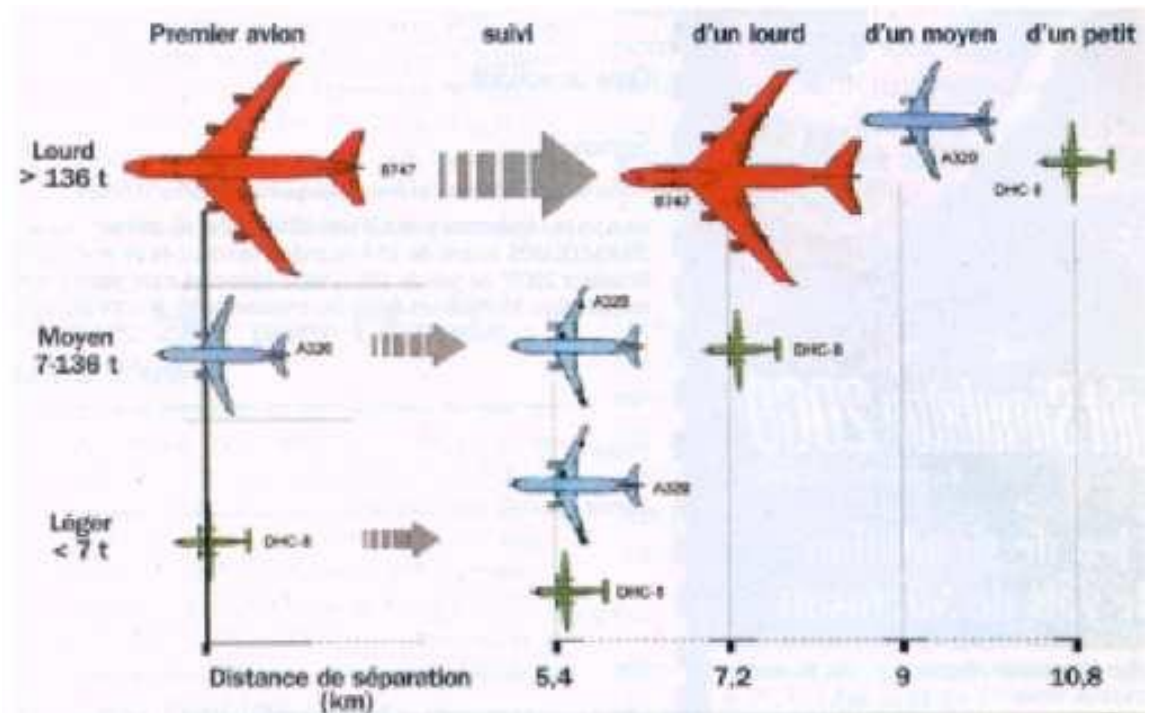


Flow around foil

Startup vortex and circulation selection



Real situation



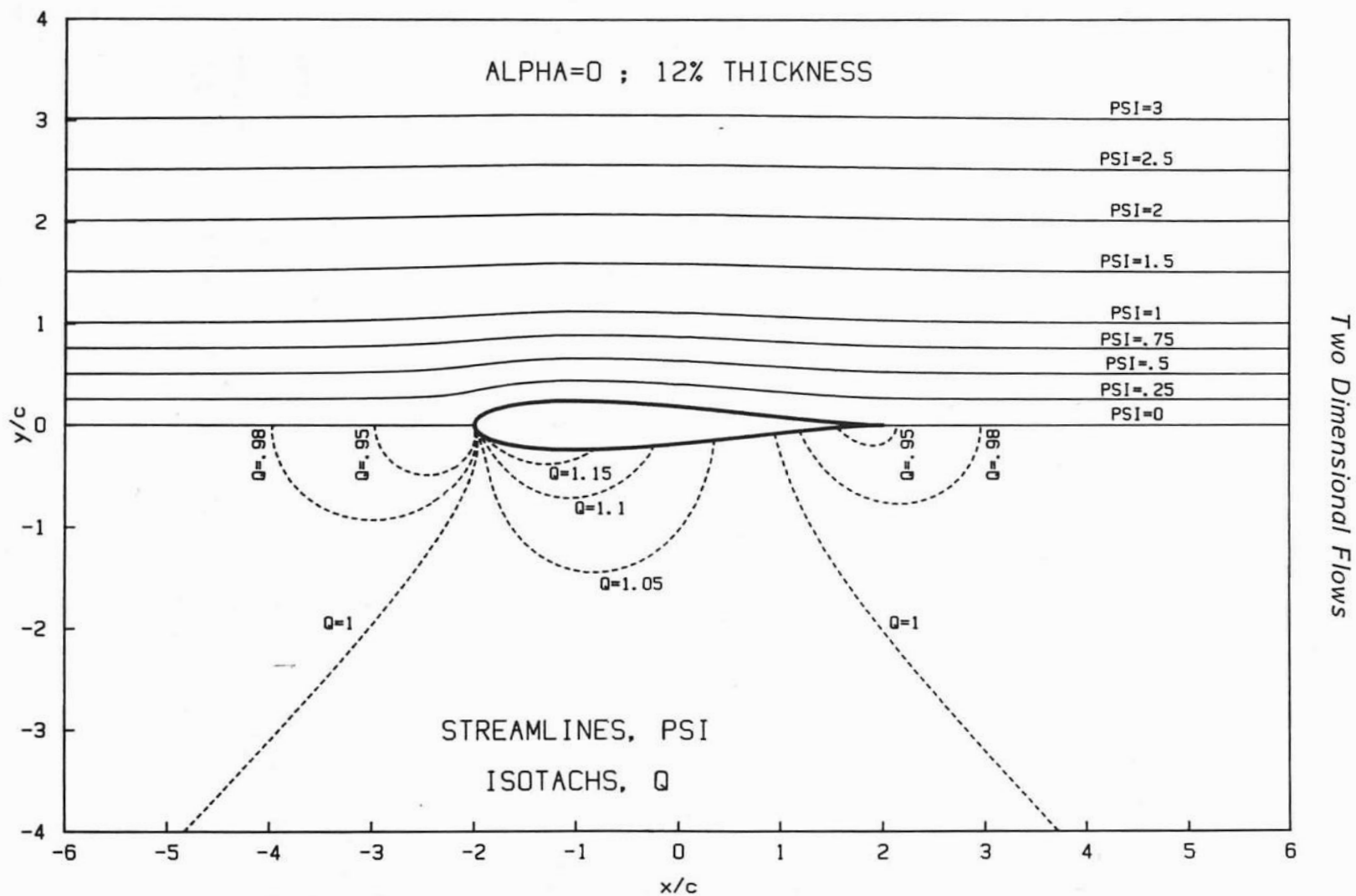


Figure 51 The symmetric Joukowski airfoil, $\alpha = 0^\circ$.

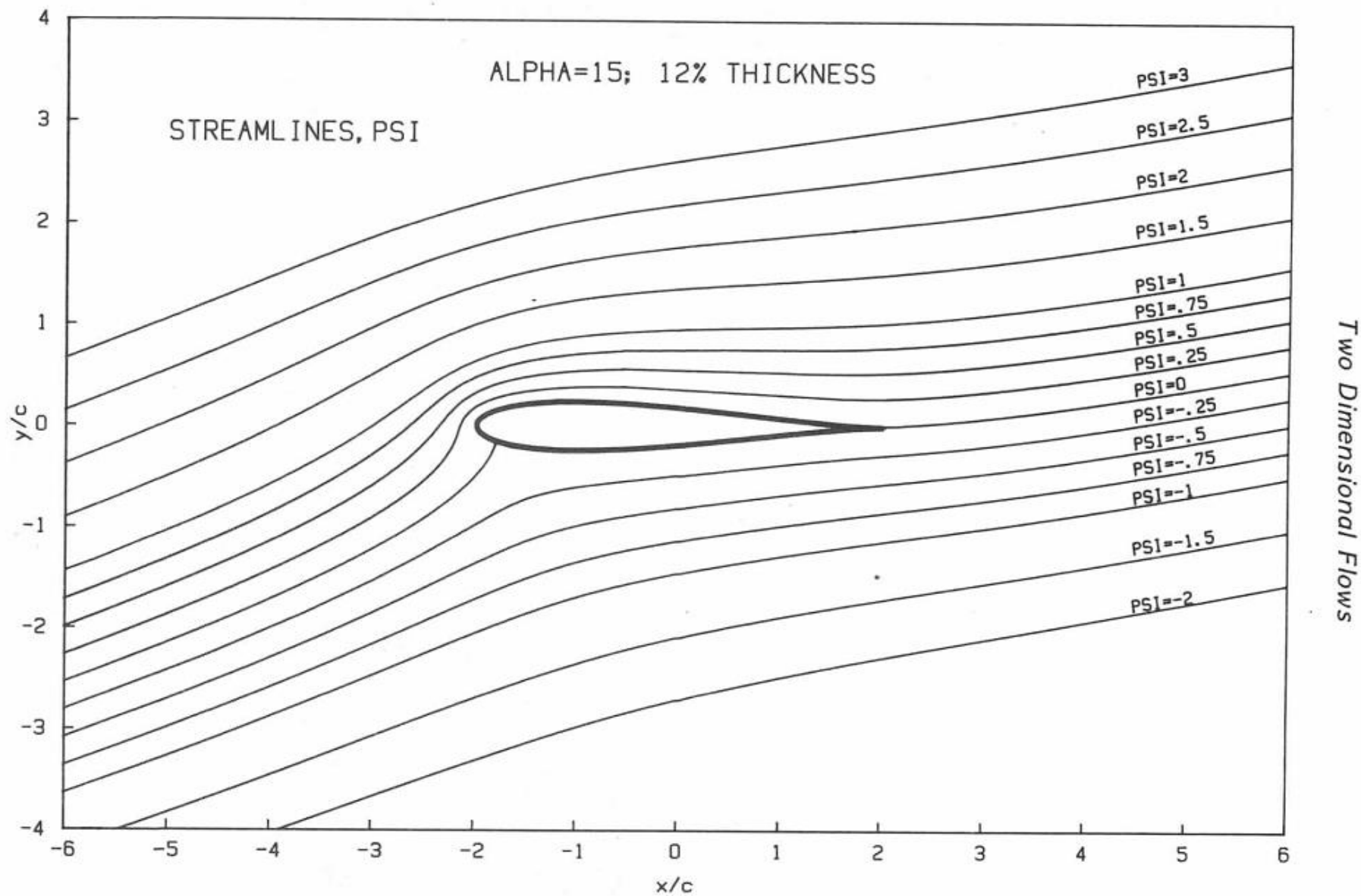
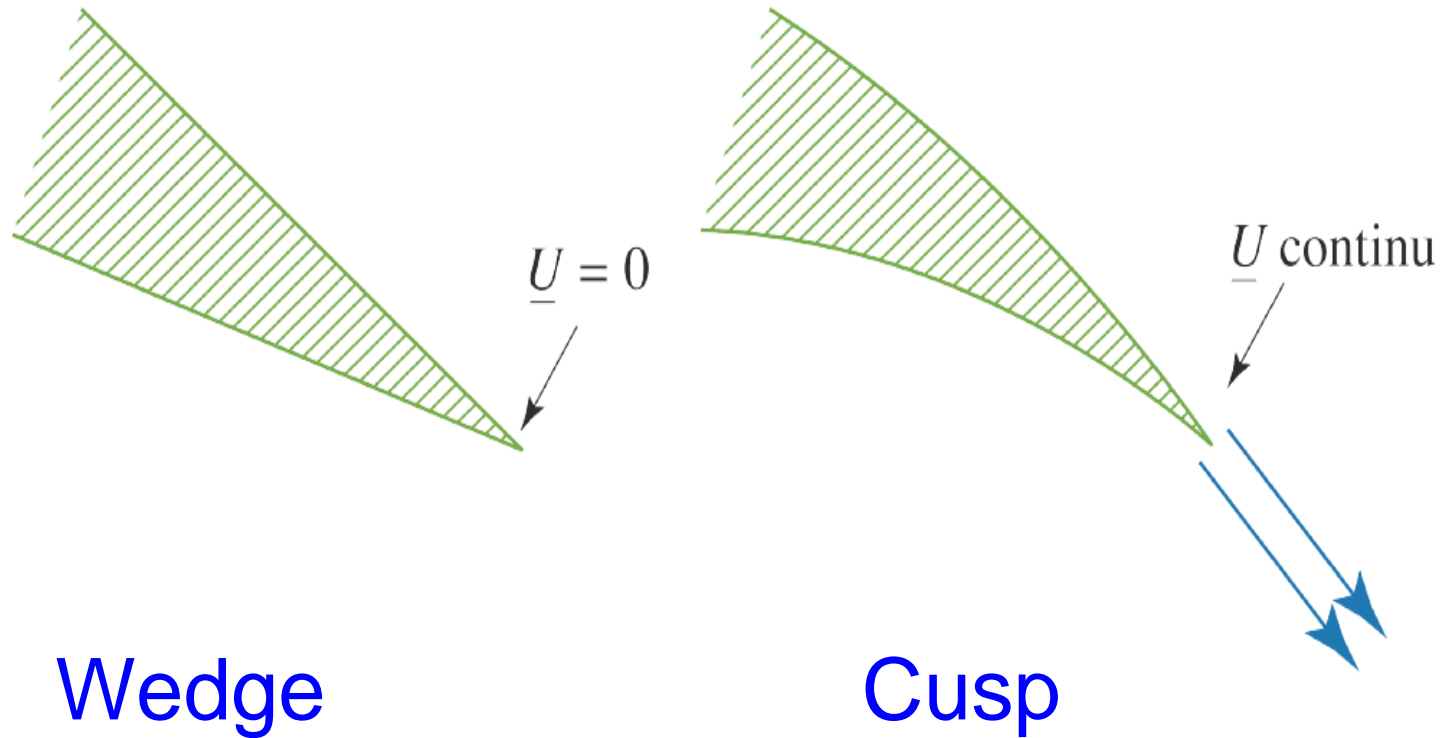


Figure 52 The symmetric Joukowski airfoil, streamlines, $\alpha = 15^\circ$.

Flow around foil

Circulation selection

By Kutta condition at trailing edge



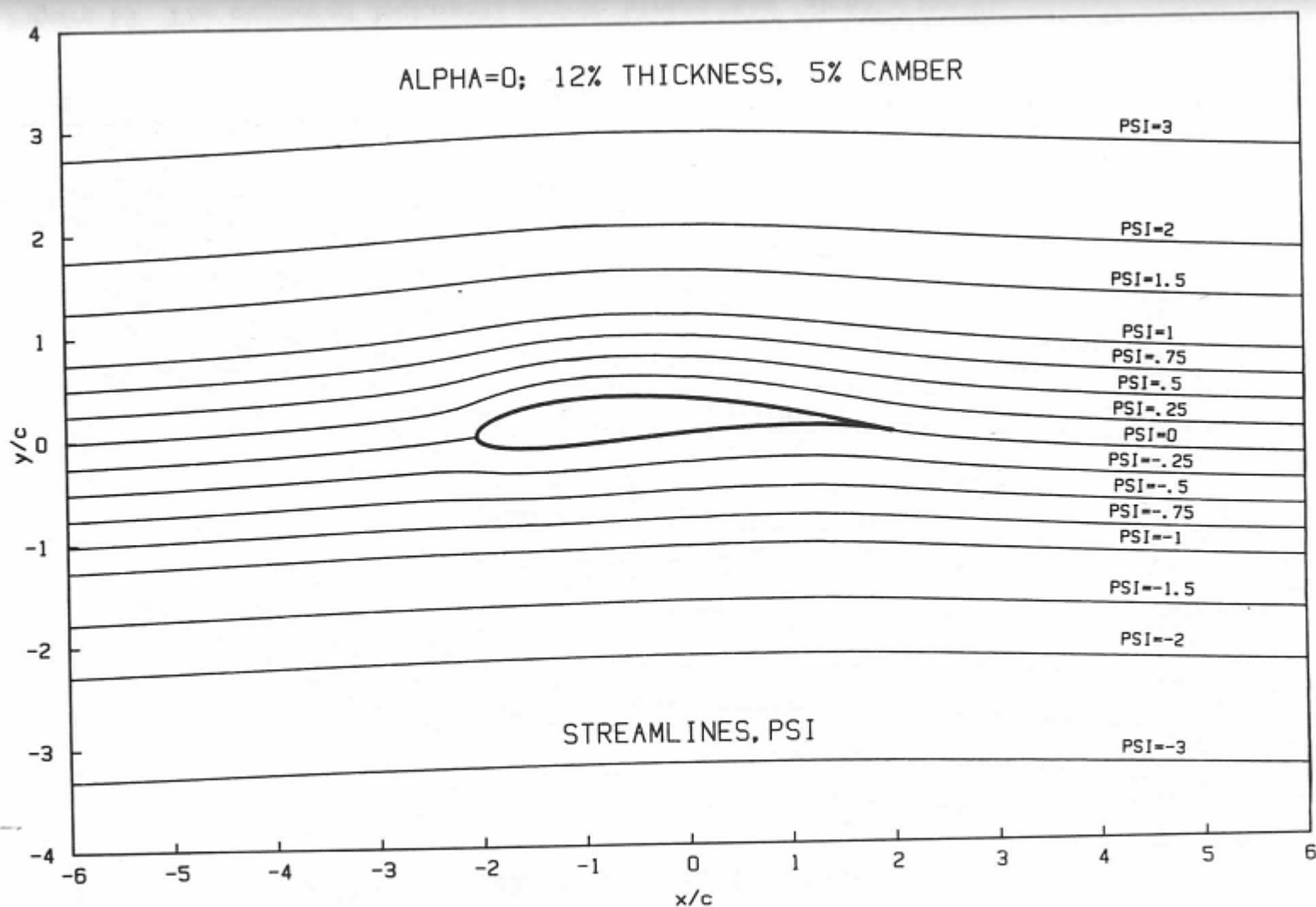


Figure 54 The cambered Joukowski airfoil, streamlines, $\alpha = 0^\circ$.

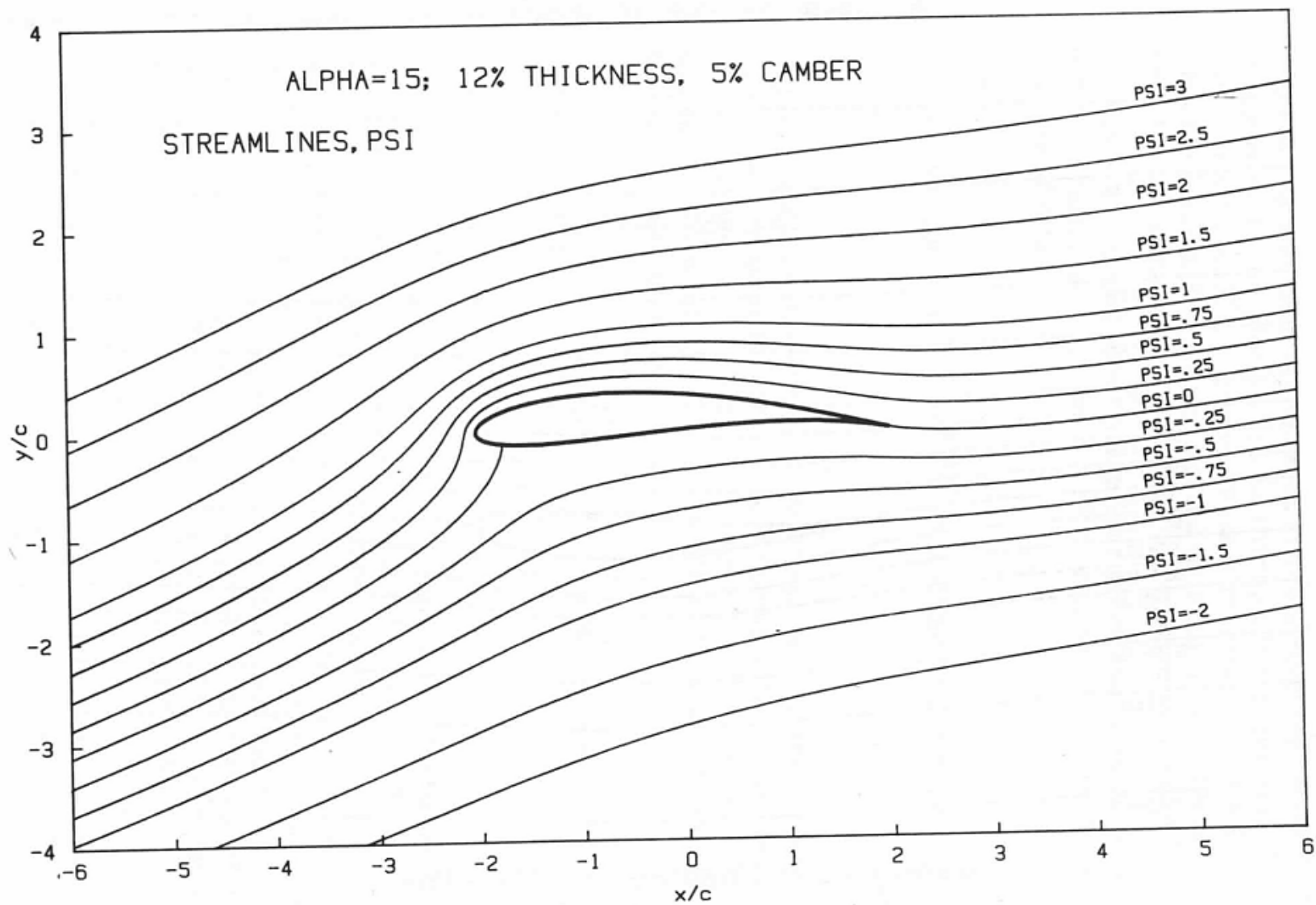


Figure 55 The cambered Joukowski airfoil, streamlines, $\alpha = 15^\circ$.

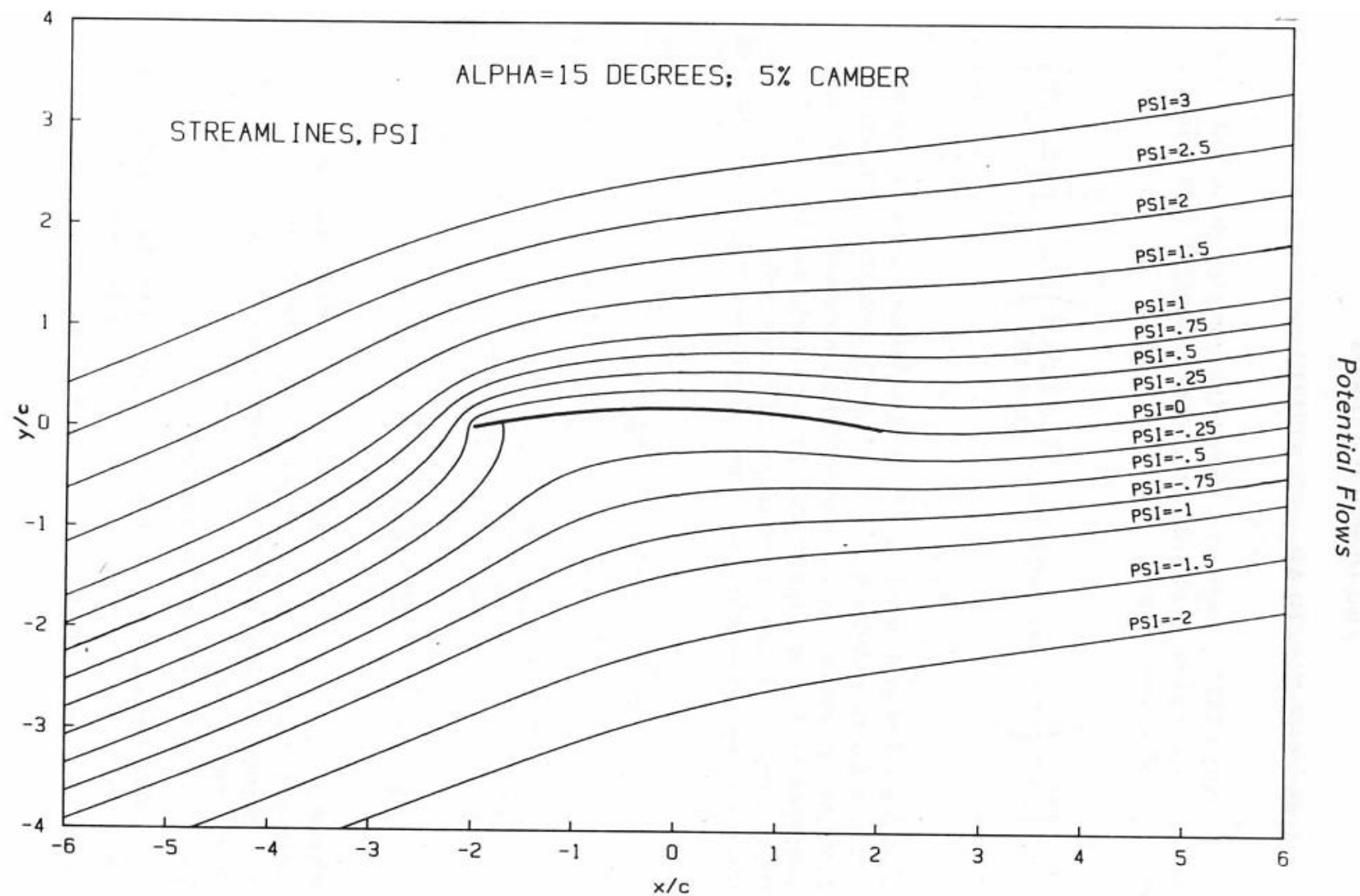


Figure 58 The circular-arc airfoil, streamlines, $\alpha = 15^\circ$.

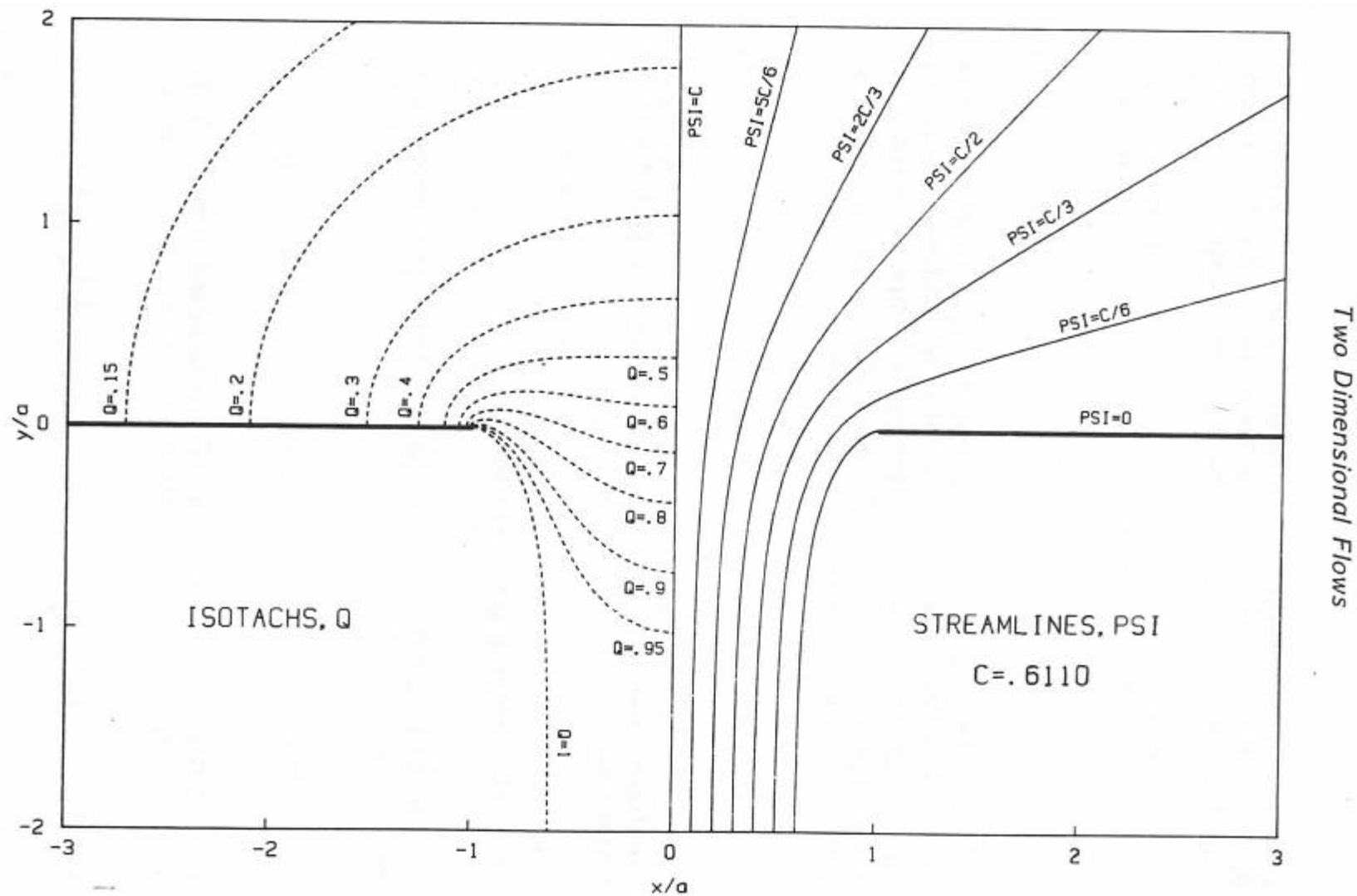


Figure 60 Efflux from an orifice.

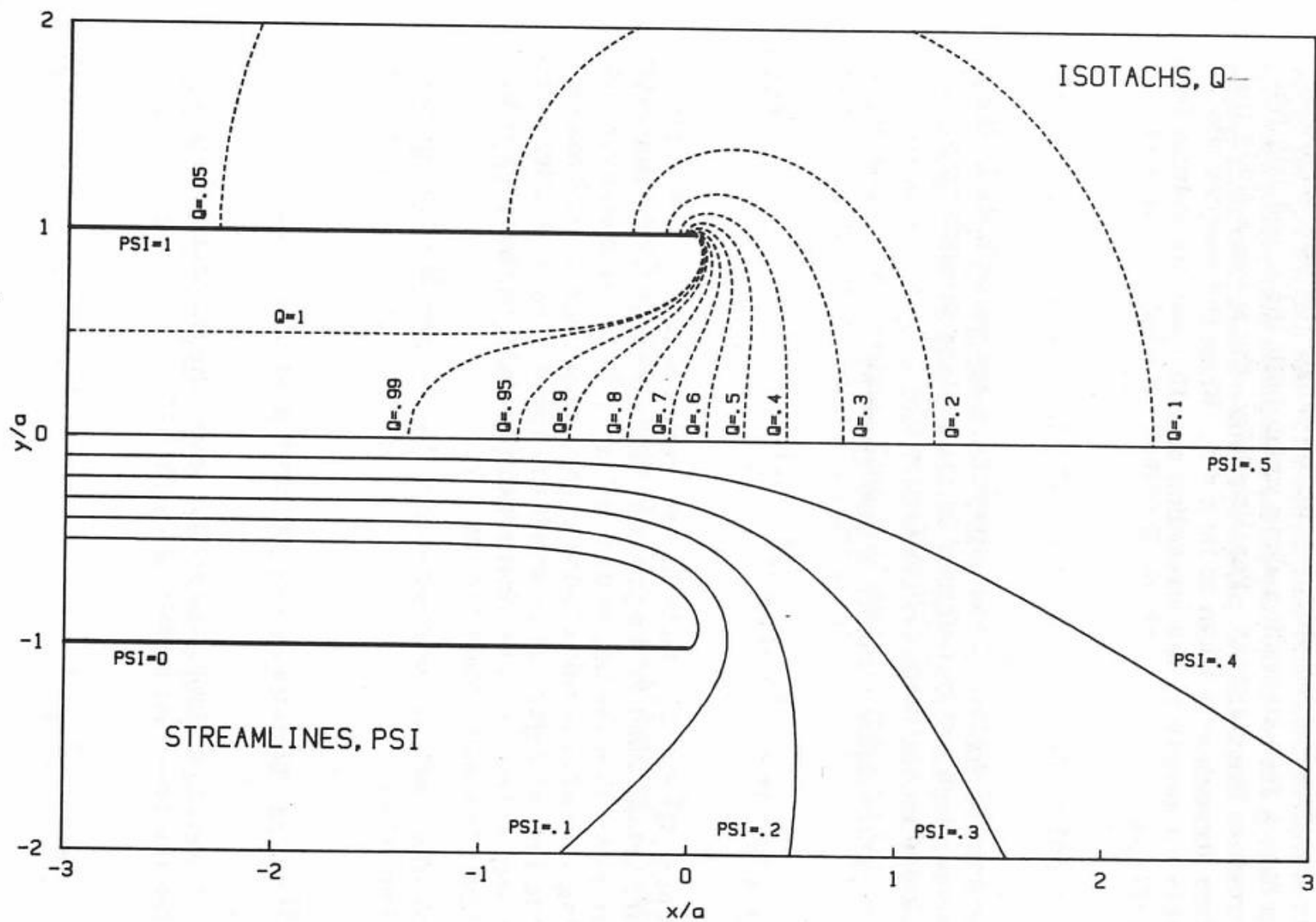
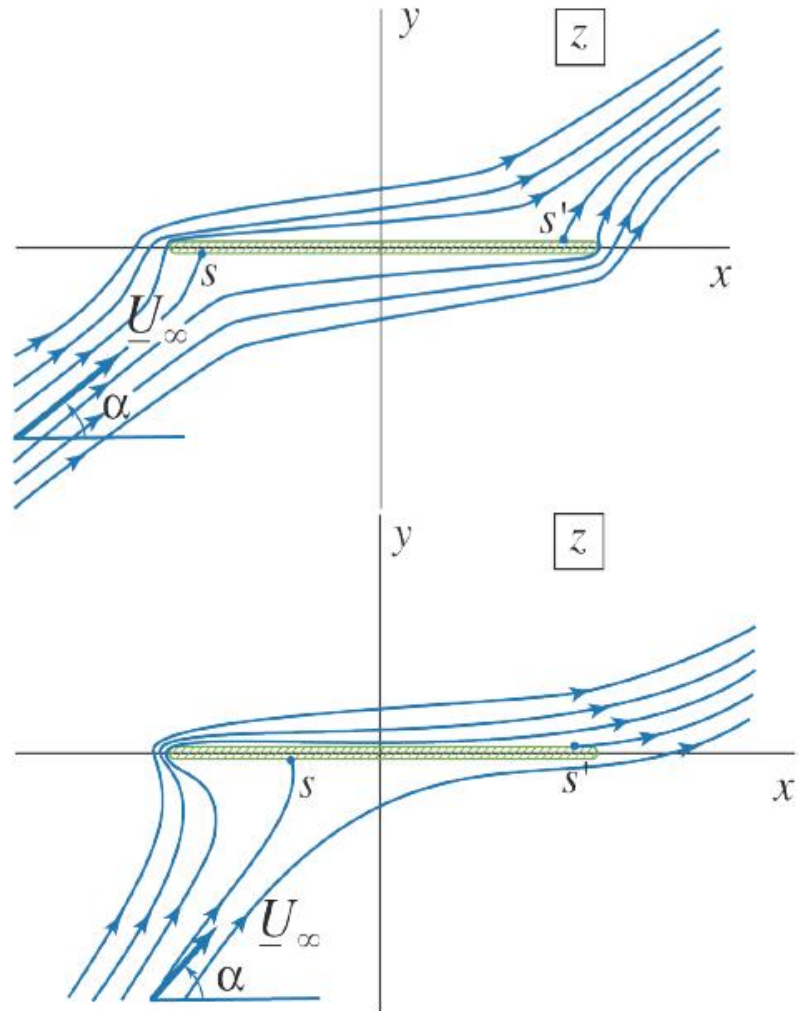
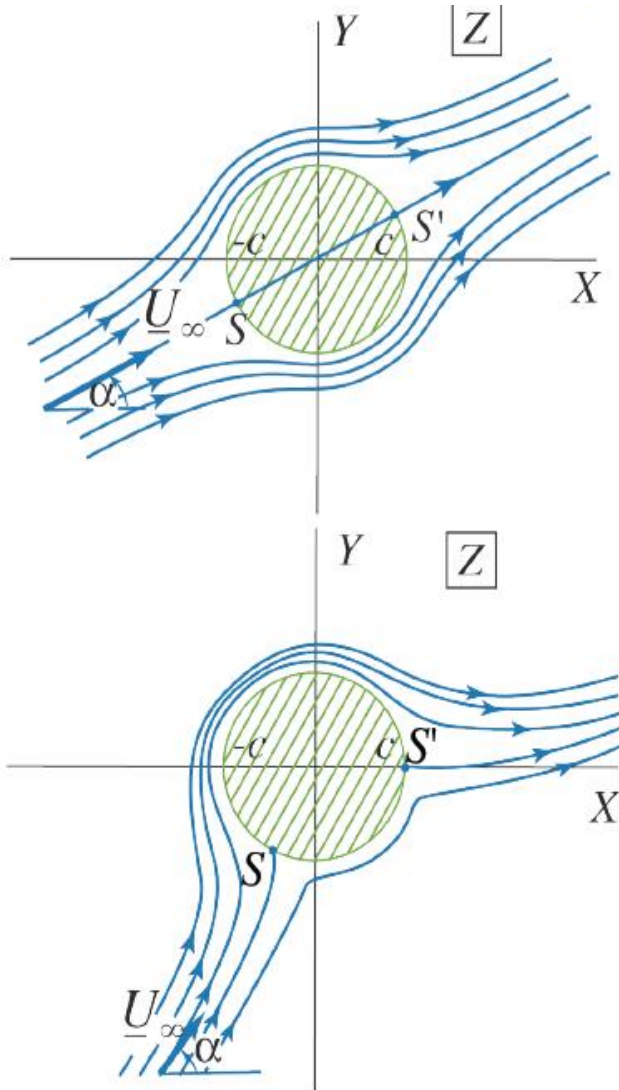


Figure 61 Borda's mouthpiece.

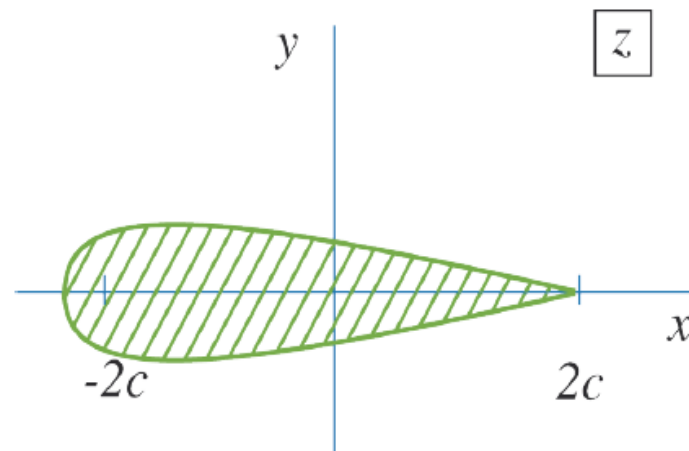
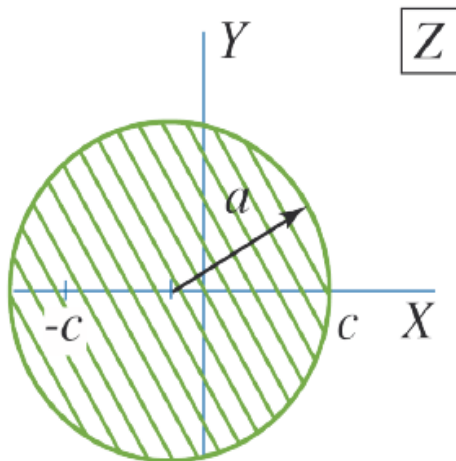
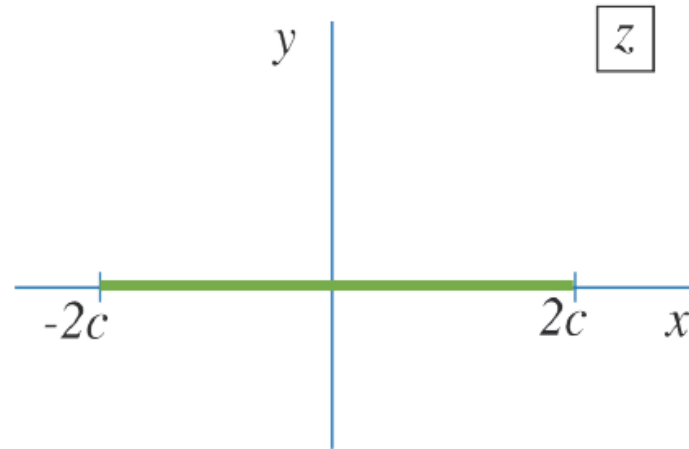
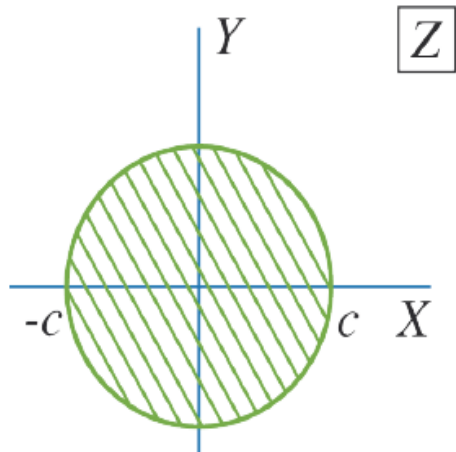
Joukowski's transform

Multiplicity of solutions



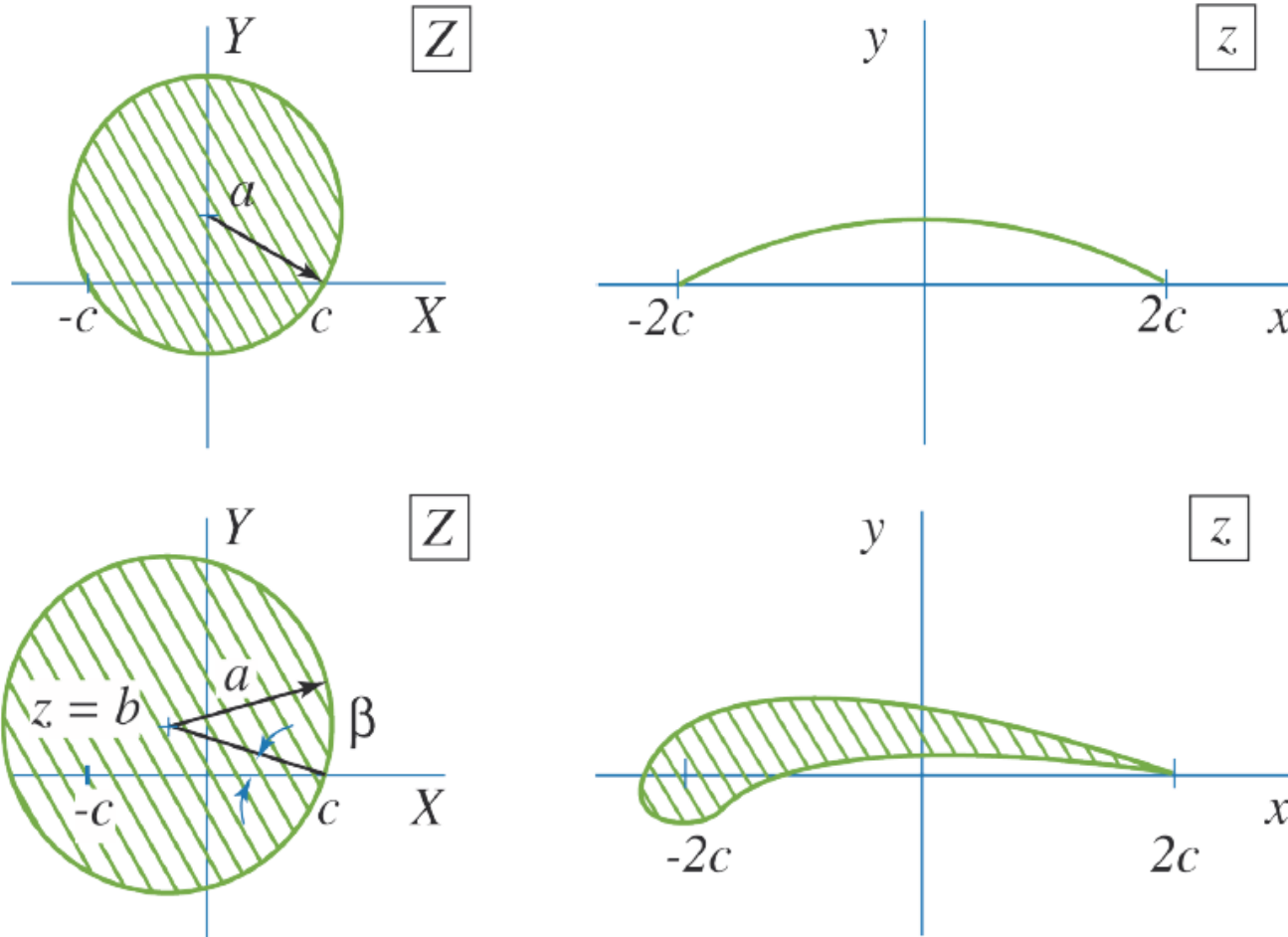
Flow around foil

Thickness effect

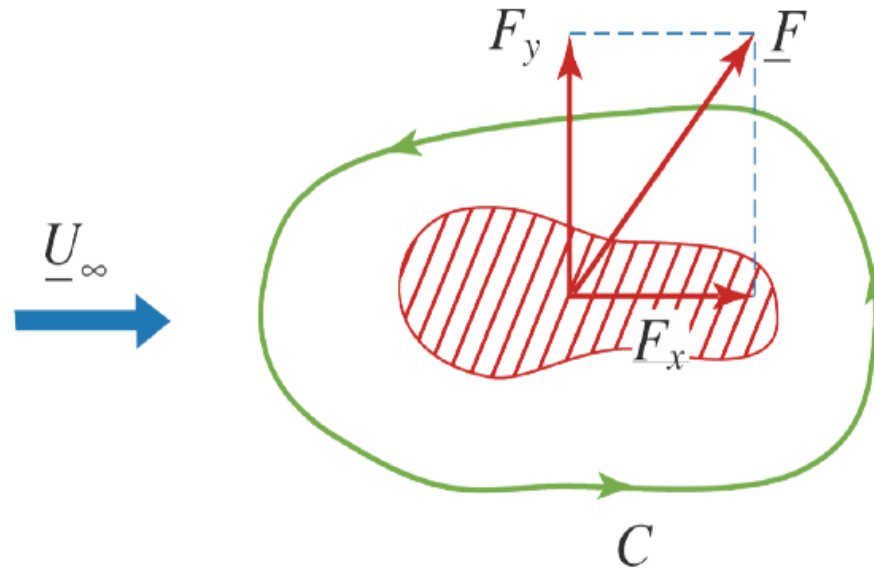


Flow around foil

Camber effect



Blasius Formula



$$F_x - iF_y = \frac{i\rho}{2} \int_C w^2(z) dz$$

$$C_0 = -\frac{\rho}{2} \Re \left\{ \int_C z w^2(z) dz \right\}$$

RÉSIDU

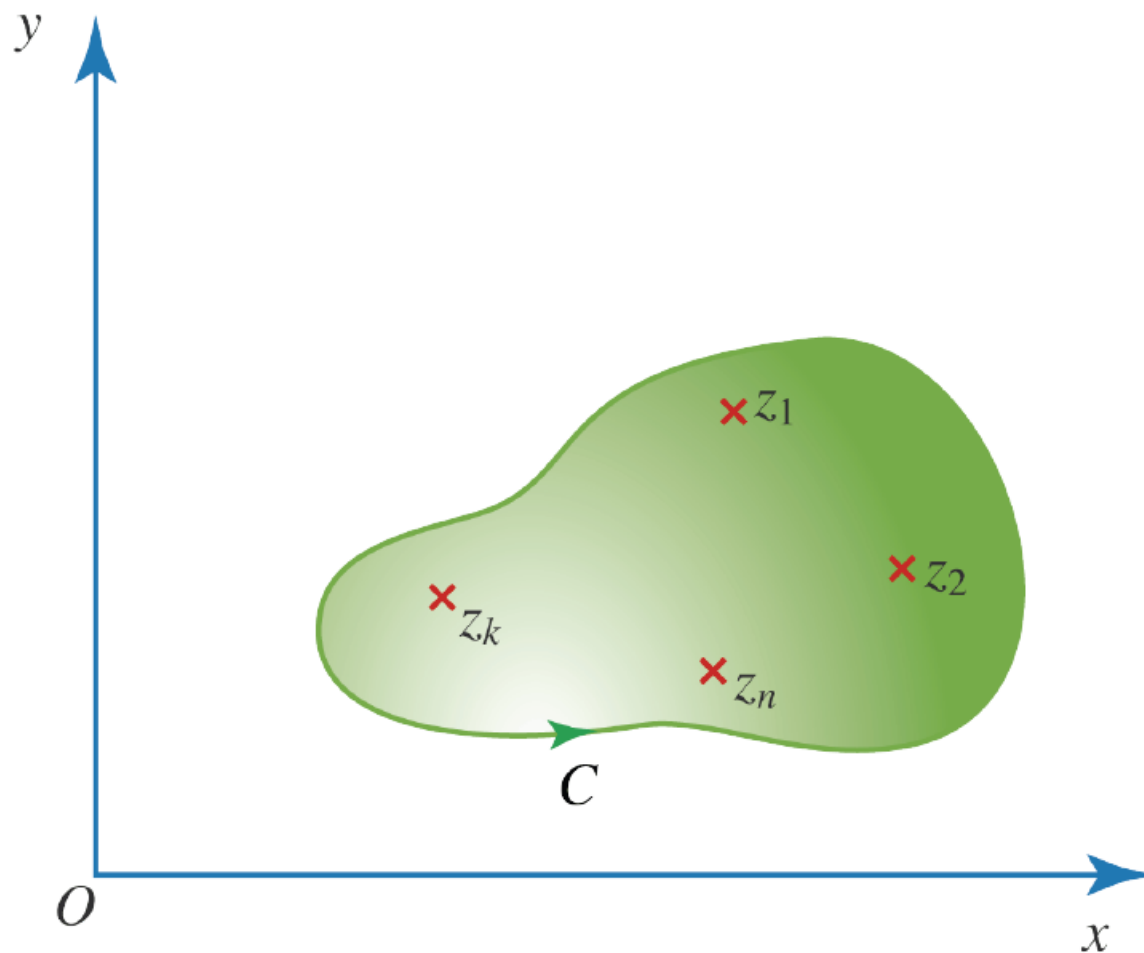
$$f(z) = \sum_{n=-\infty}^{-1} a_n (z - z_0)^n + \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

partie principale

partie régulière

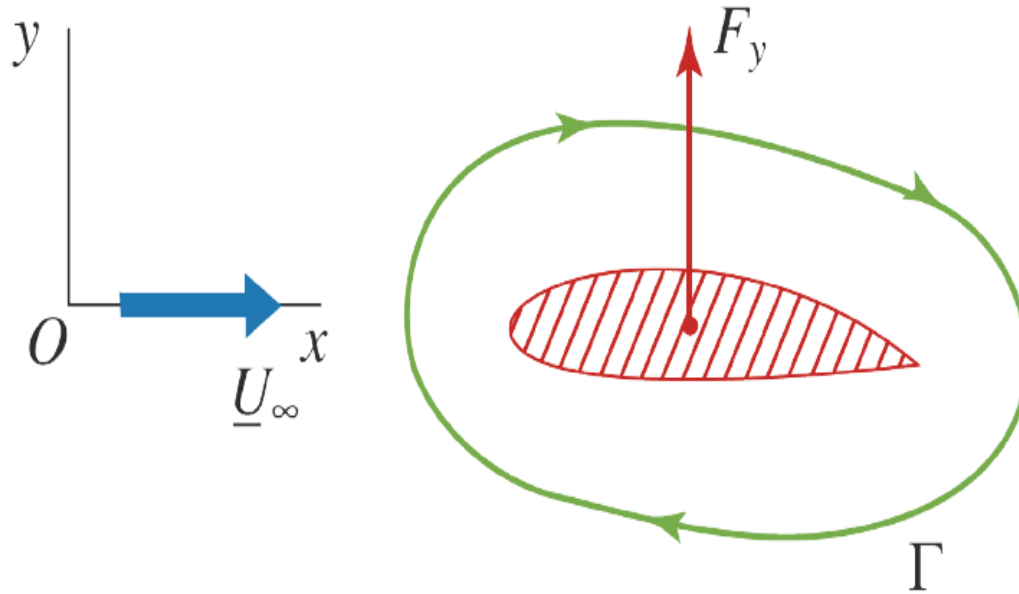
$$a_{-1} = \text{Res} (f, z_0) = \frac{1}{2\pi i} \int_C f(z) dz$$

THÉORÈME DES RÉSIDUS



$$\int_C f(z) dz = 2\pi i \sum_{k=1}^n \text{Res} (f, z_k)$$

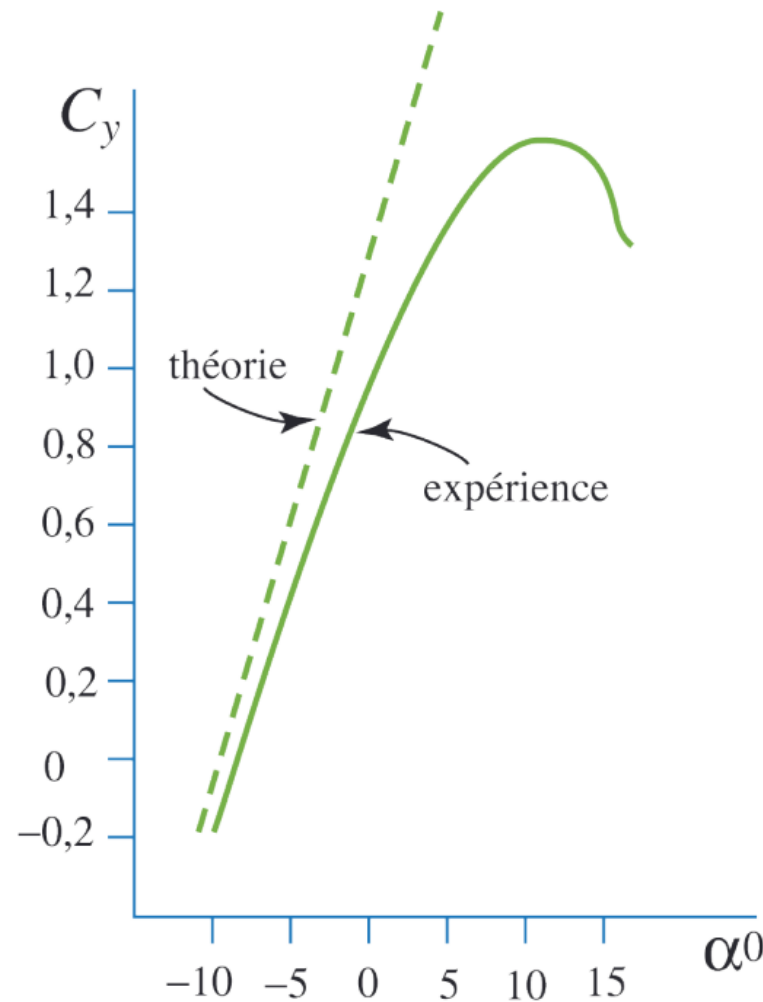
Kutta-Joukowski theorem



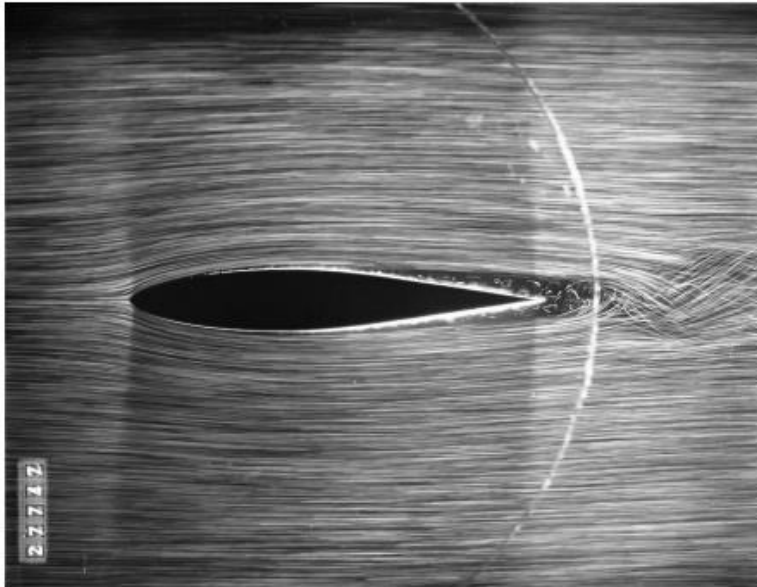
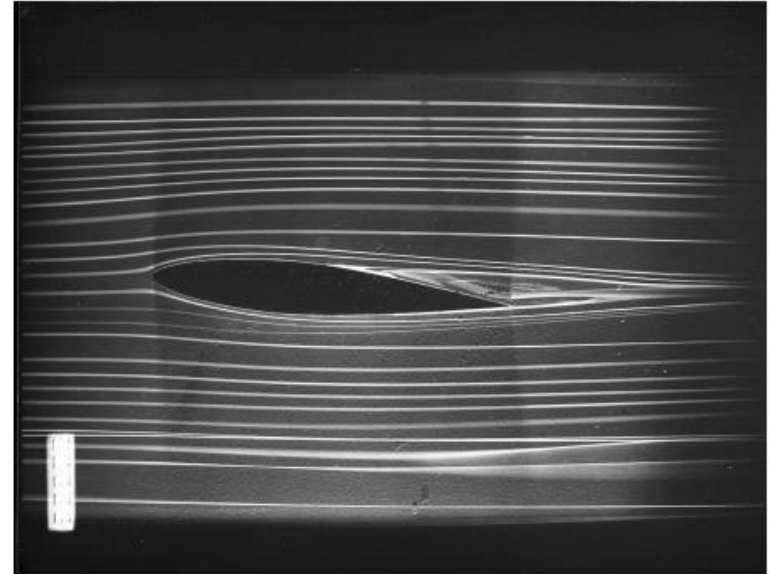
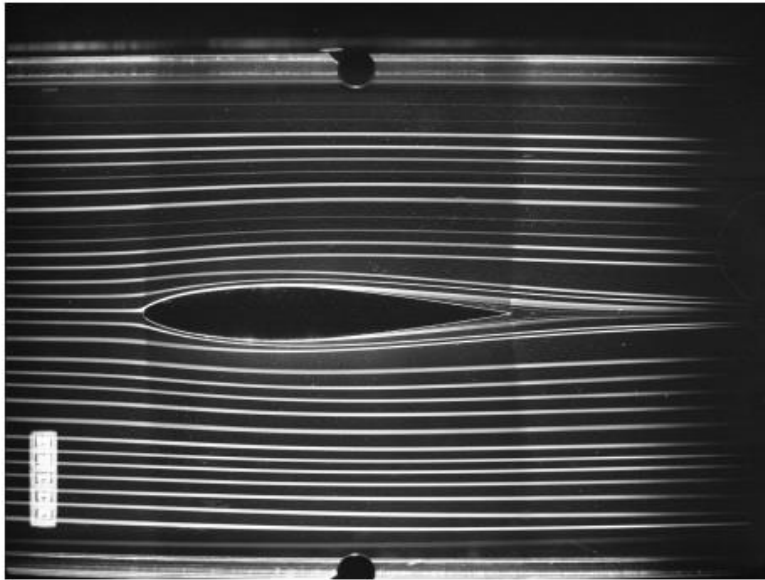
$$F_y = -\rho U_\infty \Gamma$$

Lift of a plate under incidence in incoming flow at angle

$$C_y = 2\pi(\alpha - \alpha_0)$$

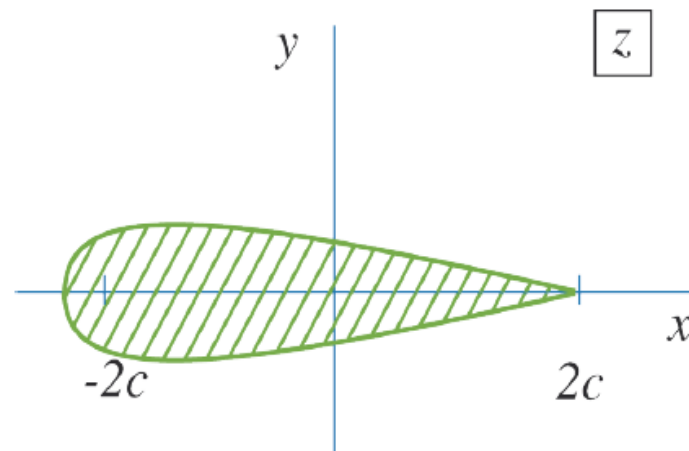
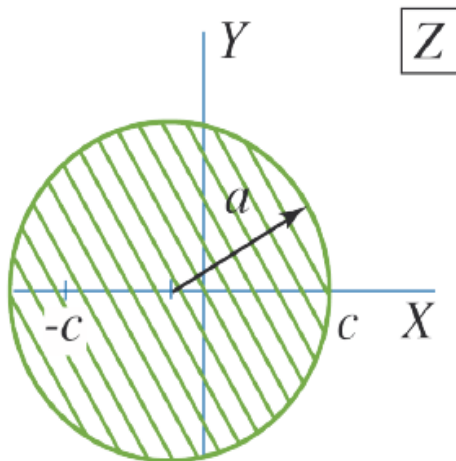
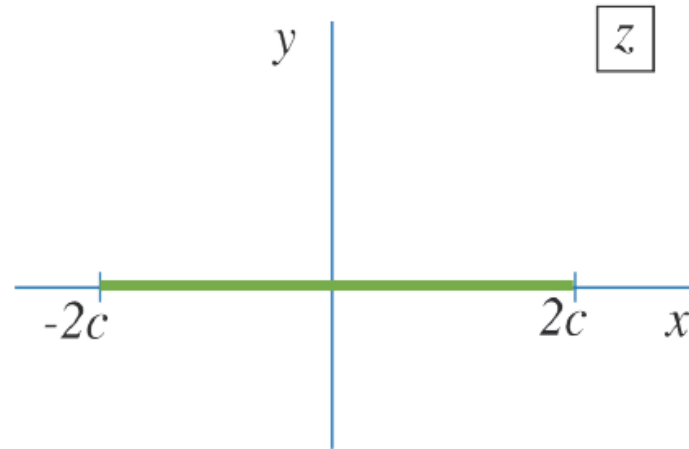
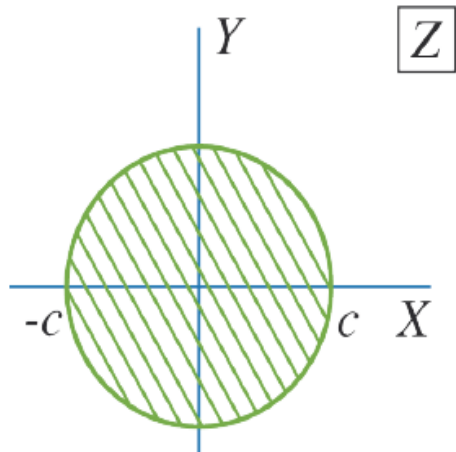


Lift of a wing under incidence



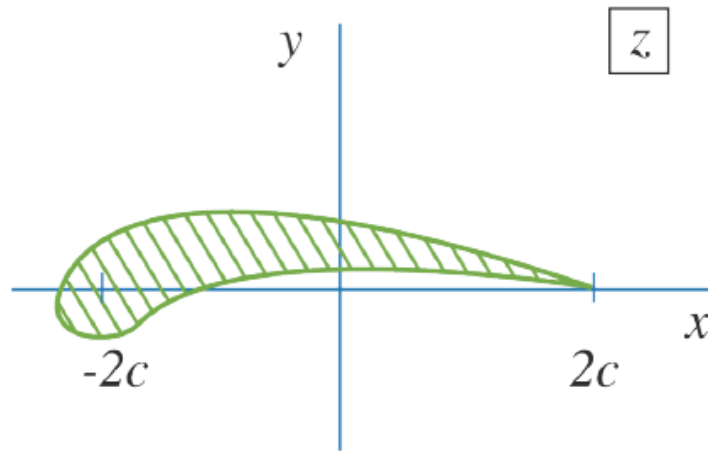
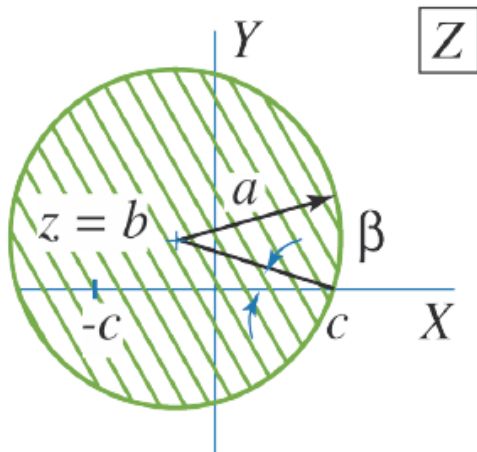
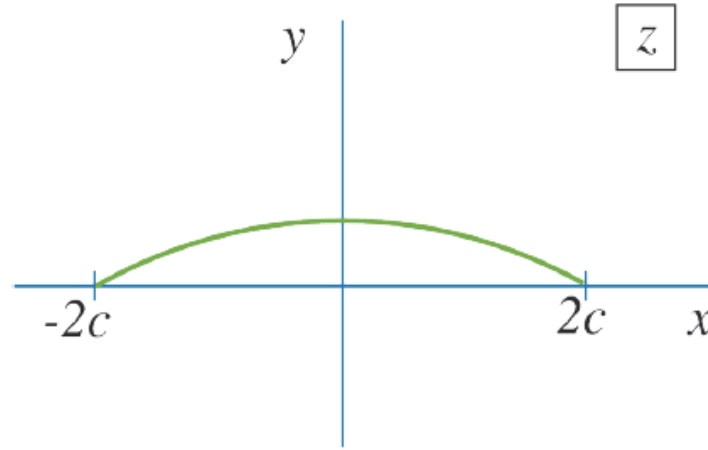
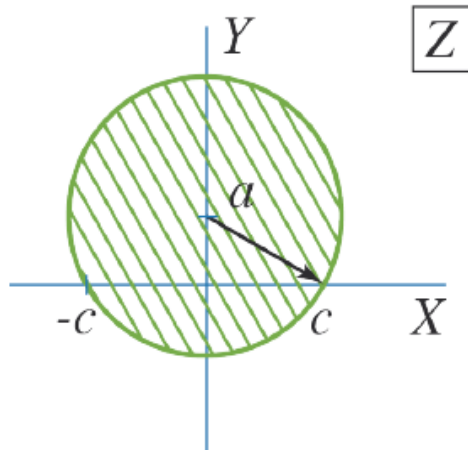
Flow around foil

Thickness effect



Flow around foil

Camber effect



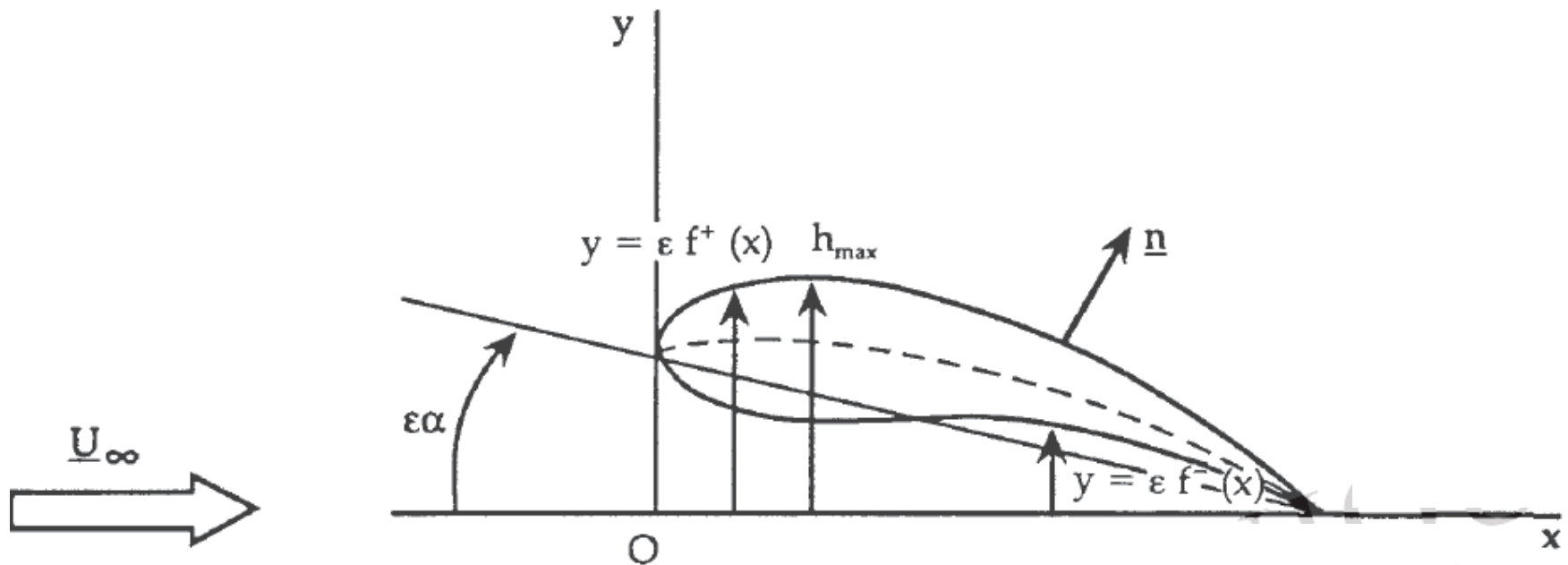
Linearized profile theory

$$y = \varepsilon f^+(x)$$

$$y = \varepsilon f^-(x)$$

$$0 \leq x \leq L$$

$$\varepsilon = \frac{h_{\max}}{L}$$



Linearized profile theory

$$\Delta\varphi = 0$$

impermeability $\underline{\text{grad}} \varphi \cdot \underline{n} = 0$

$$\frac{\partial \varphi}{\partial y} [x, \varepsilon f^\pm(x); \varepsilon] - \varepsilon \frac{df^\pm}{dx} \frac{\partial \varphi}{\partial x} [x, \varepsilon f^\pm(x); \varepsilon] = 0, 0 \leq x \leq L$$

far-field $\varphi(x, y; \varepsilon) \sim U_\infty x, |\underline{x}| \rightarrow \infty$

Linearized profile theory

$$\varphi(x, y; \varepsilon) \sim U_x x + \varepsilon \varphi_1(x, y) + \varepsilon^2 \varphi_2(x, y) \cdot$$

At order ε

$$\Delta \varphi_1 = 0 \quad ,$$

$$\frac{\partial \varphi_1}{\partial y}(x, 0^+) = U_x \frac{df^+}{dx}(x) \quad , \quad 0 \leq x \leq L \quad ,$$

$$\frac{\partial \varphi_1}{\partial y}(x, 0^-) = U_x \frac{df^-}{dx}(x) \quad , \quad 0 \leq x \leq L \quad ,$$

$$\varphi_1(x, y) \sim o(|\underline{x}|) \quad , \quad |\underline{x}| \rightarrow \infty \quad .$$

Linearized profile theory

$$\varphi(x, y; \varepsilon) \sim U_x x + \varepsilon \varphi_1(x, y) + \varepsilon^2 \varphi_2(x, y) \cdot$$

$$p(x, y; \varepsilon) \sim p_x + \varepsilon p_1(x, y) + \varepsilon p_2(x, y) +$$

$$\underline{U}(x, y; \varepsilon) \sim U_x \underline{e}_x + \varepsilon \underline{\text{grad}} \varphi_1 + \dots$$

$$p + \frac{1}{2} \rho U^2 = p_x + \frac{1}{2} \rho U_x^2$$

$$p_1 = -\rho U_x u_1 = -\rho U_x \frac{\partial \varphi_1}{\partial x}$$

$$C_p = \frac{p - p_x}{\frac{1}{2} \rho U_x^2} \sim -\frac{2\varepsilon}{U_x} u_1 = -\frac{2\varepsilon}{U_x} \frac{\partial \varphi_1}{\partial x} \cdot$$

Linearized profile theory

wing= incidence+ camber+ thickness

Camber

$$y = \varepsilon f_c(x) = \varepsilon \left[\frac{f^+(x) + f^-(x)}{2} - \alpha(L - x) \right] .$$

Thickness

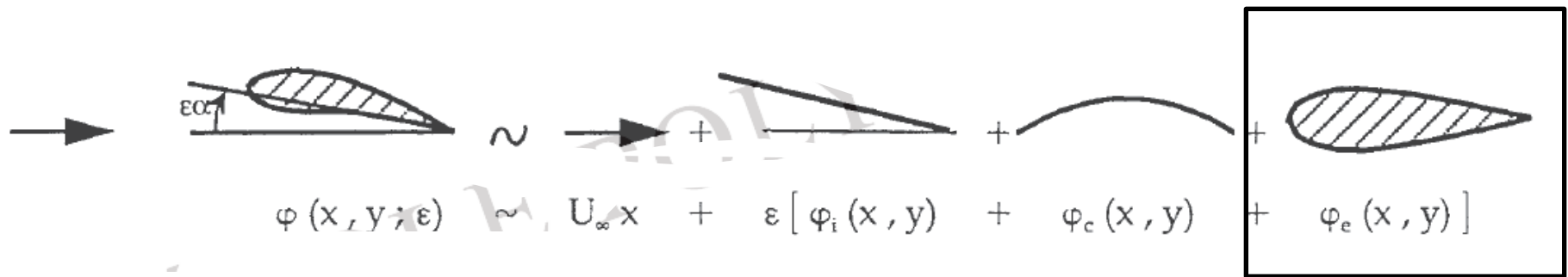
$$y = \varepsilon f_e(x) = \varepsilon \frac{f^+(x) - f^-(x)}{2}$$

$$\varepsilon f^+(x) = \varepsilon \left[\alpha(L - x) + f_c(x) + f_e(x) \right]$$

$$\varepsilon f^-(x) = \varepsilon \left[\alpha(L - x) + f_c(x) - f_e(x) \right]$$

Linearized profile theory

$$\varphi_1(x, y) = \varphi_i(x, y) + \varphi_c(x, y) + \varphi_e(x, y)$$



No lift by symmetry

Incidence

Find vorticity distribution $\gamma(x)$

$$f(z) = \varphi_i(x, y) + i \psi_i(x, y) = -\frac{i}{2\pi} \int_0^L \gamma(x') \log(z - x') dx'$$

such that impermeability

$$\frac{\partial \varphi_i}{\partial y}(x, 0^\pm) = -U_\infty \alpha, \quad 0 \leq x \leq L,$$

$$w(z) = \frac{df}{dz} = u_i(x, y) - i v_i(x, y) = -\frac{i}{2\pi} \int_0^L \frac{\gamma(x')}{z - x'} dx'$$

$$v_i(x, 0^\pm) = \frac{1}{2\pi} \text{v.p.} \int_0^L \frac{\gamma(x')}{x - x'} dx'$$

$$\text{v.p.} \int_0^L \frac{\gamma(x')}{x - x'} dx' = \lim_{\delta \rightarrow 0} \left[\int_0^{x-\delta} \frac{\gamma(x')}{x - x'} dx' + \int_{x+\delta}^L \frac{\gamma(x')}{x - x'} dx' \right]$$

$$U_\infty \alpha + \frac{1}{2\pi} \text{v.p.} \int_0^L \frac{\gamma(x')}{x - x'} dx' = 0$$

Incidence

Kutta!

$$u_i(L, 0^+) = u_i(L, 0^-)$$

$$\llbracket u_i(x, y) \rrbracket_{y=0^-}^{y=0^+} = -\gamma(x), \quad 0 < x < L$$

$$\gamma(L) = 0$$

On peut démontrer que

$$\gamma(x) = -2U_\infty \alpha \sqrt{\frac{L-x}{x}}$$

Incidence

Distribution de C_p

$$C_p(x, 0^\pm) = \mp 2 \varepsilon \alpha \sqrt{\frac{L-x}{x}} .$$

Théorème de Kutta-Joukowski

$$F_y = -\rho U_\infty \Gamma = -\rho U_\infty \int_0^L \gamma(x) dx = 2\rho U_\infty^2 \alpha \int_0^L \sqrt{\frac{L-x}{x}} dx$$

$$F_y = \pi \alpha \rho L U_\infty^2 .$$

$$C_y = \frac{F_y}{\frac{1}{2} \rho U_\infty^2 L} = 2 \pi \alpha .$$

Skeleton (camber)

$$\gamma(x) = \frac{1}{\sqrt{x(L-x)}} \left[C + \frac{2U_\infty}{\pi} \text{v.p.} \int_0^L \frac{df_c}{dx} \sqrt{x'(L-x')} \frac{dx'}{x-x'} \right]$$

$$\gamma(L) = 0$$

to be integrated numerically

$$C_y = -2\pi \alpha_0$$

Linear superposition

$$C_y = 2\pi (\alpha - \alpha_0) ,$$