Outline

1. Intro+kinematics
2. Dynamics
3. Dimensional Analysis
4. Low Reynolds number flow/ Stokes eq.
5. Stokes drag
6. Lubrification-Hele Shaw-Pipe Flows

8. Unsteady flows

9. Boundary layer

10. Invisicd fluid- Bernoulli-potential flow

11. Potential flow, lift

12. Flow separation and detachment

13. Waves

14. Wave drag

Vorticity, inviscid and potential flow

Vorticity equation on plane

$$\frac{\partial}{\partial t}\omega + u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} = \nu\left[\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}\right]$$

Advection-diffusion equation

Vorticity equation: axisymmetric case

$$\frac{\partial}{\partial t}\omega + u\frac{\partial \omega}{\partial z} + v\frac{\partial \omega}{\partial r} = \nu\left[\frac{\partial^2 \omega}{\partial z^2} + \frac{\partial^2 \omega}{\partial r^2} + \frac{1}{r}\frac{\partial \omega}{\partial r} - \frac{\omega}{r^2}\right]$$

Advection-diffusion equation

Streamfunction and (u, v) through vorticity Biot-Savart induction

Assume ω(x,y) is given, then

$$\Psi(x,y) = -\frac{1}{4\pi}\iint_D \omega(x',y')\log\big((x'-x)^2+(y'-y)^2\big)\mathrm{d}x'\mathrm{d}y'.$$

and u, v are given by

$$u(x,y) = \partial_y\Psi = \frac{1}{2\pi}\iint_D \frac{(y'-y)\omega(x',y')}{(x'-x)^2+(y'-y)^2}\,\mathrm{d}x'\mathrm{d}y',$$

$$v(x,y) = -\partial_x\Psi = -\frac{1}{2\pi}\iint_D \frac{(x'-x)\omega(x',y')}{(x'-x)^2+(y'-y)^2}\,\mathrm{d}x'\mathrm{d}y'.$$

Point Vortex flow

$$\omega(x, y) = k\delta(x - x_1)\,\delta(y - y_1),$$

$$u(x, y) = -\frac{k}{2\pi}\,\frac{y - y_1}{(x - x_1)^2 + (y - y_1)^2},$$

$$v(x, y) = \frac{k}{2\pi}\,\frac{x - x_1}{(x - x_1)^2 + (y - y_1)^2},$$

Vortex ring

$$\omega_\phi(x', r') = \gamma \delta(x')\, \delta(r' - R)$$

$$\Psi_\gamma(x, r) = \frac{\gamma}{4\pi} rR \int_0^{2\pi} \mathrm{d}\phi \, \frac{\cos\phi}{\left[x^2 + r^2 - 2rR\cos\phi + R^2\right]^{1/2}}$$

Vortex ring

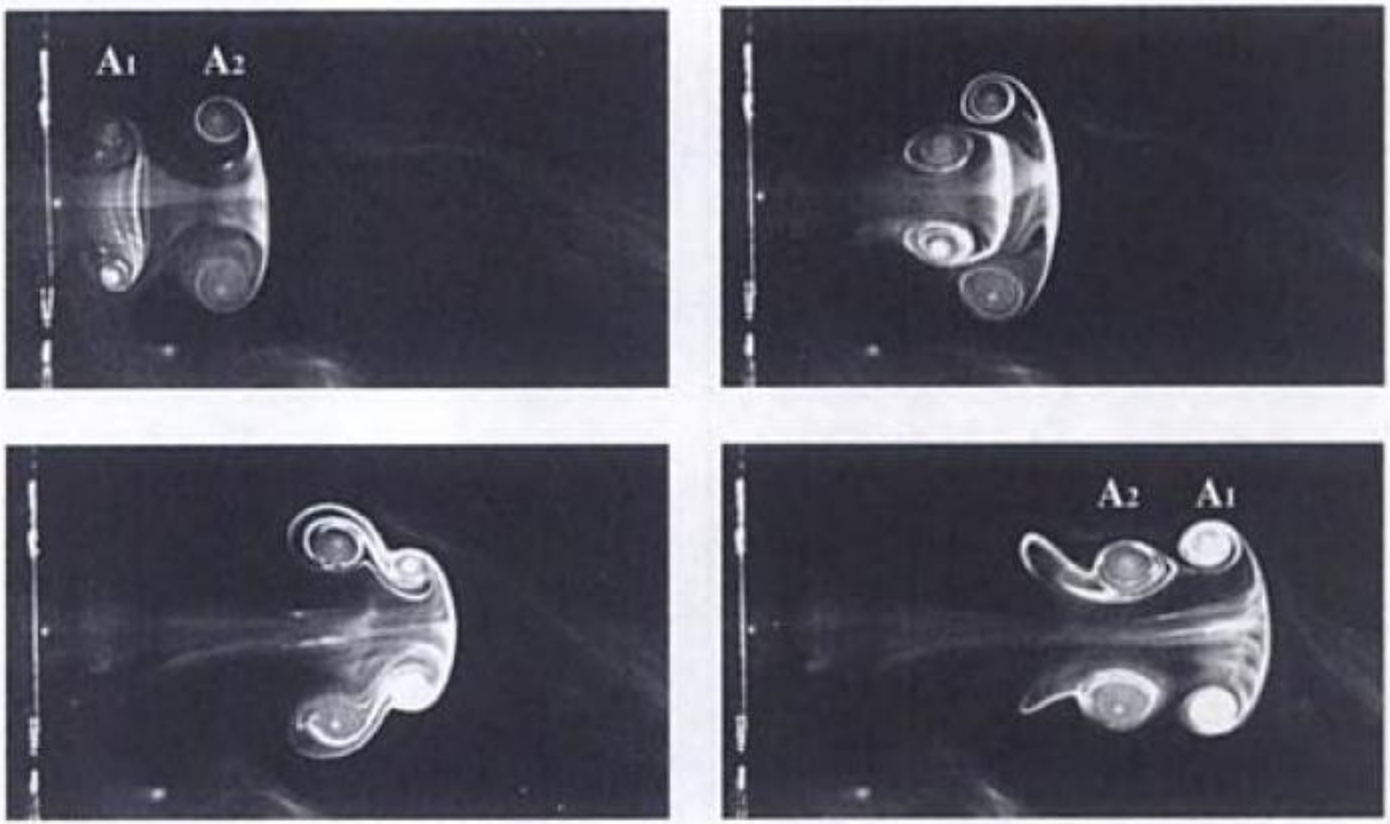


FIG. 7.23 – *Le mouvement relatif « en saute-mouton » de deux anneaux tourbillons coaxiaux est décrit par cette séquence de quatre photos (extraite de* An Album of Fluid Motion *de M. Van Dyke).*

For 2D problem:

$(u,v,p) \longrightarrow (\omega, \psi)$

Winning: two variables instead of three

Losses: difficulties with boundary conditions for streamfunction

For 3D problem: generalized Helmholtz equation

$$\frac{\partial \omega_x}{\partial t} + v\frac{\partial \omega_x}{\partial x} + u\frac{\partial \omega_x}{\partial y} + w\frac{\partial \omega_x}{\partial z} = \omega_x \frac{\partial u}{\partial x} + \omega_y \frac{\partial u}{\partial y} + \omega_z \frac{\partial u}{\partial z} + \nu \Delta \omega_x \quad ,$$

$$\frac{\partial \omega_y}{\partial t} + v\frac{\partial \omega_y}{\partial x} + u\frac{\partial \omega_y}{\partial y} + w\frac{\partial \omega_y}{\partial z} = \omega_x \frac{\partial v}{\partial x} + \omega_y \frac{\partial v}{\partial y} + \omega_z \frac{\partial v}{\partial z} + \nu \Delta \omega_y \quad ,$$

$$\frac{\partial \omega_z}{\partial t} + v\frac{\partial \omega_z}{\partial x} + u\frac{\partial \omega_z}{\partial y} + w\frac{\partial \omega_z}{\partial z} = \omega_x \frac{\partial w}{\partial x} + \omega_y \frac{\partial w}{\partial y} + \omega_z \frac{\partial w}{\partial z} + \nu \Delta \omega_z \quad ,$$

where $\Delta = \partial^2 / \partial x^2 + \partial^2 / \partial y^2 + \partial^2 / \partial z^2$

$$\vec{\omega} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

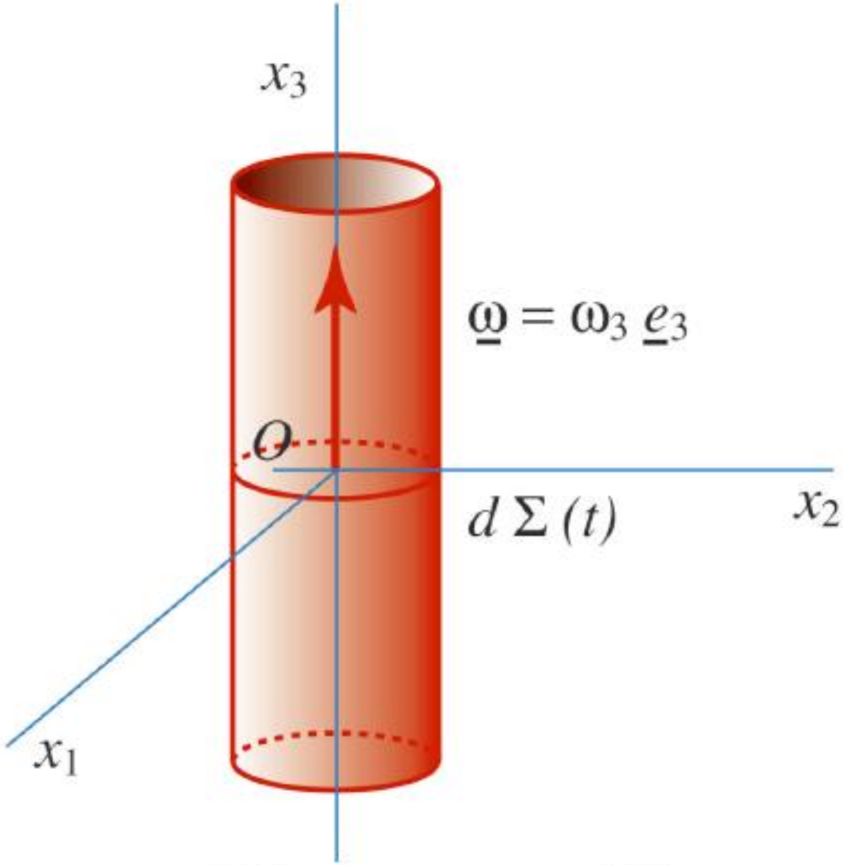
For 3D problem we can not introduce streamfuction like for 2D problem.

$$\frac{d\underline{\omega}}{dt} = \underline{\underline{\mathrm{grad}\ U}} \cdot \underline{\omega} - \underline{\omega}\ \mathrm{div}\ \underline{U} \qquad + \underline{\mathrm{rot}}\left(\frac{1}{\rho}\ \mathrm{div}\ \underline{\underline{\tau}}\right)$$

Tilting/stretching

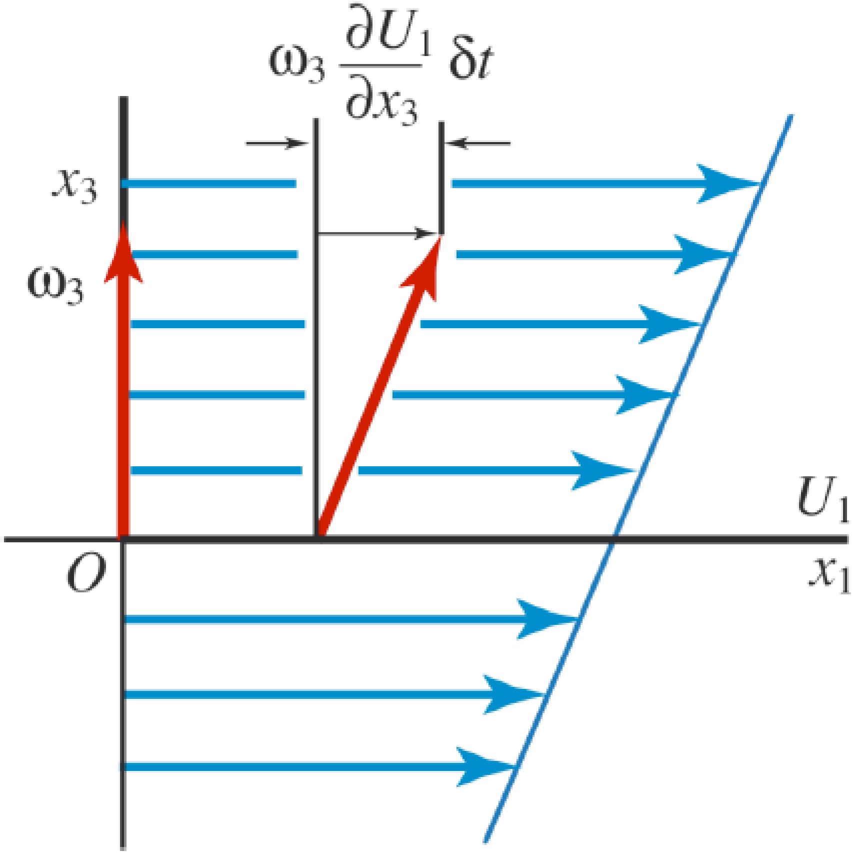
Diffusion

Tilting/stretching of vorticity tubes



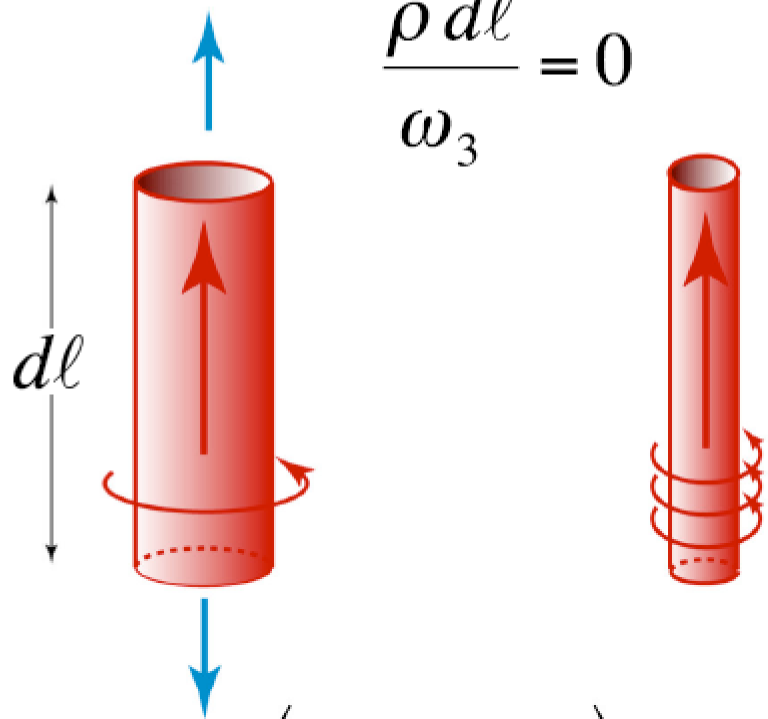
$$\frac{d\underline{\omega}}{dt} = \omega_3 \frac{\partial U_1}{\partial x_3} \underline{e}_1 + \omega_3 \frac{\partial U_2}{\partial x_3} \underline{e}_2 - \omega_3 \left(\frac{\partial U_1}{\partial x_1} + \frac{\partial U_2}{\partial x_2} \right) \underline{e}_3$$

Tilting of vorticity tubes



Stretching of vorticity tubes

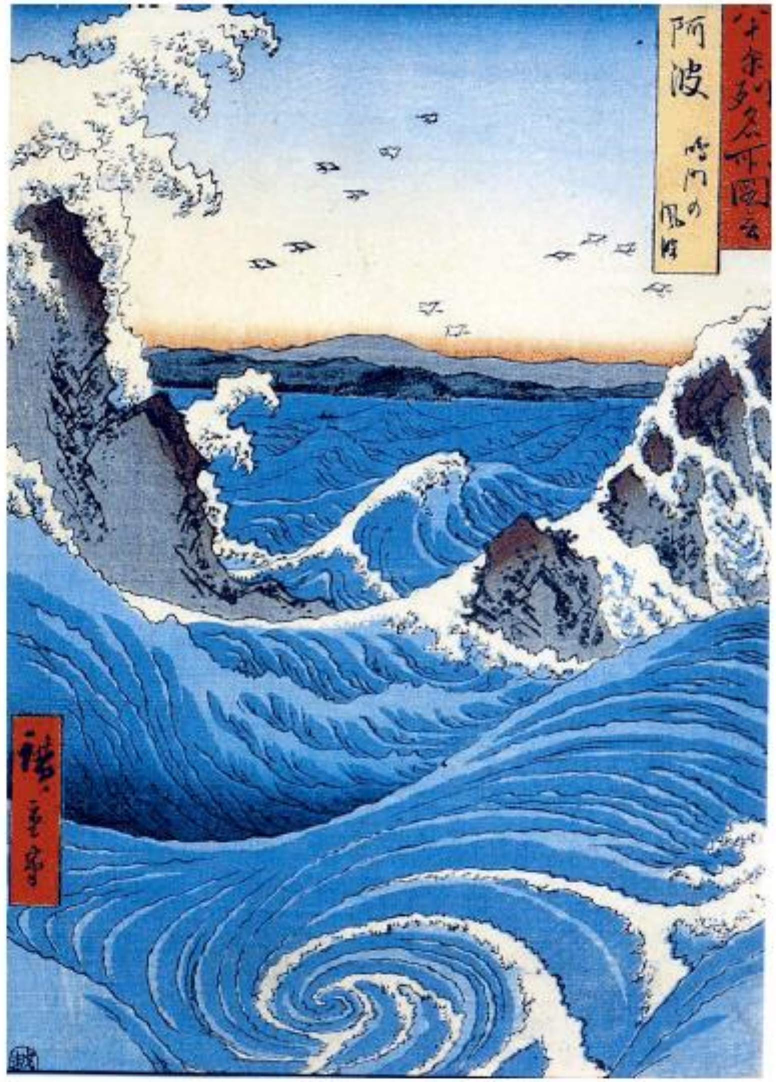
$$\frac{\dot{\overbrace{\rho\, d\ell}}}{\omega_3} = 0$$



$$\frac{1}{\omega_3}\frac{d\omega_3}{dt} = -\left(\frac{\partial U_1}{\partial x_1} + \frac{\partial U_2}{\partial x_2}\right) = -\mathrm{div}\underline{U} + \frac{\partial U_3}{\partial x_3}$$

$$\frac{1}{\omega_3}\frac{d\omega_3}{dt} = \frac{1}{\rho}\frac{d\rho}{dt} + \frac{1}{d\ell}\frac{d\ell}{dt}$$

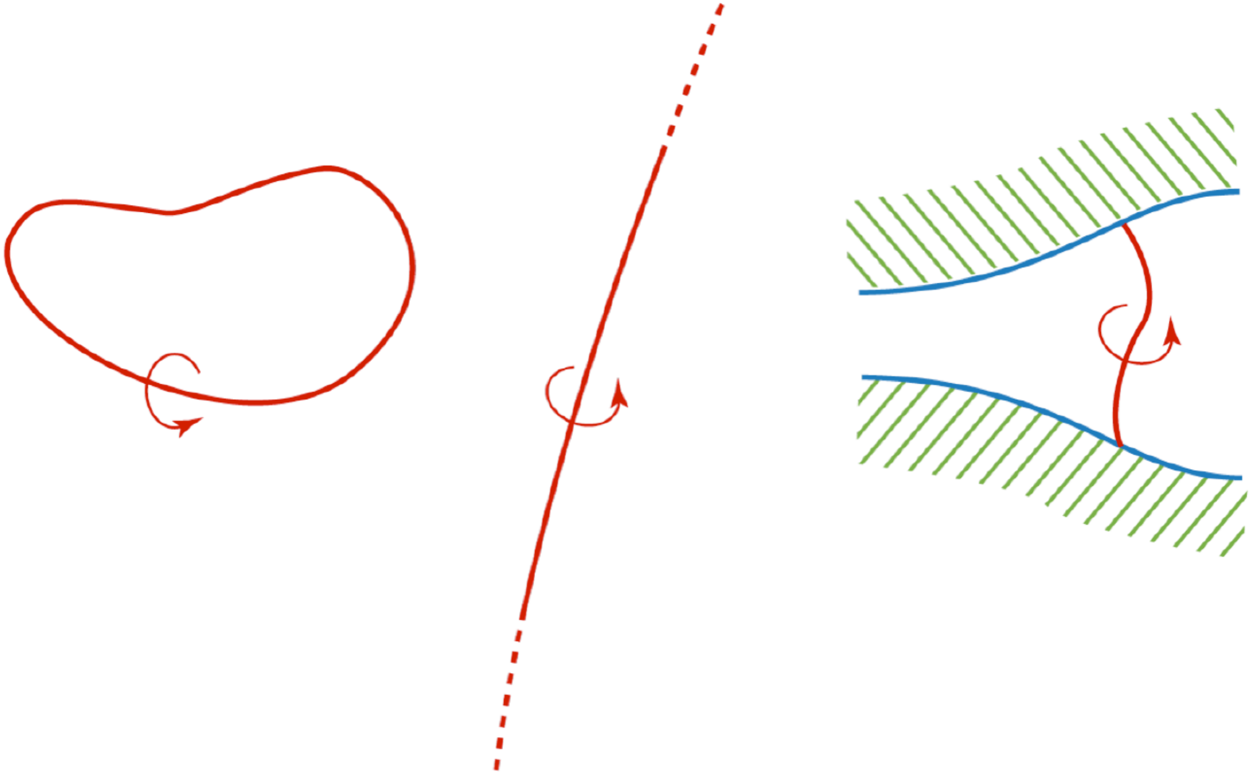
Le tourbillon de Naruto



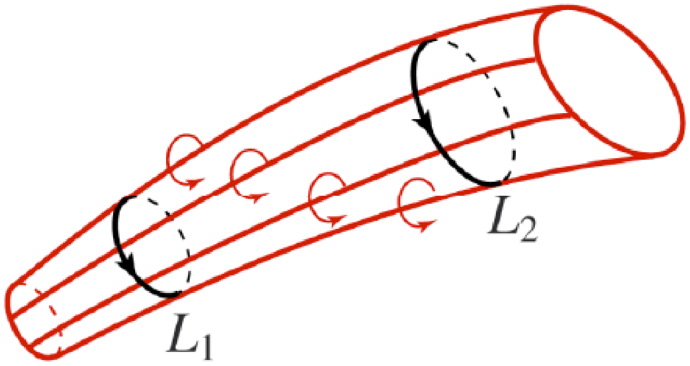
Hiroshige (1855)

Kinematic properties of the vorticity field

$$\operatorname{div} \underline{\omega} = 0$$



Circulation



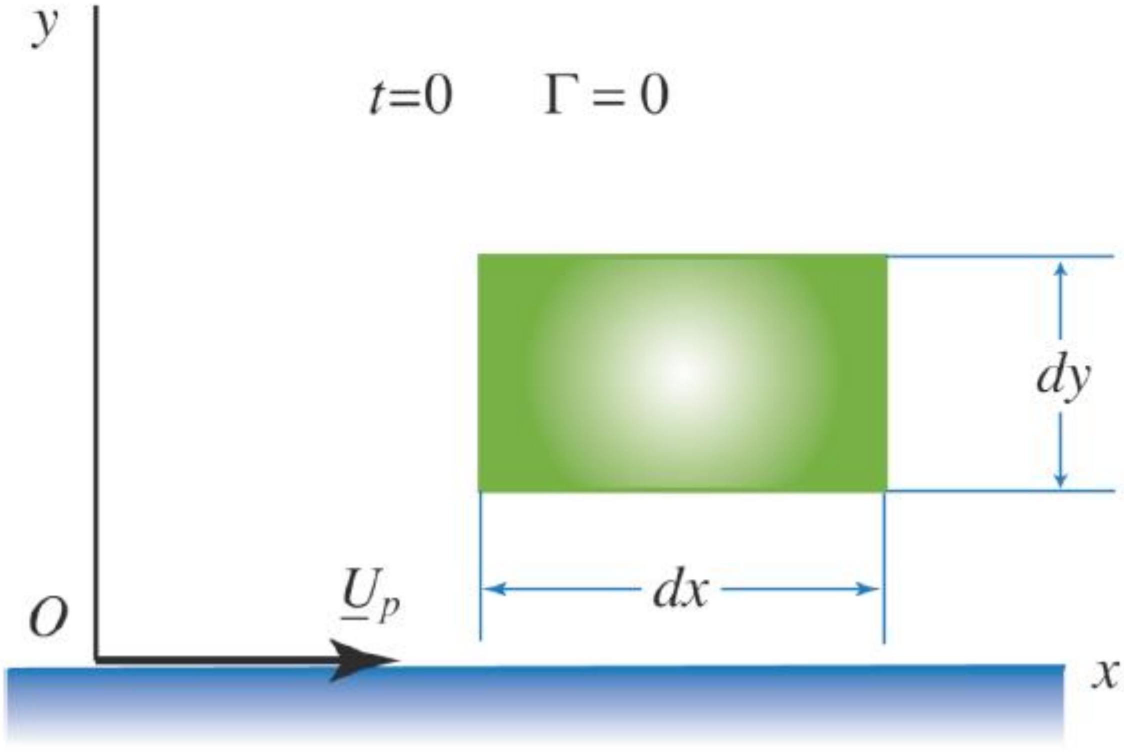
$$\Gamma(t) = \int_{L_i} \underline{U} \cdot \underline{d\ell}$$

$$\int_{\Sigma_1} \underline{\omega} \cdot \underline{n}\, da = \int_{\Sigma_2} \underline{\omega} \cdot \underline{n}\, da$$

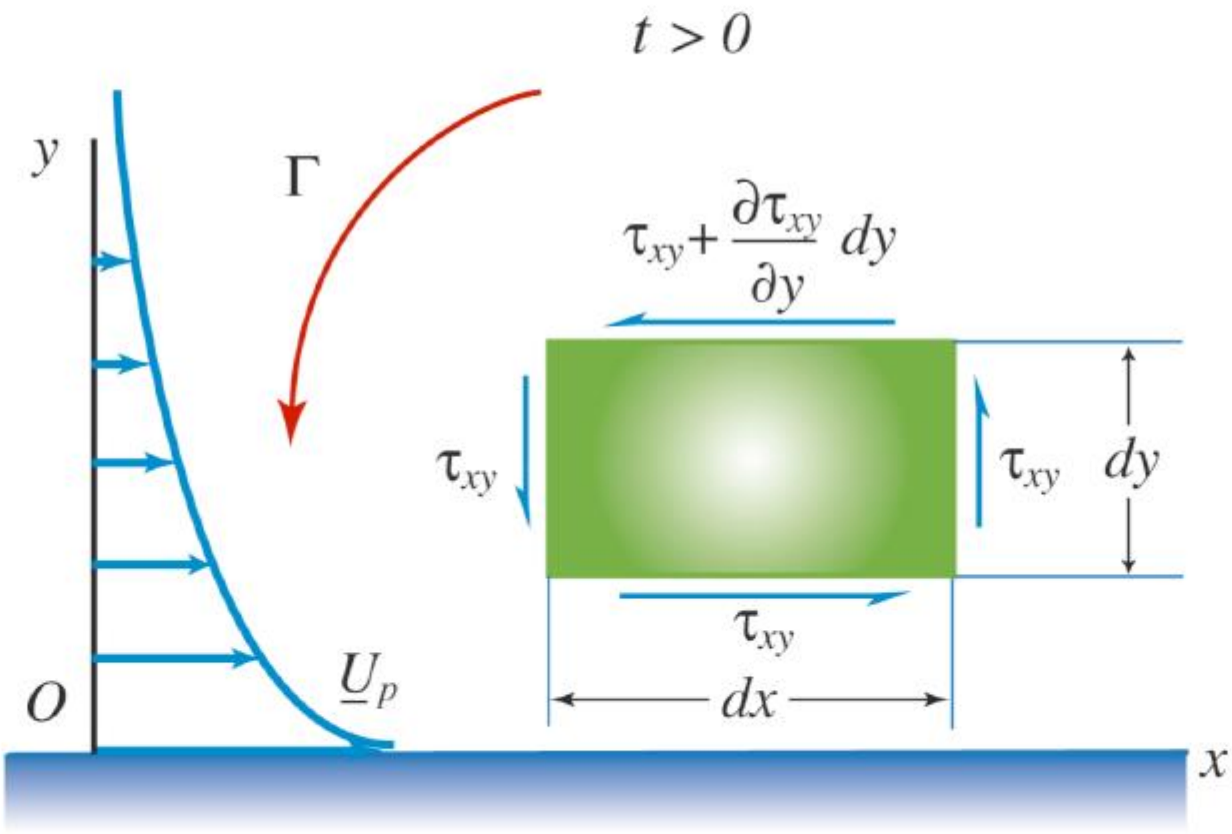
$$\int_{L_1} \underline{U} \cdot \underline{d\ell} = \int_{L_2} \underline{U} \cdot \underline{d\ell}$$

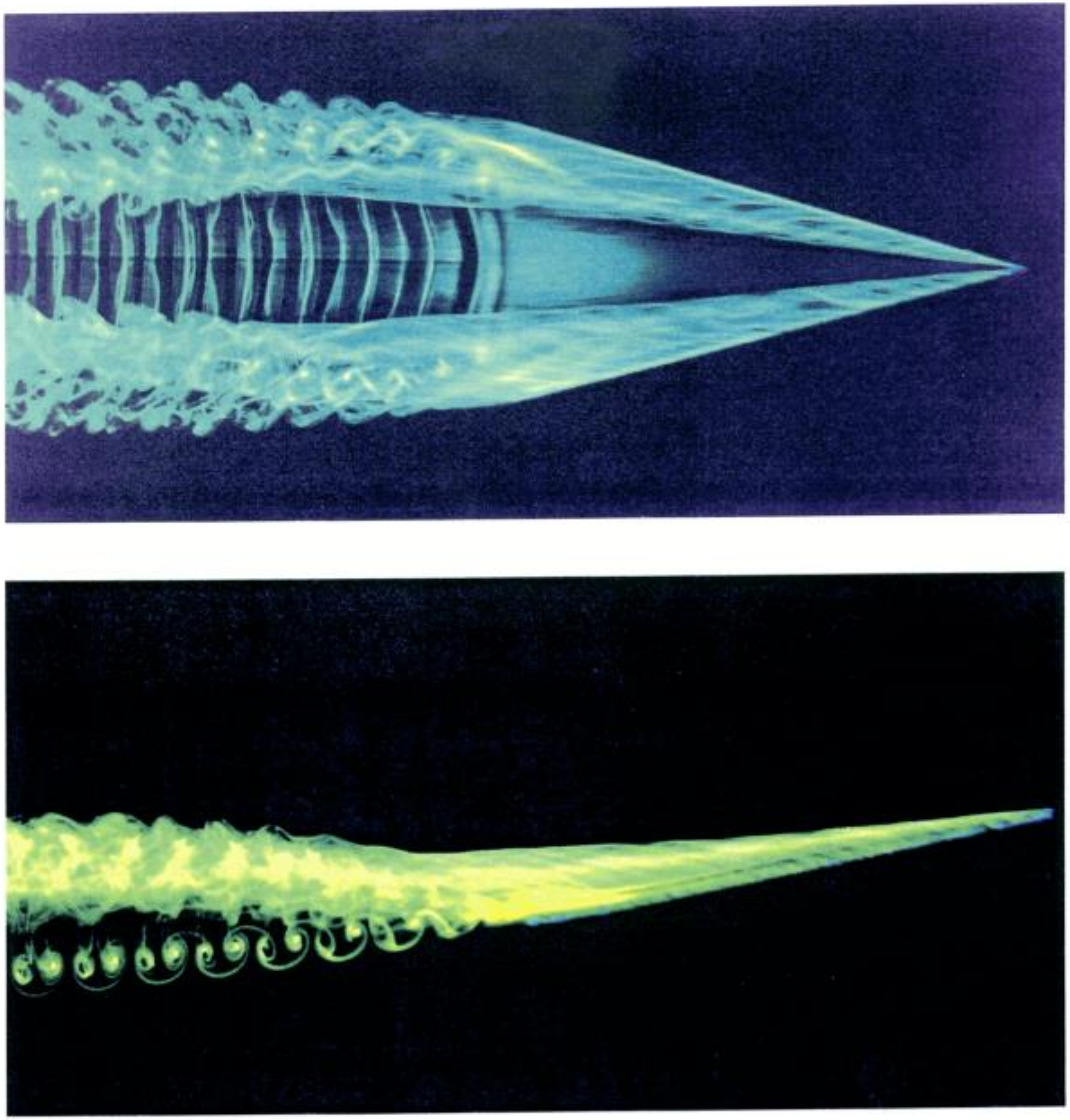
$$\Gamma(t) = \int_{L} \underline{U} \cdot \underline{d\ell}$$

Circulation production due to viscous stress



Circulation production due to viscous stress





C.Williamson

Kelvin theorem

Inviscid flow $\underline{\underline{\tau}} = \underline{q} = 0$

Barotropic fluid : $p = p(\rho)$

Conservative forces : $\underline{F} = -\underline{\text{grad}}\ \Phi$

$$\frac{d}{dt}\int_{L(t)} \underline{U}\cdot\underline{d\ell} = \frac{d}{dt}\int_{\Sigma(t)} \underline{\omega}\cdot\underline{n}\, d\,\Sigma = 0$$

Lagrange theorem

Inviscid flow, Barotropic fluid

Conservative forces

$$\underline{\omega}(\underline{x},0)=0 \quad \forall \underline{x}$$

$$\Downarrow$$

$$\underline{\omega}(\underline{x},t)=0 \quad \forall \underline{x}\ ,\ \forall t$$

Inviscid flow

Daniel Bernoulli 1700-1782

Leonhard Euler 1707-1783

Inviscid flow: Euler equations

- Continuity

$$\frac{d\rho}{dt} + \rho \,\mathrm{div}\, \underline{U} = 0$$

- Momentum conservation

$$\rho \frac{d\underline{U}}{dt} = \rho \underline{F} - \underline{\mathrm{grad}}\, p$$

Assume conservative volumn forces

$$\underline{F} = -\ \underline{\text{grad}}\ \Phi$$

Vectorial identity

$$\underline{\underline{\text{grad}\ U}} \cdot \underline{U} = \underline{\omega} \wedge \underline{U} + \underline{\text{grad}} \left(\frac{U^2}{2} \right)$$

Vorticity ω=rot(U)

Inviscid flow: Euler equations

- *Continuité*

$$\frac{d\rho}{dt} + \rho \,\mathrm{div}\, \underline{\mathrm{U}} = 0$$

- *Loi fondamentale de la dynamique*

$$\frac{\partial \underline{U}}{\partial t} + \underline{\mathrm{grad}}\left(\frac{U^2}{2} + \Phi\right) + \underline{\omega} \wedge \underline{U} = -\frac{1}{\rho}\,\underline{\mathrm{grad}}\, p$$

- *Énergie interne*

$$\rho \frac{de}{dt} = -p \,\mathrm{div}\, \underline{\mathrm{U}}$$

- *Équations d'état*

$$p = p(\rho, T)$$

$$e = e(\rho, T)$$

Conservation of enthalpy
1st Bernoulli theorem

Assumptions

- Steady flow
- inviscid,
- conservative volumn forces

$$H = h + \frac{U^2}{2} + \Phi = \text{ const. sur ligne de courant}$$

$$h = e + \frac{p}{\rho}$$

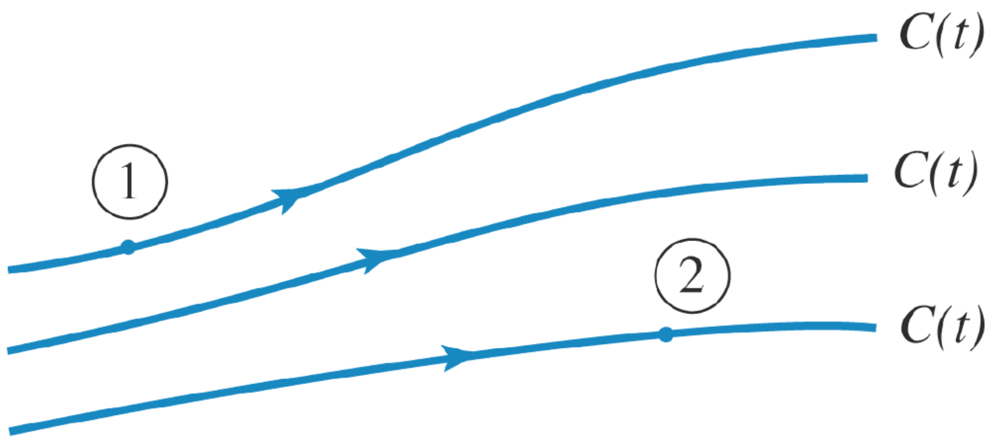
2nd Bernoulli theorem

Assumptions

- Irrotational flow
- inviscid
- Barotropic ρ(p) only
- conservative volumn forces

$$\underline{U} = \underline{\text{grad}}\ \varphi$$

$$\frac{\partial \varphi}{\partial t} + \int \frac{dp}{\rho} + \frac{U^2}{2} + \Phi = C(t)$$



Incompressible flow

e = constant on a streamline

1er théorème de B.

$$\frac{p}{\rho} + \frac{U^2}{2} + \Phi = \text{constant on a streamline}$$

2è théorème de B.

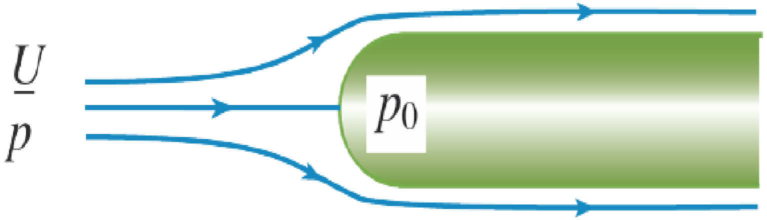
$$\frac{\partial \varphi}{\partial t} + \frac{p}{\rho} + \frac{U^2}{2} + \Phi = C(t)$$

Steady, inviscid, incompressible flow

$$\underline{F} = 0$$

$$p + \frac{1}{2}\rho U^2 = p_0$$

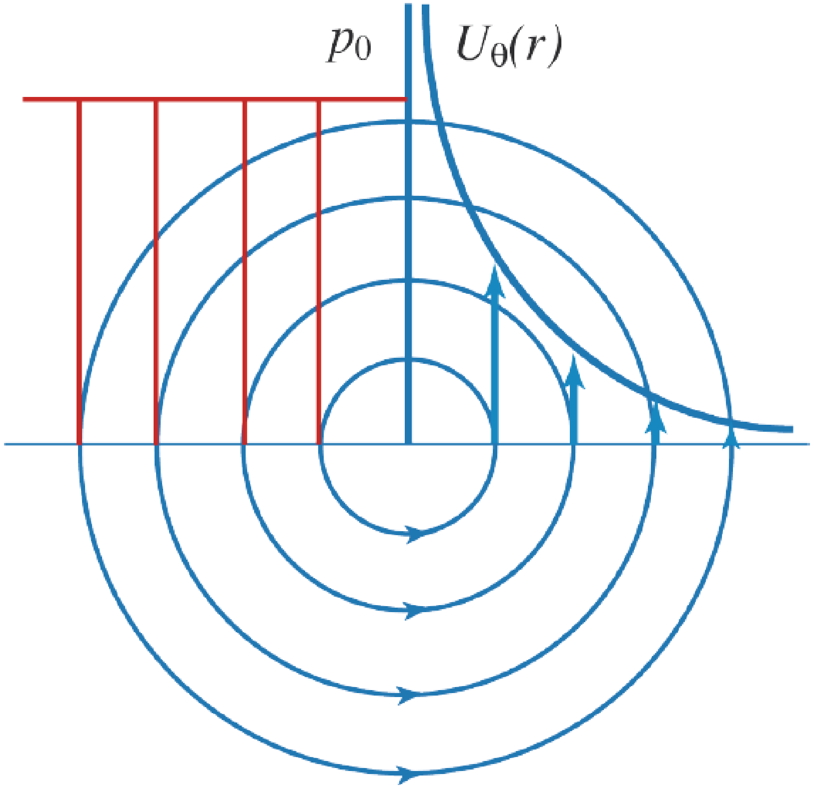
pression statique + pression dynamique = pression totale



Pitot tube

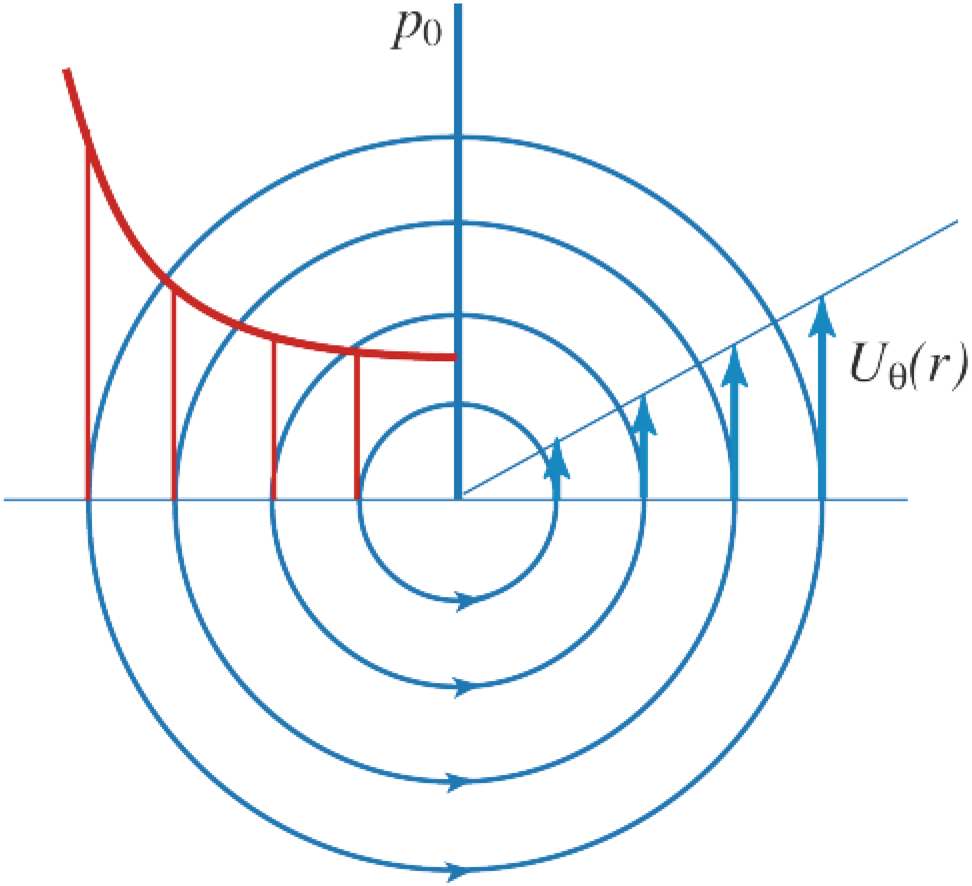
Point vortex

$$U_\theta(r) = \frac{\Gamma}{2\pi r}$$

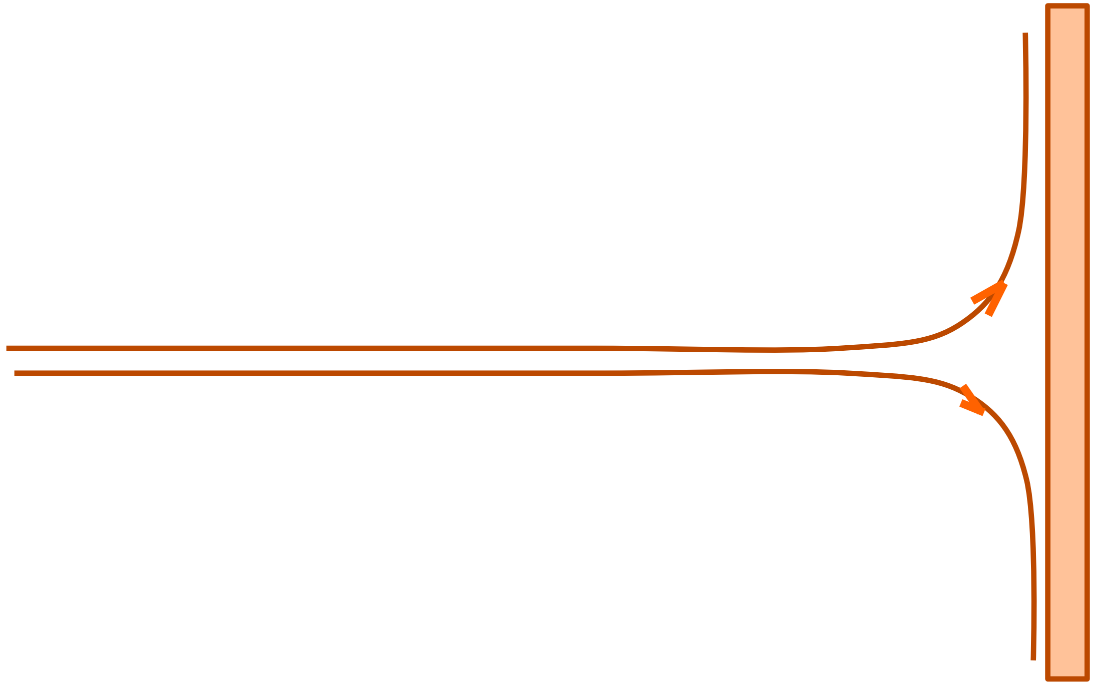


Solid body rotation

$$U_\theta(r) = 2\omega r$$

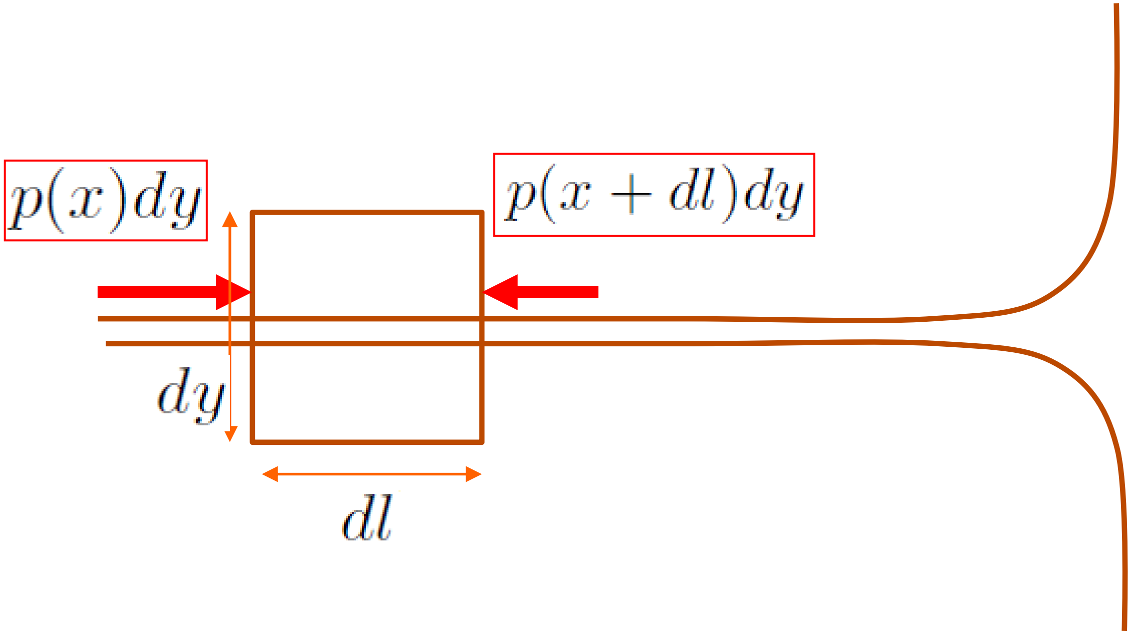


When the Reynolds number tends to infinity (inviscid flow), the pressure only serves to accelerate or decelerate the flow



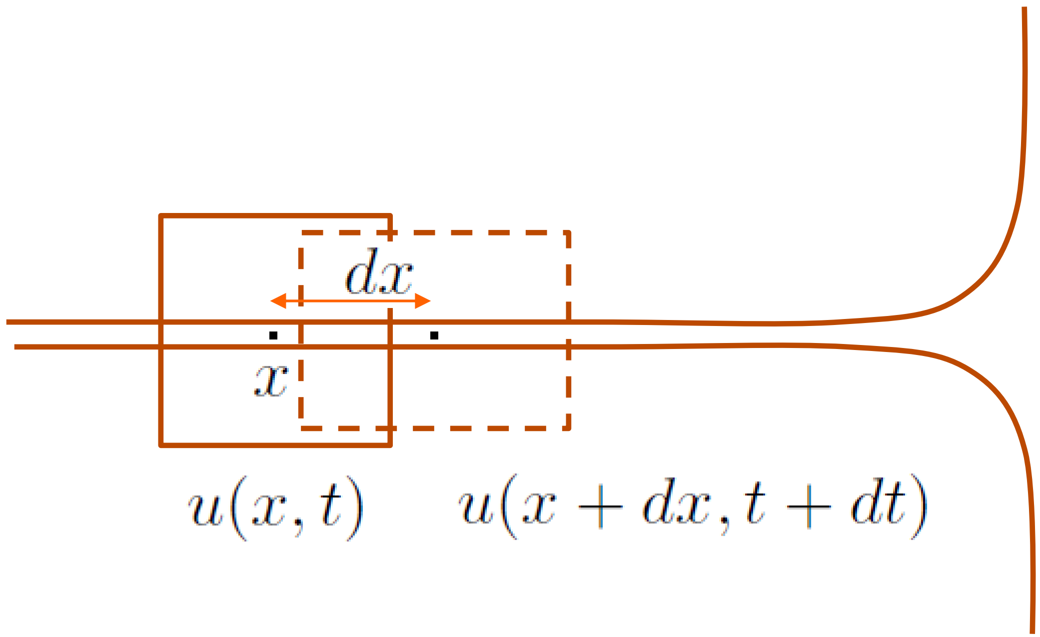
Example: stagnation point flow

Fluid parcel in a stagnation point flow



For Re= ∞ viscous forces become negligible

During dt, the particle goes from x to x+dx and accelerates



$$u(x+dx, t+dt) = u(x,t) + \frac{du}{dx}dx$$
$$= u(x,t) + \frac{du}{dx}udt$$

ma =sum of forces

$$\rho a dl dy = p(x)dy - p(x+dl)dy$$

$$\rho \frac{1}{2}\frac{du^2}{dx} dl dy = -\frac{dp}{dx} dl dy$$

$$\frac{u^2}{2} + \frac{p}{\rho} = C$$

(le long d'une ligne de courant)

Stagnation point flow

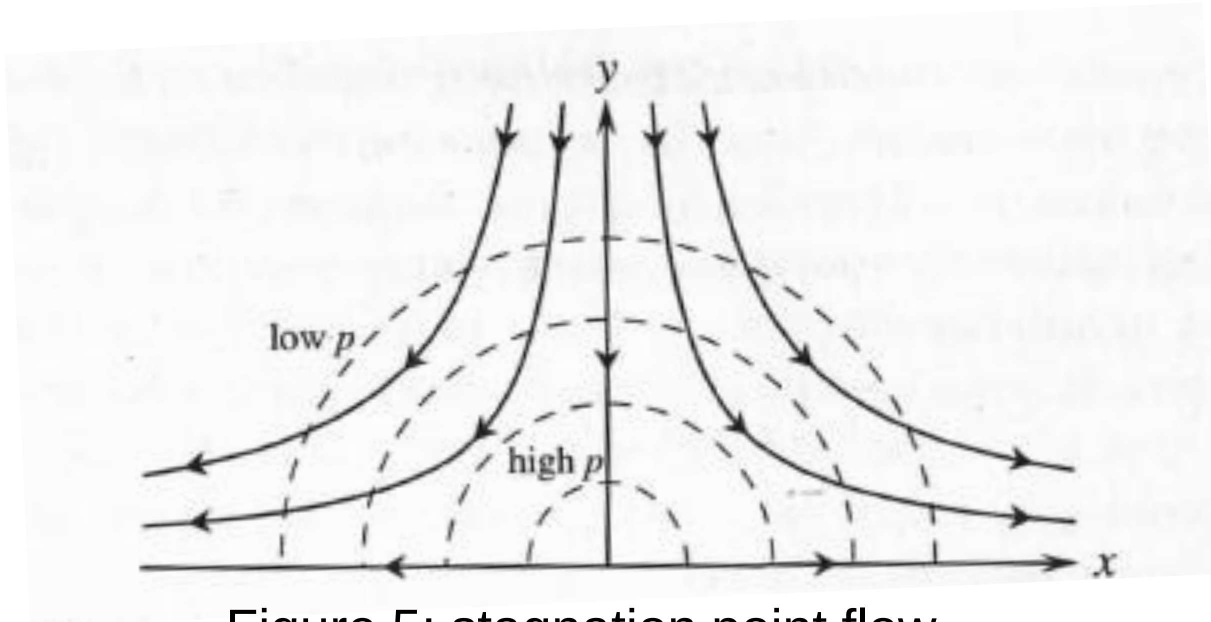
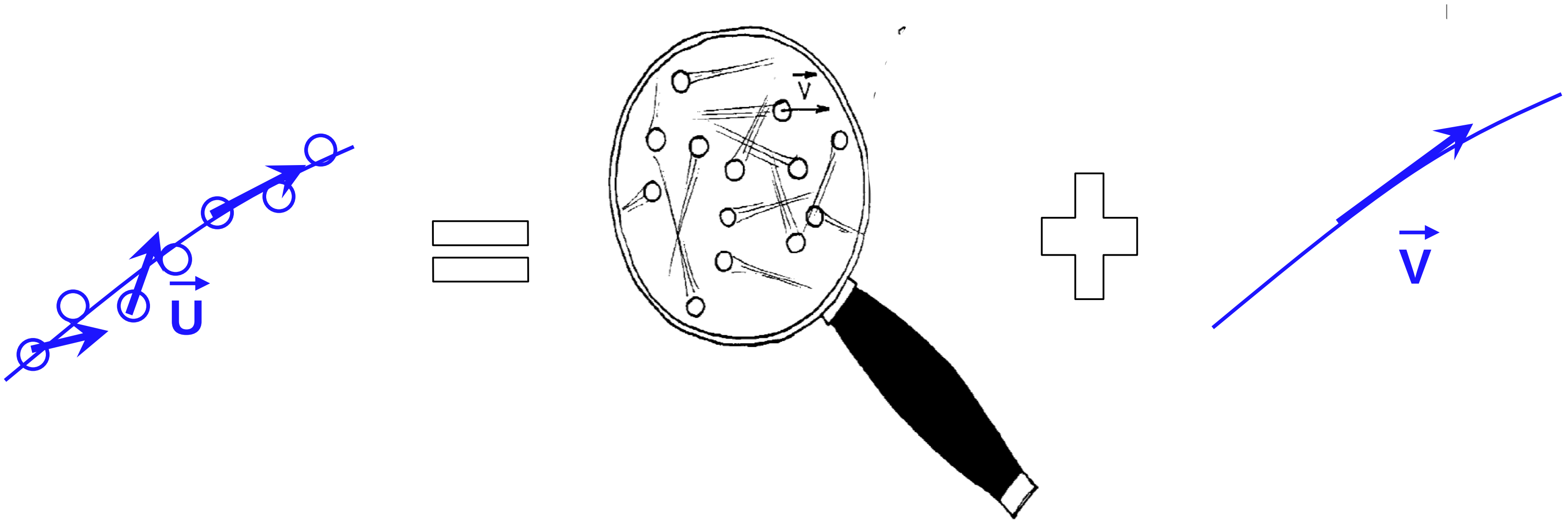


Figure 5: stagnation point flow

Beware, the flow is not going from high pressures to low pressures. By going from the low pressure regions to the high pressure regions, it slows down.

Vitesse d’une particule fluide



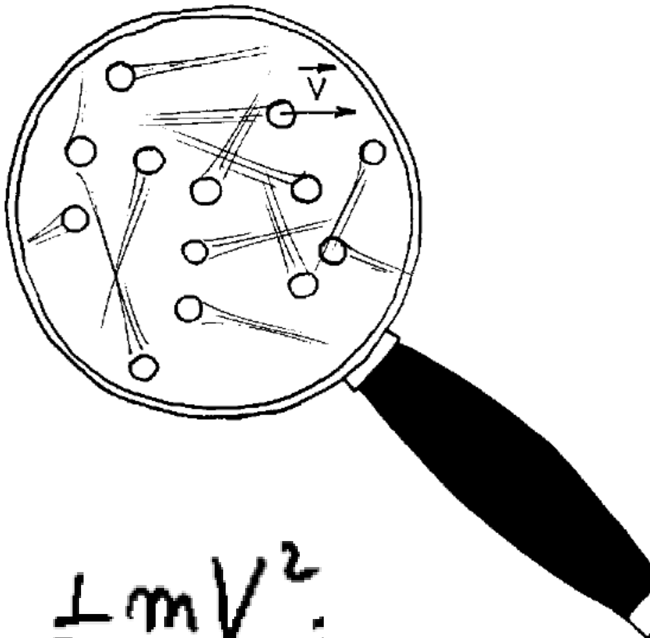
Vitesse = fluctuation + vitesse moyenne

$$\vec{U} = \vec{v} + \vec{V}$$

340 m/s

Conservation de l'énergie

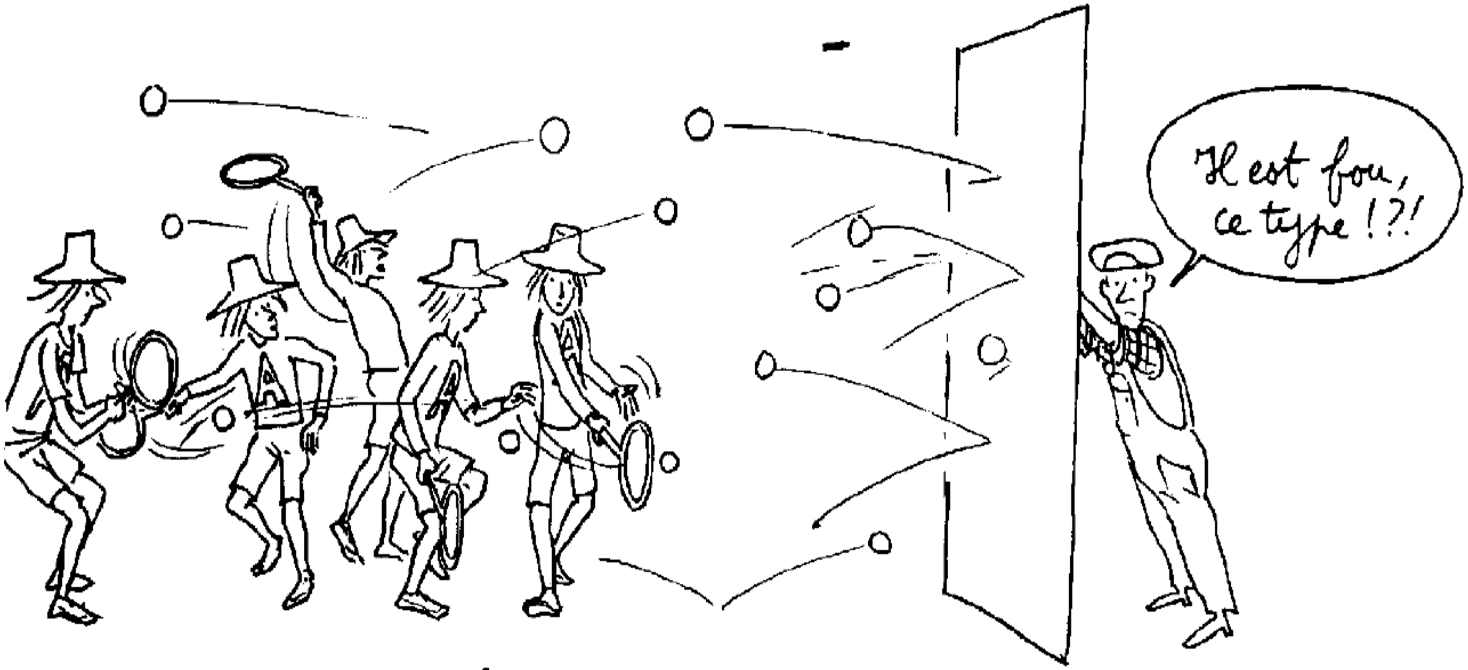
L'ÉNERGIE THERMIQUE



+ L'ÉNERGIE CINÉTIQUE $\frac{1}{2}mV^2$

= constante

La pression



Ce sont les innombrables chocs moléculaires qui se produisent sur une paroi qui créent ce phénomène qu'on nomme PRESSION.

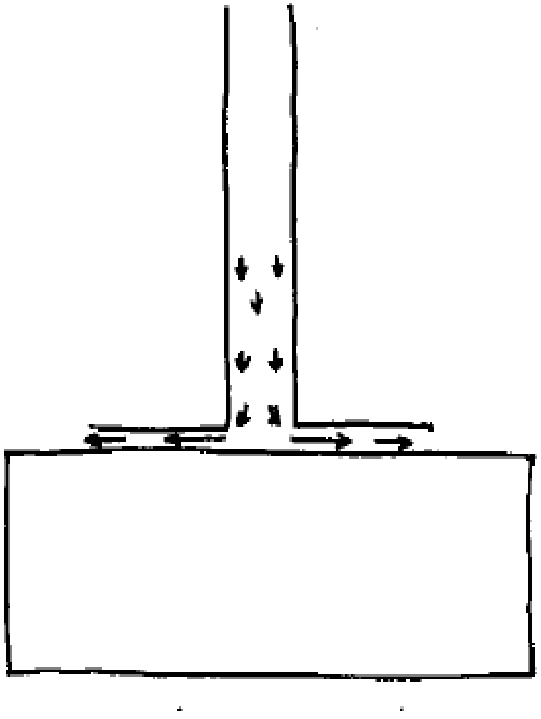
LOi DE BERNOULLi :

Pression et vitesse
varient inversement.

Daniel Bernoulli (1700-1782)

How can one exert a succion force while blowing?

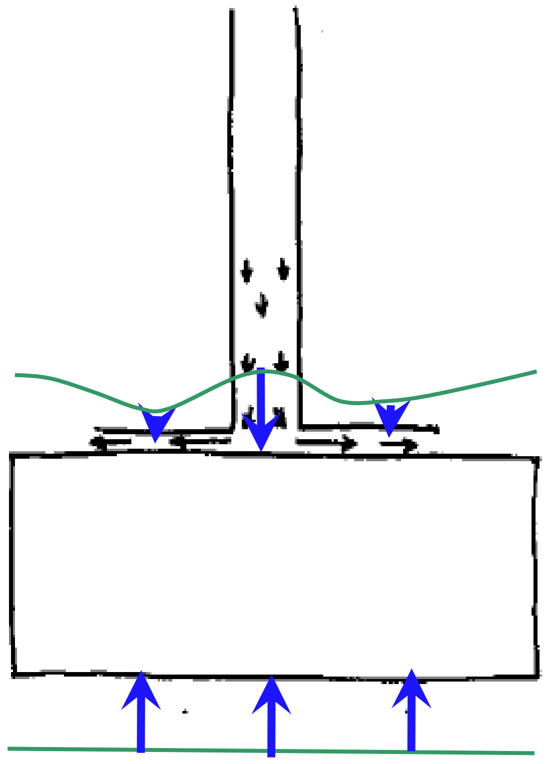
velocity excess



LOi DE BERNOULLi :

Pression et vitesse
varient inversement.

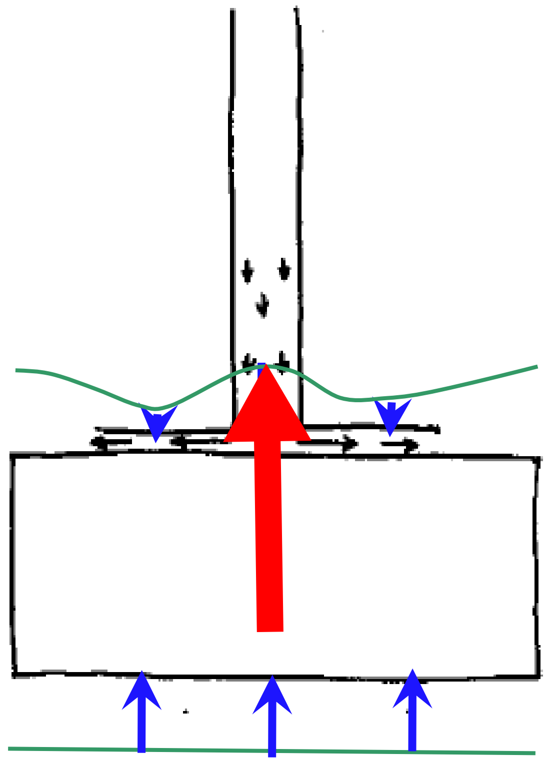
velocity excess
⇨ pressure deficit



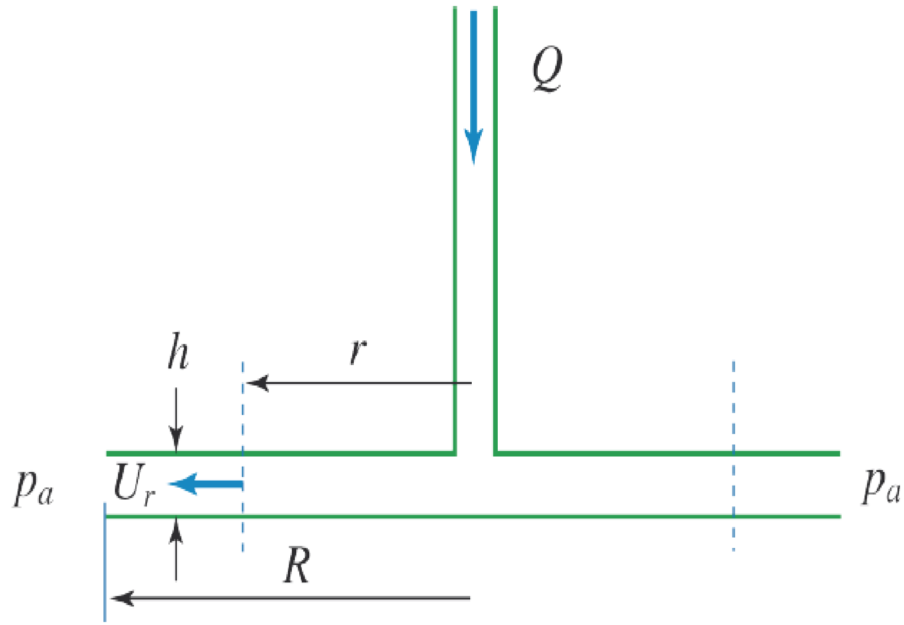
LOI DE BERNOULLI :

Pression et vitesse
varient inversement.

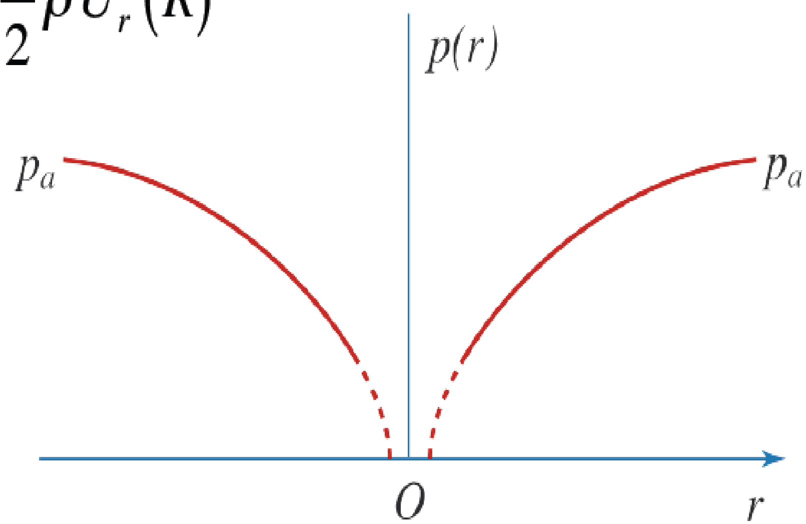
velocity excess
⇨ pressure deficit
⇨ lift



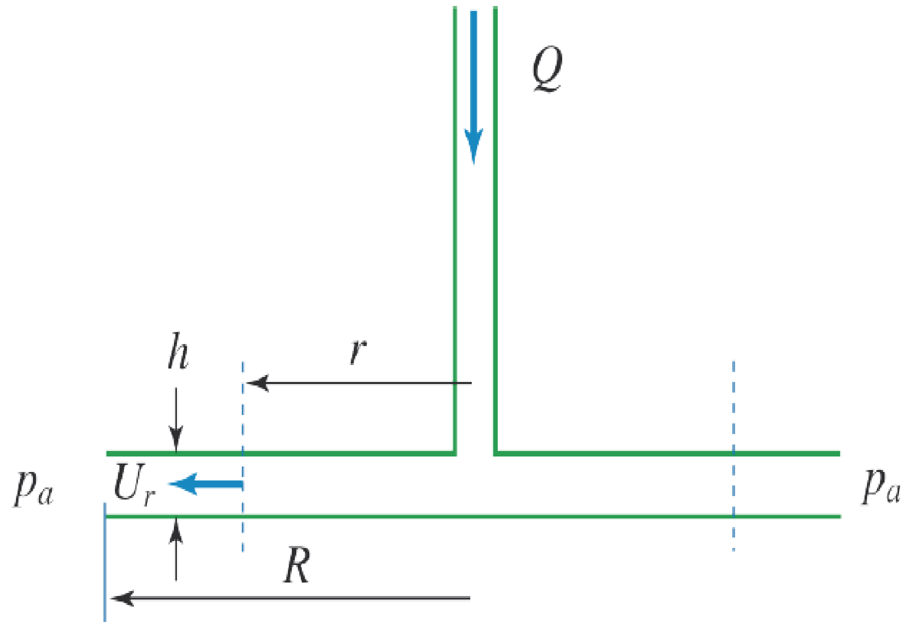
$$U_r(r) = \frac{Q}{2\pi r h}$$



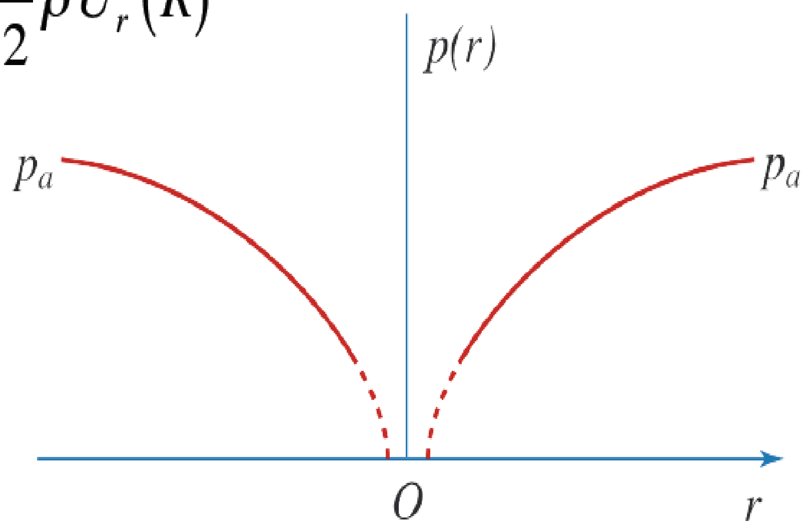
$$p(r) + \frac{1}{2}\rho U_r^2(r) = p_a + \frac{1}{2}\rho U_r^2(R)$$



$$U_r(r) = \frac{Q}{2\pi r h}$$

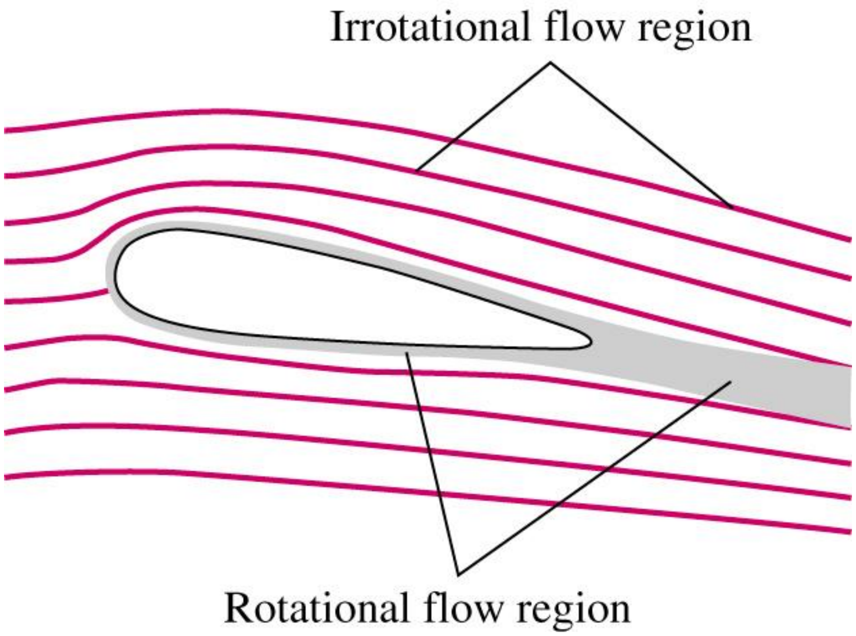


$$p(r) + \frac{1}{2}\rho U_r^2(r) = p_a + \frac{1}{2}\rho U_r^2(R)$$



Potential flow

Irrotational Flow Approximation



- Irrotational approximation: vorticity is negligibly small

$$\vec{\omega} = \nabla \times \vec{V} \cong 0$$

- In general, inviscid regions are also irrotational, but there are situations where inviscid flow are rotational, e.g., solid body rotation.

Irrotational Flow Approximation
2D Flows

- For 2D flows, we can also use the streamfunction
- Recall the definition of streamfunction for planar (x-y) flows

$$U = \frac{\partial \psi}{\partial y}, \quad V = -\frac{\partial \psi}{\partial x}$$

- Since vorticity is zero,

$$\boldsymbol{\omega} = \frac{\partial V}{\partial x} - \frac{\partial U}{\partial y} = 0$$

$$\frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial x^2} = 0$$

- This proves that the Laplace equation holds for the streamfunction

Potential flow: Irrotational, inviscid, homogeneous, incompressible flow, conservative forces

- Continuity

$$\mathrm{div}\ \underline{U} = 0$$

- Euler

$$\frac{\partial \underline{U}}{\partial t} + \underline{\mathrm{grad}}\left(\frac{U^2}{2} + \Phi\right) = -\frac{1}{\rho}\ \underline{\mathrm{grad}}\ \mathrm{p}$$

- Irrotational flow

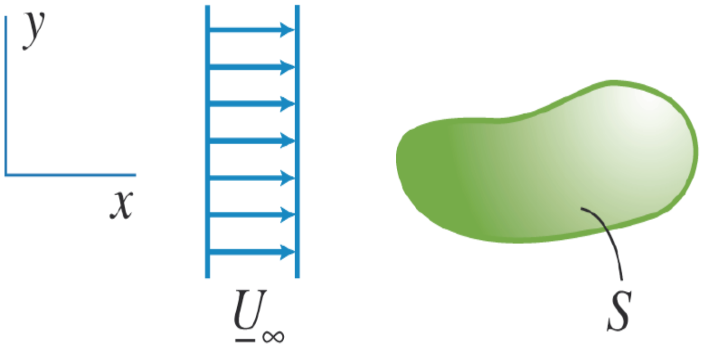
$$\underline{\mathrm{rot}}\ \underline{U} = 0$$

Potential flow

$$\Delta\psi = 0$$

$$u = \frac{\partial\psi}{\partial y} \qquad v = -\frac{\partial\psi}{\partial x}$$

Boundary conditions



$$\psi(x,y) = \text{const.} \quad \text{sur} \;\; S$$

$$\psi \sim U_\infty y \qquad , \qquad |\underline{X}| \to \infty$$

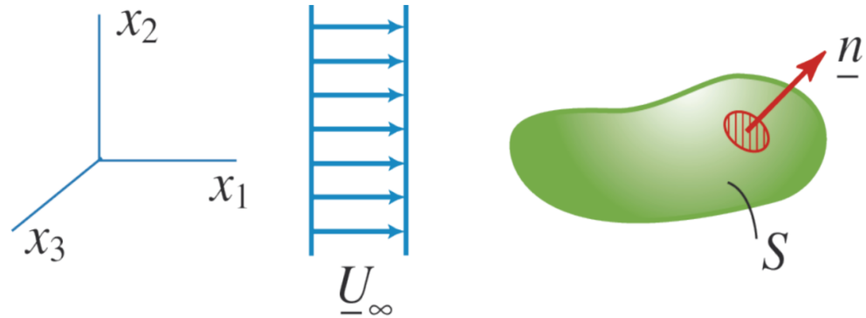
Potential flow

$$\Delta \varphi = 0$$

$$\underline{U} = \underline{\text{grad}}\ \varphi$$

$$\frac{\partial \varphi}{\partial t} + \frac{p}{\rho} + \frac{U^2}{2} + \Phi = C(t)$$

Boundary conditions

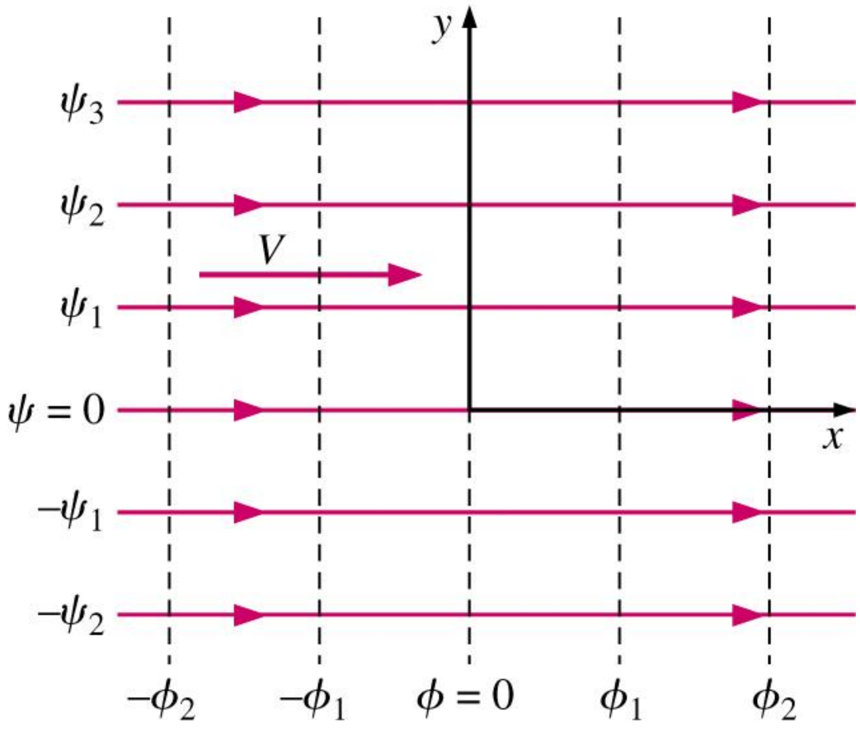


$$\underline{n} \,.\, \underline{\text{grad}}\ \varphi = 0 \quad \text{sur S}$$

$$\varphi \sim U_\infty x_1 \quad , \quad |\underline{x}| \to \infty$$

Elementary Planar Irrotational Flows

Uniform Stream



- In Cartesian coordinates

$$\phi = Vx, \quad \psi = Vy$$

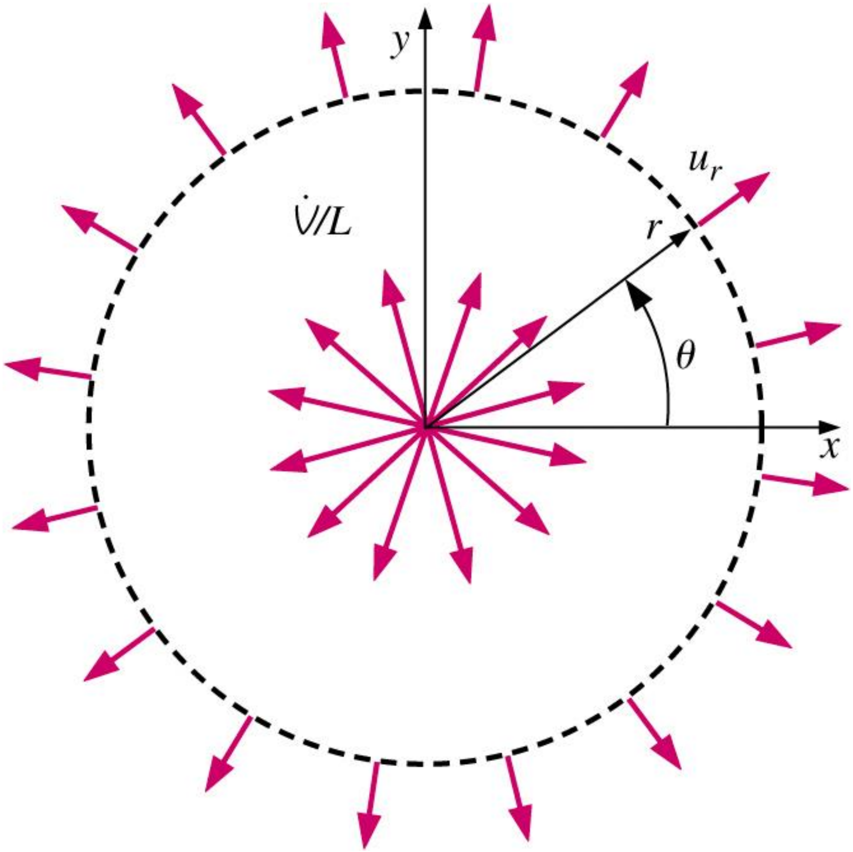
- Conversion to cylindrical coordinates can be achieved using the transformation

$$x = rcos\theta, \quad y = rsin\theta$$

$$\phi = Vrcos\theta, \quad \psi = Vrsin\theta$$

Elementary Planar Irrotational Flows

Line Source/Sink

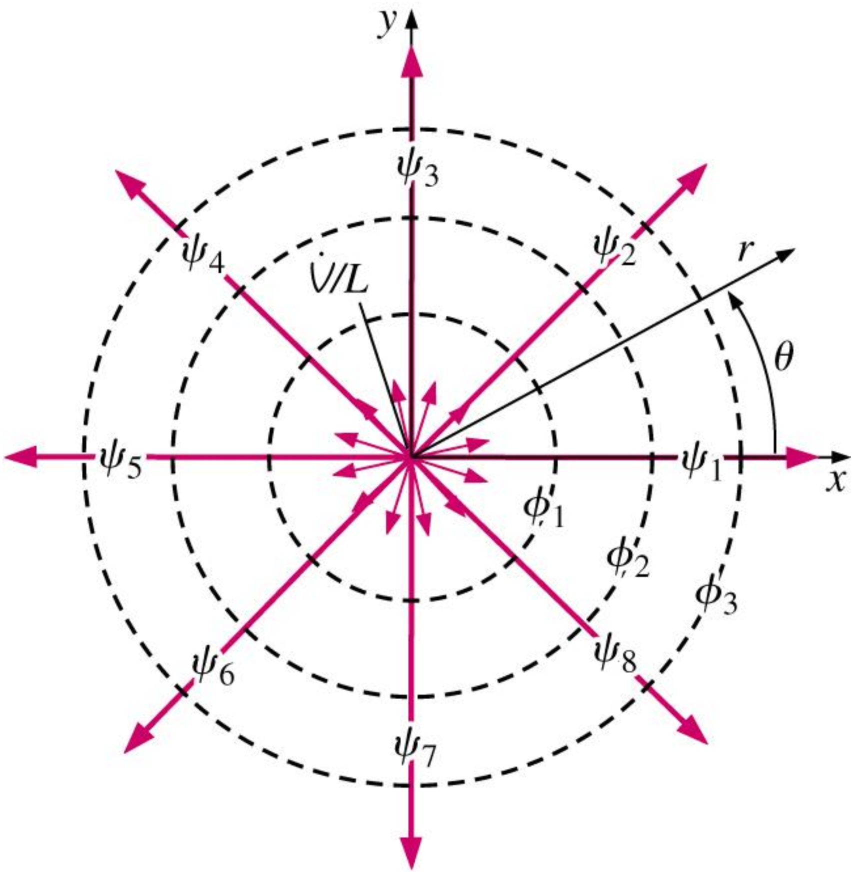


- Potential and streamfunction are derived by observing that volume flow rate across any circle is $\dot{\mathcal{V}}/L$
- This gives velocity components

$$U_r = \frac{\dot{\mathcal{V}}/L}{2\pi r}, \quad U_\theta = 0$$

Elementary Planar Irrotational Flows

Line Source/Sink



- Using definition of (U_r, U_θ)

$$U_r = \frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{\dot{\mathcal{V}}/L}{2\pi r}$$

$$U_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r} = 0$$

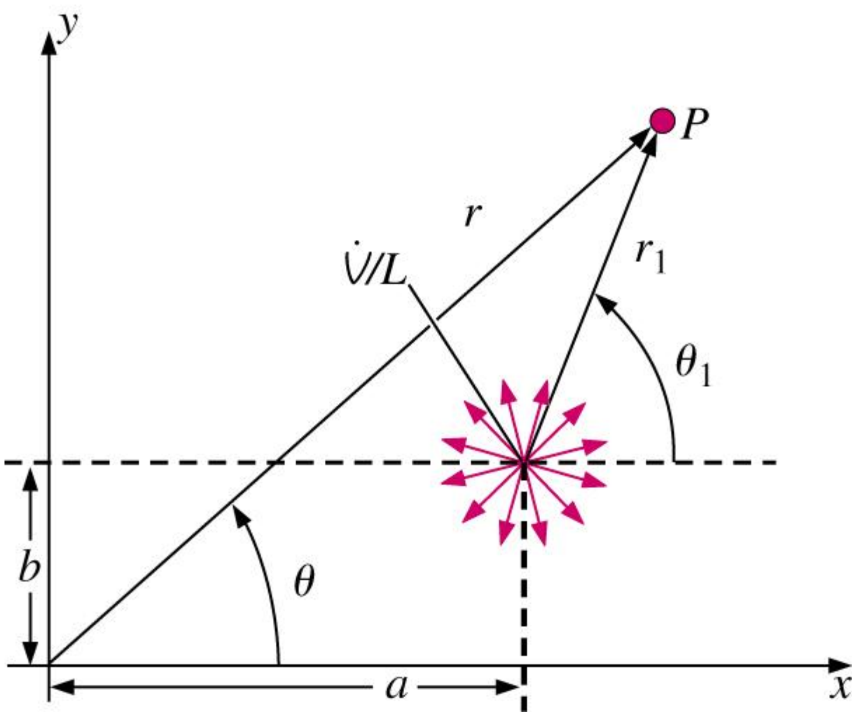
- These can be integrated to give ϕ and ψ

$$\phi = \frac{\dot{\mathcal{V}}/L}{2\pi}\ln r \qquad \psi = \frac{\dot{\mathcal{V}}/L}{2\pi}\theta$$

Equations are for a source/sink at the origin

Elementary Planar Irrotational Flows

Line Source/Sink



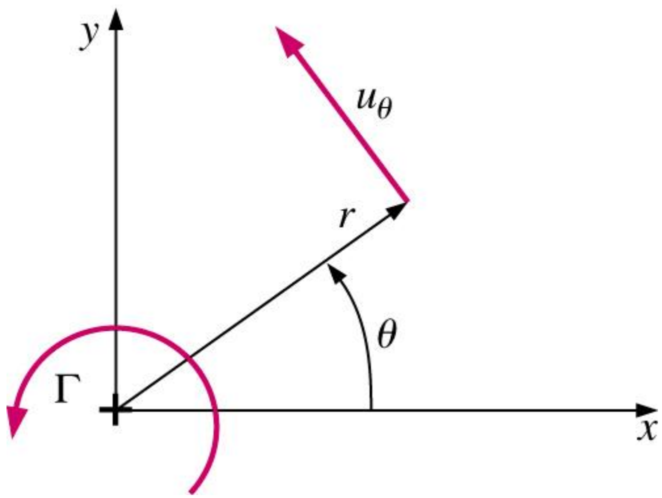
- If source/sink is moved to (x,y) = (a,b)

$$\phi = \frac{\dot{\mathcal{V}}/L}{2\pi}\ln r_1 = \frac{\dot{\mathcal{V}}/L}{2\pi}\ln\sqrt{(x-a)^2+(y-b)^2}$$

$$\psi = \frac{\dot{\mathcal{V}}/L}{2\pi}\theta_1 = \frac{\dot{\mathcal{V}}/L}{2\pi}\tan^{-1}\left(\frac{y-b}{x-a}\right)$$

Elementary Planar Irrotational Flows

Line Vortex



- Vortex at the origin. First look at velocity components

$$U_r = \frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = 0$$

$$U_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r} = \frac{\Gamma}{2\pi r}$$

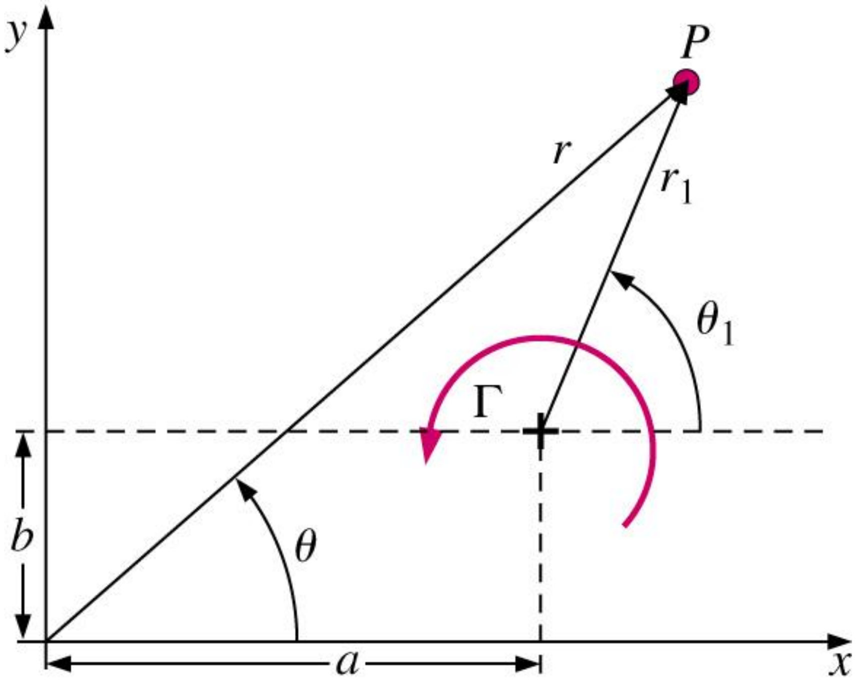
- These can be integrated to give ϕ and ψ

$$\phi = \frac{\Gamma}{2\pi}\theta \qquad \psi = -\frac{\Gamma}{2\pi}\ln r$$

Equations are for a point vortex at the origin

Elementary Planar Irrotational Flows

Line Vortex



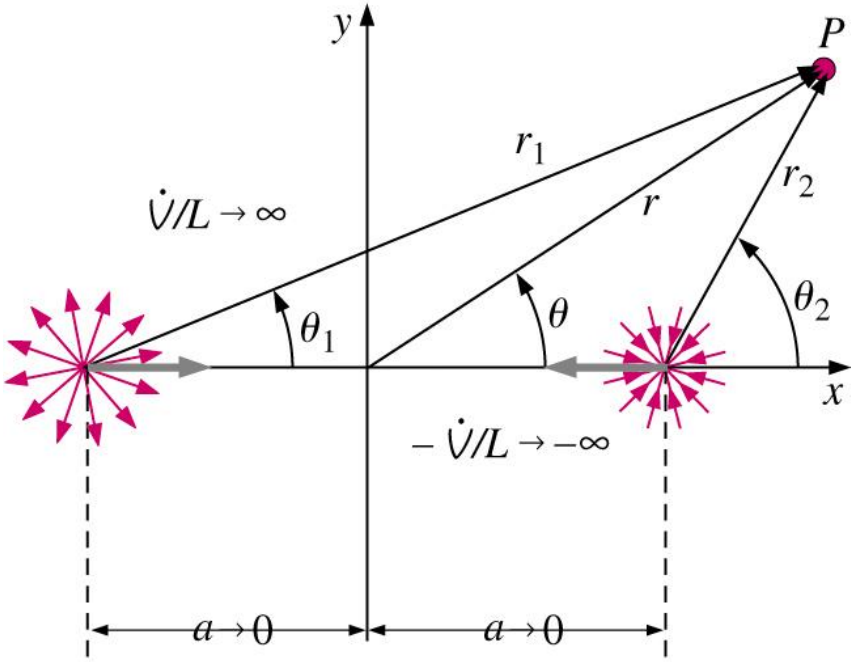
- If vortex is moved to (x,y) = (a,b)

$$\phi = \frac{\Gamma}{2\pi}\theta_1 = \frac{\Gamma}{2\pi}\tan^{-1}\left(\frac{y-b}{x-a}\right)$$

$$\psi = -\frac{\Gamma}{2\pi}\ln r_1 = -\frac{\Gamma}{2\pi}\ln\sqrt{(x-a)^2+(y-b)^2}$$

Elementary Planar Irrotational Flows

Doublet



- A doublet is a combination of a line sink and source of equal magnitude
- Source

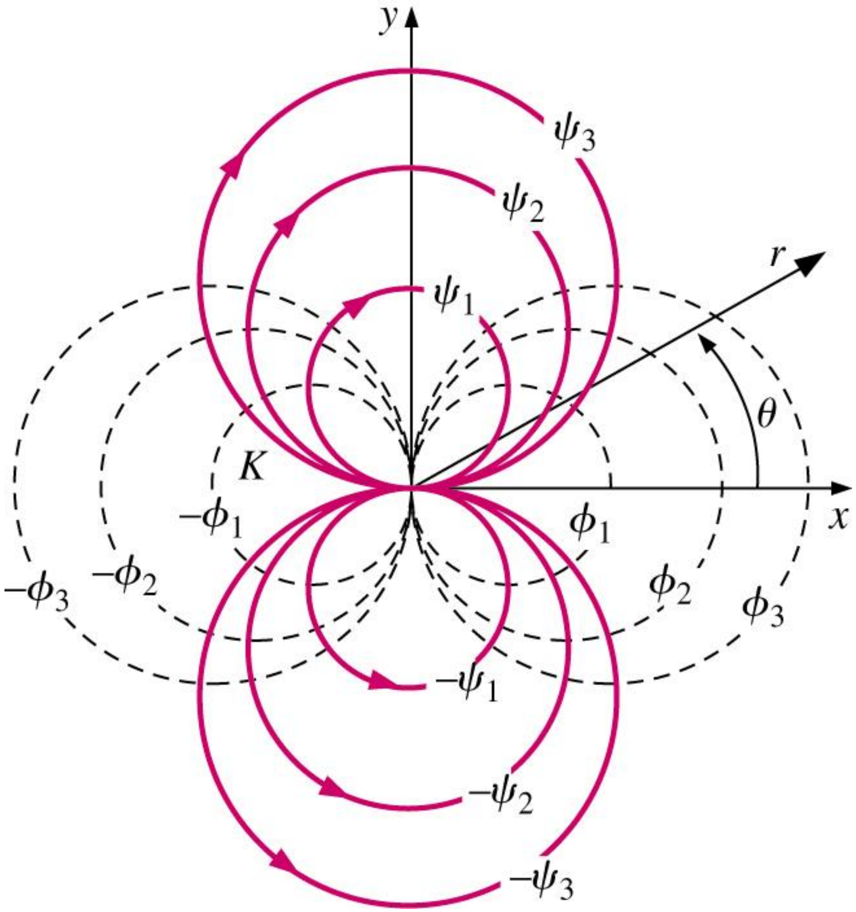
$$\psi = \frac{\dot{\mathcal{V}}/L}{2\pi}\theta_1 \qquad \theta_1 = \tan^{-1}\left(\frac{y}{x+a}\right)$$

- Sink

$$\psi = -\frac{\dot{\mathcal{V}}/L}{2\pi}\theta_2 \qquad \theta_2 = \tan^{-1}\left(\frac{y}{x-a}\right)$$

Elementary Planar Irrotational Flows

Doublet



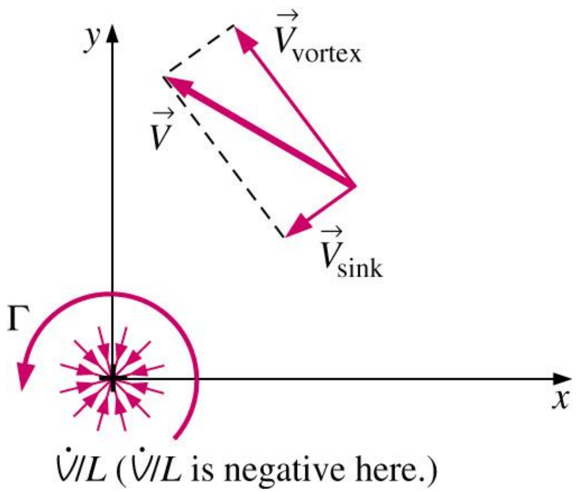
- Adding ψ_1 and ψ_2 together, performing some algebra, and taking $a \rightarrow 0$ gives

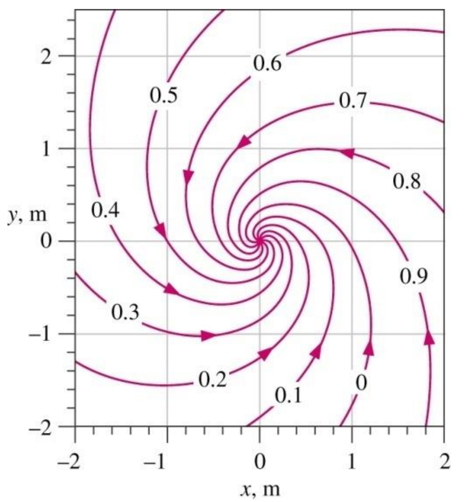
$$\psi = -K\frac{sin\theta}{r}$$

$$\phi = K\frac{cos\theta}{r}$$

K is the doublet strength

Examples of Irrotational Flows Formed by Superposition





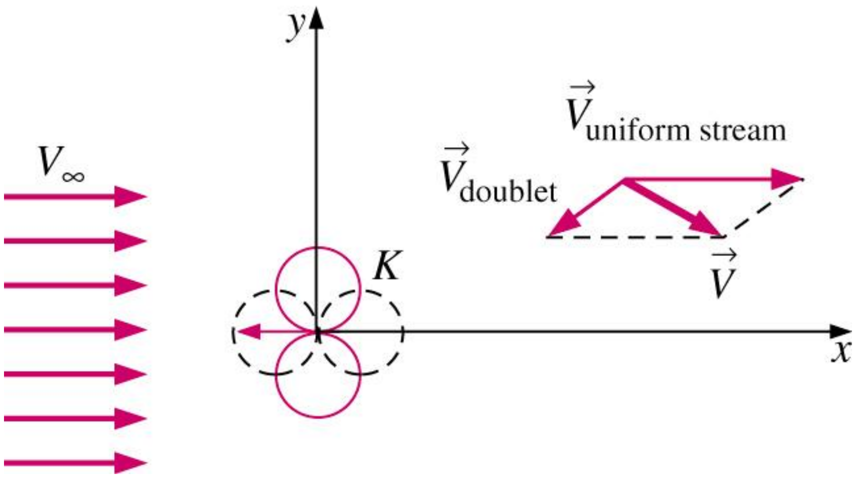
- Superposition of sink and vortex : bathtub vortex

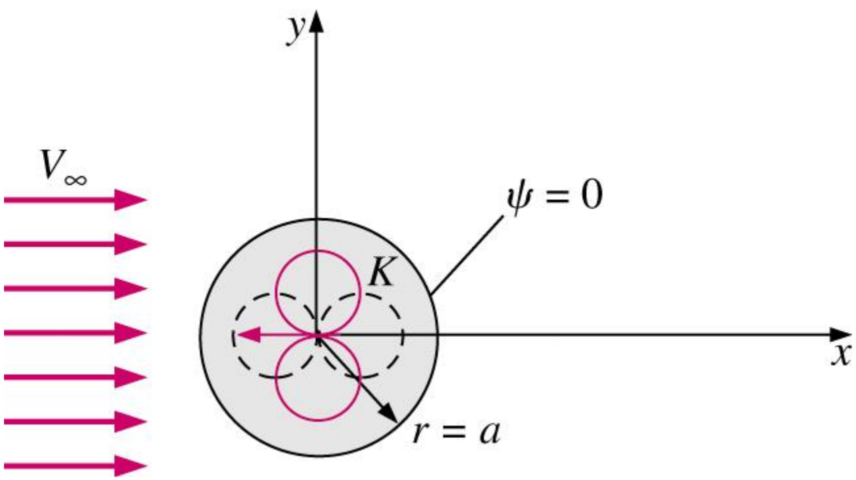
$$\psi = \underbrace{\frac{\dot{\mathcal{V}}/L}{2\pi}\theta}_{\textbf{Sink}} - \underbrace{\frac{\Gamma}{2\pi}\ln r}_{\textbf{Vortex}}$$

$$U_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{\dot{\mathcal{V}}/L}{2\pi r}$$

$$U_\theta = -\frac{\partial \psi}{\partial r} = \frac{\Gamma}{2\pi r}$$

Examples of Irrotational Flows Formed by Superposition





- Flow over a circular cylinder: Free stream + doublet

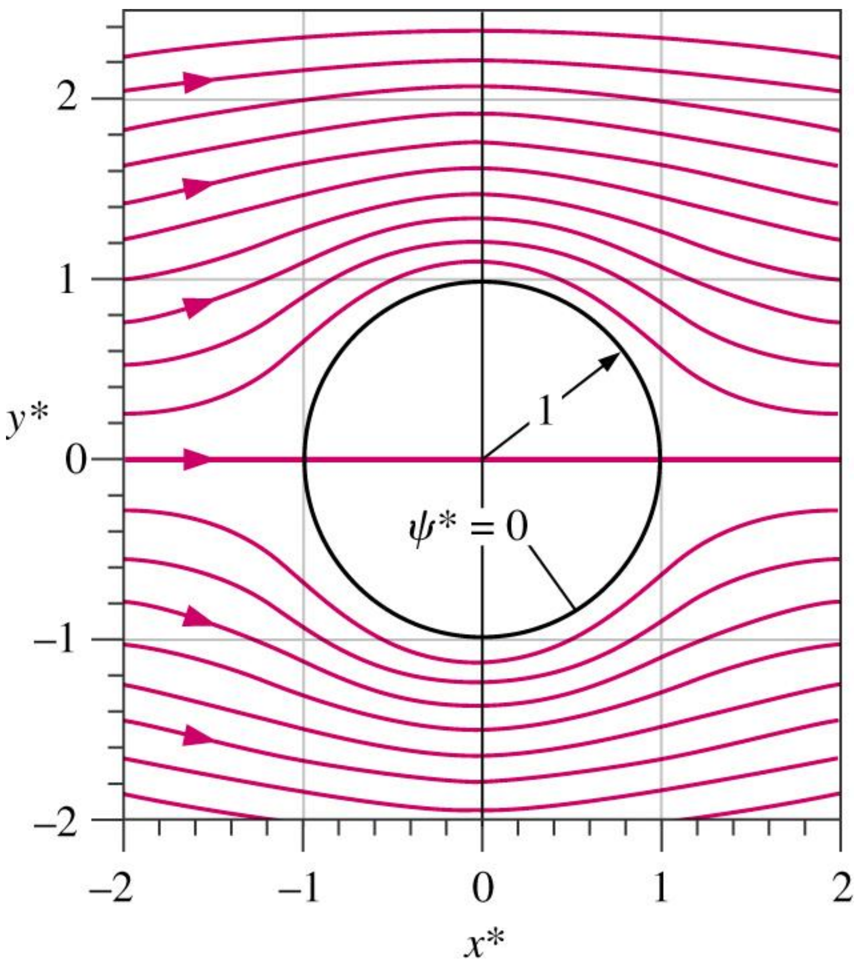
$$\phi = V r \cos\theta + K \frac{\cos\theta}{r}$$

$$\psi = V r \sin\theta - K \frac{\sin\theta}{r}$$

- Assume body is $\psi = 0$ ($r = a$) $\Rightarrow K = Va^2$

$$\psi = V \sin\theta \left(r - a^2/r\right)$$

Examples of Irrotational Flows Formed by Superposition



- Velocity field can be found by differentiating streamfunction

$$U_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = V\cos\theta\left(1 - a^2/r^2\right)$$

$$U_\theta = -\frac{\partial \psi}{\partial r} = -V\sin\theta\left(1 + a^2/r^2\right)$$

- On the cylinder surface (r=a)

$$U_r = 0, \quad U_\theta = -2V\sin\theta$$

Normal velocity (U_r) is zero, Tangential velocity (U_θ) is non-zero ⇒slip condition.