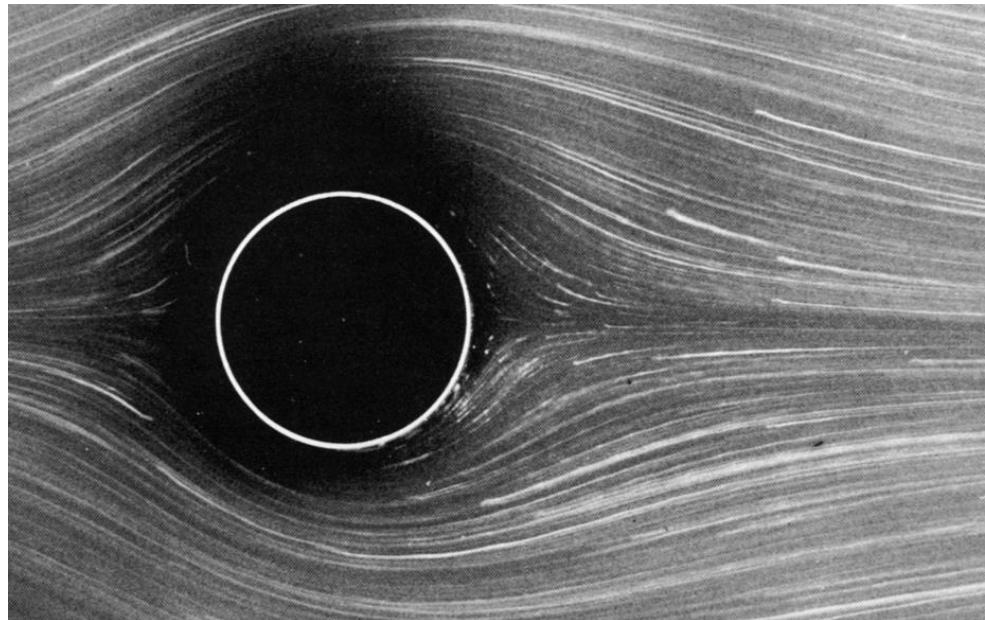


# Chapter 6: Lubrication

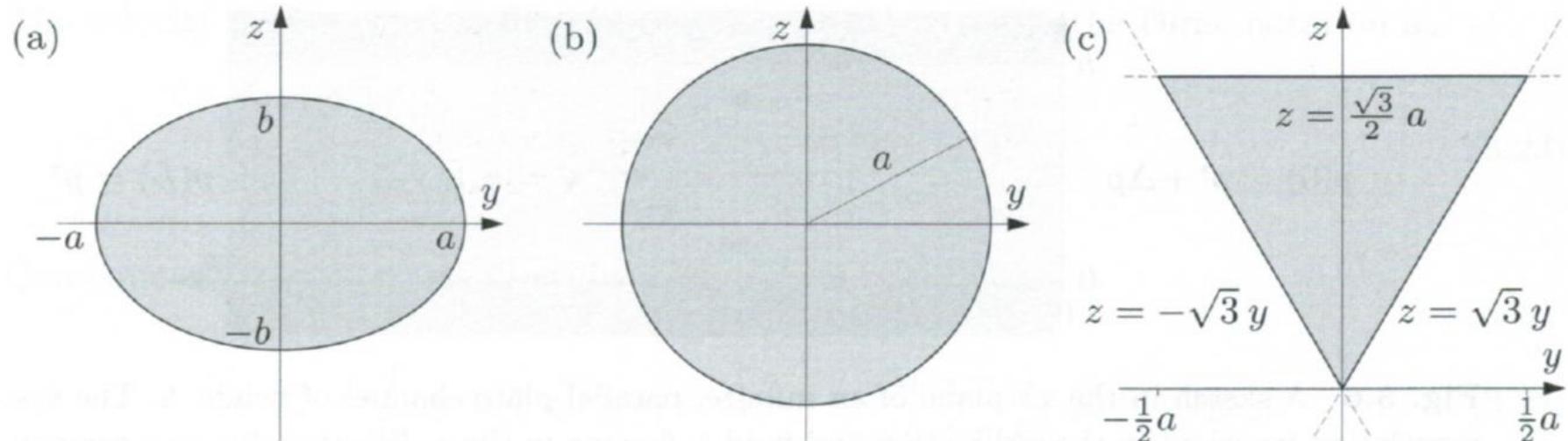


# Outline and course structure

1. Introduction- Eulerian-Lagrangian
2. Local momentum balance, stress modeling (Newtonian model), Navier-Stokes
3. Dimensional analysis
4. Stokes equations (creeping flow)
5. Stokes drag for a sphere, Hele-Shaw cells, flow in pipes of various cross-sections
6. Lubrication

# Pipe flows

- In strictly parallel flows, the inertial terms disappear as the consequence of geometry



Parallel flows in a) elliptical duct b) cylindrical duct c)  
triangular duct



# Flow in a rectangular duct

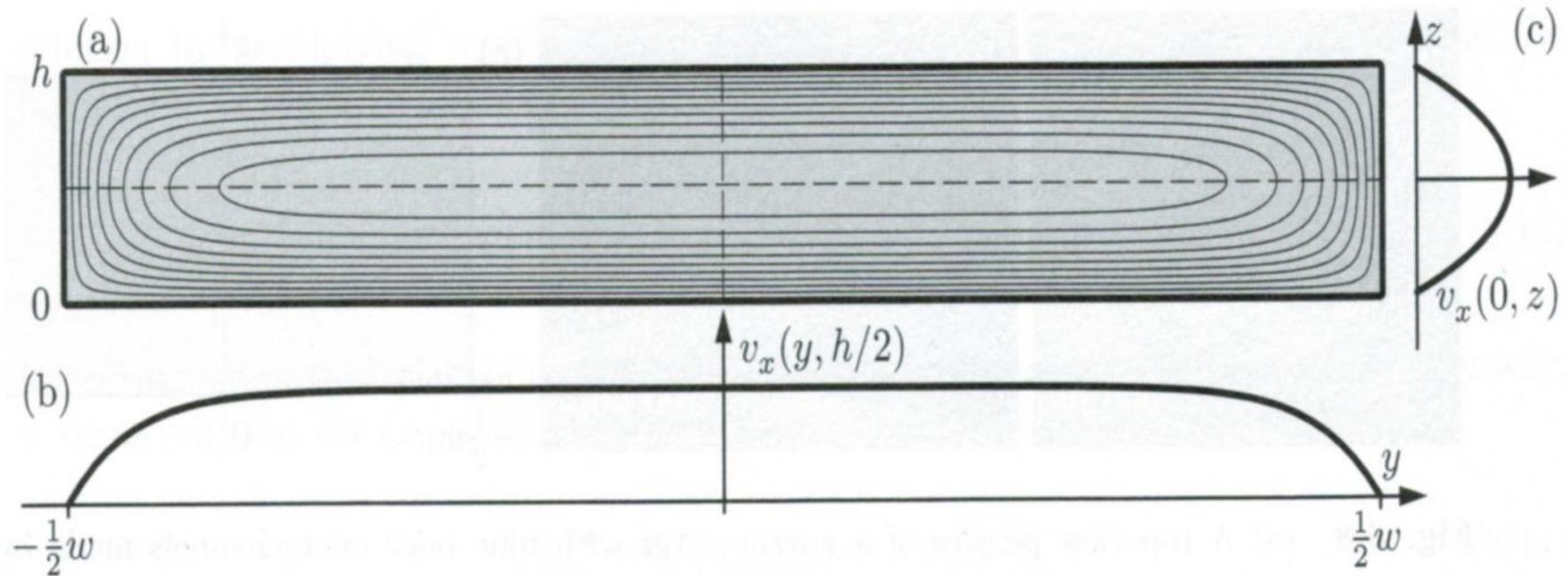


Figure 23: Flow in a rectangular duct

$$v_x(y, z) = \frac{4h^2 \Delta p}{\pi^3 \mu L} \sum_{n, \text{odd}}^{\infty} \frac{1}{n^3} \left[ 1 - \frac{\cosh(n\pi \frac{y}{h})}{\cosh(n\pi \frac{w}{2h})} \right] \sin\left(n\pi \frac{z}{h}\right)$$

# Flow in a rectangular duct

- Velocity

$$v_x(y, z) = \frac{4h^2 \Delta p}{\pi^3 \mu L} \sum_{n, \text{odd}}^{\infty} \frac{1}{n^3} \left[ 1 - \frac{\cosh(n\pi \frac{y}{h})}{\cosh(n\pi \frac{w}{2h})} \right] \sin\left(n\pi \frac{z}{h}\right)$$

- Integrating to find flow rate

$$\begin{aligned} Q &= 2 \int_0^{\frac{1}{2}w} dy \int_0^h dz v_x(y, z) \\ &= \frac{4h^2 \Delta p}{\pi^3 \mu L} \sum_{n, \text{odd}}^{\infty} \frac{1}{n^3} \frac{2h}{n\pi} \left[ w - \frac{2h}{n\pi} \tanh\left(n\pi \frac{w}{2h}\right) \right] \\ &= \frac{8h^3 w \Delta p}{\pi^4 \mu L} \sum_{n, \text{odd}}^{\infty} \left[ \frac{1}{n^4} - \frac{2h}{\pi w} \frac{1}{n^5} \tanh\left(n\pi \frac{w}{2h}\right) \right] \\ &= \frac{h^3 w \Delta p}{12 \mu L} \left[ 1 - \sum_{n, \text{odd}}^{\infty} \frac{1}{n^5} \frac{192}{\pi^5} \frac{h}{w} \tanh\left(n\pi \frac{w}{2h}\right) \right] \end{aligned}$$

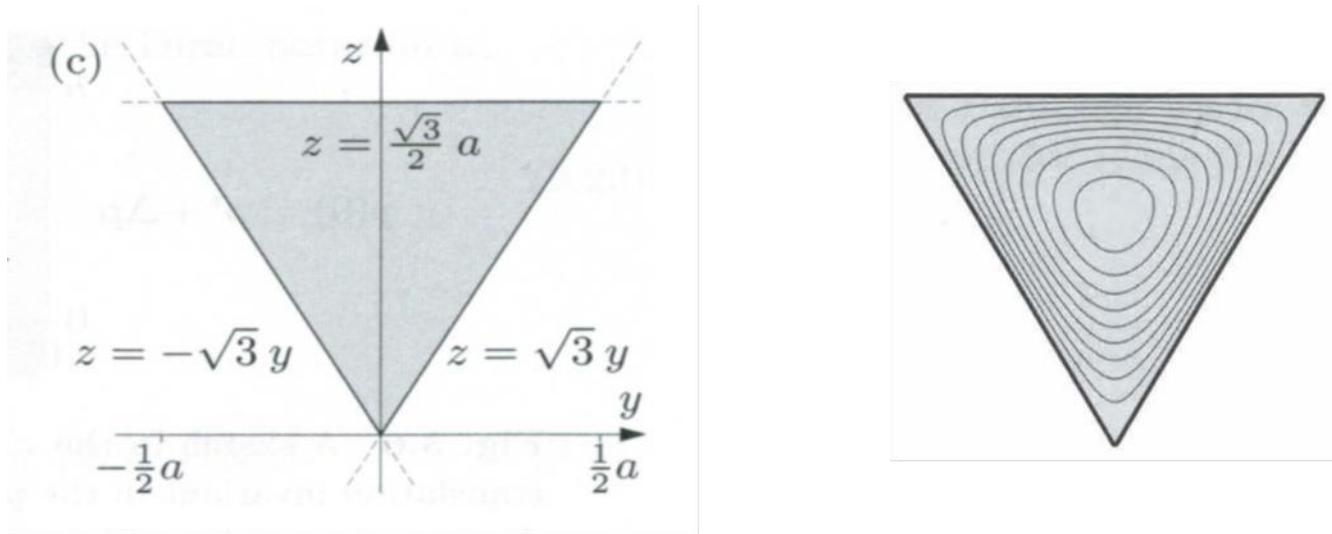
# Flow in a rectangular duct

With  $\sum_{n, \text{odd}}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{96}$  , one finds

$$Q \approx \frac{h^3 w \Delta p}{12 \mu L} \left[ 1 - 0.630 \frac{h}{w} \right], \text{ for } h < w$$

13% error for square, 0.2% for aspect ratio 2

# Flow in a triangular duct



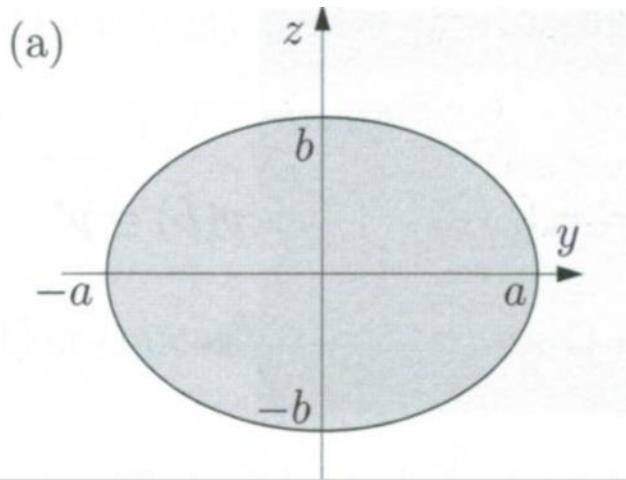
Flow in a triangular duct

$$v_x(y, z) = \frac{v_0}{a^3} \left( \frac{\sqrt{3}}{2} a - z \right) (z - \sqrt{3}y) (z + \sqrt{3}y) = \frac{v_0}{a^3} \left( \frac{\sqrt{3}}{2} a - z \right) (z^2 - 3y^2)$$

$$v_0 = \frac{1}{2\sqrt{3}} \frac{\Delta p}{\mu L} a^2$$

# Flow in an elliptical duct

(a)

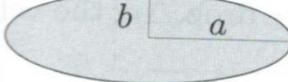
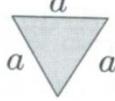
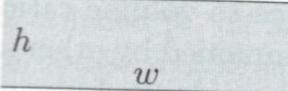
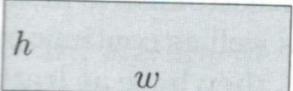
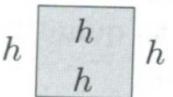
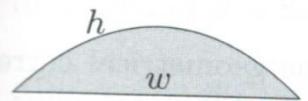
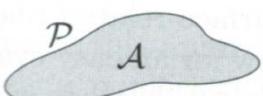


Flow in an elliptical duct

$$v_x(y, z) = v_0 \left( 1 - \frac{y^2}{a^2} - \frac{z^2}{b^2} \right)$$

$$v_0 = \frac{\Delta p}{2\mu L} \frac{a^2 b^2}{a^2 + b^2}$$

# Hydraulic resistance

	$\frac{8}{\pi} \eta L \frac{1}{a^4}$
	$\frac{4}{\pi} \eta L \frac{1 + (b/a)^2}{(b/a)^3} \frac{1}{a^4}$
	$\frac{320}{\sqrt{3}} \eta L \frac{1}{a^4}$
	$12 \eta L \frac{1}{h^3 w}$
	$\frac{12 \eta L}{1 - 0.63(h/w)} \frac{1}{h^3 w}$
	$28.4 \eta L \frac{1}{h^4}$
	$\frac{105}{4} \eta L \frac{1}{h^3 w}$
	$\approx 2 \eta L \frac{P^2}{A^3}$

$$R_{\text{hyd.}} \cdot Q = \Delta p$$

Hydraulic resistance for various duct configuration  
Bruus (2008)

# Lubrication

How does tape glue?

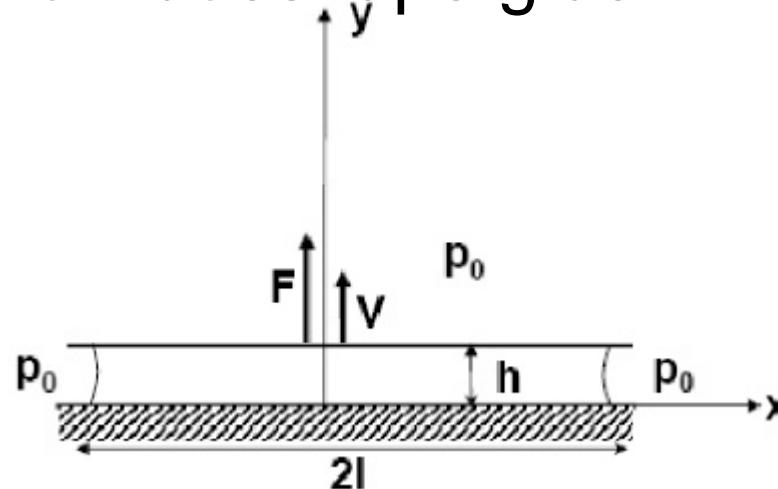


Figure1: Duck tape with glue layer on a plane surface,  
force  $F$  and pulling velocity  $V$

What is the intensity of  $F$  that should be applied to pull at velocity  $V$ ?

# Stokes equations

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial p}{\partial x} - \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = 0$$

$$\frac{\partial p}{\partial y} - \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) = 0$$

# Dimensional analysis ... and more

Introduce following scales

Variables :  $x, y, u, v, p$

Dimensionless variables :  $\tilde{x}, \tilde{y}, \tilde{u}, \tilde{v}, \tilde{p}$

Gauges (scales) :  $l, h, U, V, P$

$$x = l\tilde{x}$$

$$y = h\tilde{y}$$

$$u = U\tilde{u}$$

$$v = V\tilde{v}$$

$$p = P\tilde{p}$$

$V, P$  are unknown at this stage!

# Scaling analysis

Continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Dominant balance=  
keep terms when possible + throw them away if necessary

# Scaling analysis

Continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \rightarrow \frac{U}{l} \frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{V}{h} \frac{\partial \tilde{v}}{\partial \tilde{y}} = 0$$

Dominant balance=

keep terms when possible + throw them away if necessary

$$V = \frac{Uh}{l}, \quad h \ll l$$
$$\Rightarrow \quad V \ll U$$

$$\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{v}}{\partial \tilde{y}} = 0$$

# Scaling analysis

Momentum equation along Ox

$$\frac{\partial p}{\partial x} - \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = 0$$



P is still free, let us chose it according to dominant balance.

# Scaling analysis

Momentum equation along Ox

$$\frac{\partial p}{\partial x} - \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = 0$$



$$\frac{P}{l} \frac{\partial \tilde{p}}{\partial \tilde{x}} - \mu \left( \frac{U}{l^2} \frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} + \frac{U}{h^2} \frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2} \right) = 0$$

$$\frac{P}{l}$$

$$\mu \frac{U}{l^2} \ll \mu \frac{U}{h^2}$$

$$(h \ll l)$$

P is still free, let us chose it according to dominant balance.

$$\rightarrow P = \frac{\mu U l}{h^2} \rightarrow \frac{\partial \tilde{p}}{\partial \tilde{x}} = \frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}$$

# Scaling analysis

## Momentum balance along Oy

$$\frac{\partial p}{\partial y} - \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) = 0$$

$$P = \frac{\mu U l}{h^2}$$
$$V = \frac{U h}{l}$$

# Scaling analysis

## Momentum balance along Oy

$$\frac{\partial p}{\partial y} - \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) = 0$$



$$\frac{P}{h} \frac{\partial \tilde{p}}{\partial \tilde{y}} - \mu \left( \frac{V}{l^2} \frac{\partial^2 \tilde{v}}{\partial \tilde{x}^2} + \frac{V}{h^2} \frac{\partial^2 \tilde{v}}{\partial \tilde{y}^2} \right) = 0$$

$$\left( \frac{P}{h} \right)$$

$$\left( \mu \frac{V}{l^2} \right) \ll \left( \mu \frac{V}{h^2} \right)$$

$(h \ll l)$

$$\frac{\mu U}{lh} \frac{\cancel{l^2}}{\cancel{h^2}} = \frac{\mu Ul}{h^3}$$

$\gg 1$

$$\gg \frac{\mu U}{lh}$$

$$P = \frac{\mu Ul}{h^2}$$

$$V = \frac{Uh}{l}$$

$$\frac{\partial \tilde{p}}{\partial \tilde{y}} = 0$$

# Lubrication equations

Hypothesis  $h \ll l$

starting from Stokes

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial p}{\partial x} - \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = 0$$

$$\frac{\partial p}{\partial y} - \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) = 0$$

$$\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{v}}{\partial \tilde{y}} = 0$$

$$\frac{\partial \tilde{p}}{\partial \tilde{x}} = \frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}$$

$$\frac{\partial \tilde{p}}{\partial \tilde{y}} = 0$$

Equations de lubrification:

←(without dimensions)

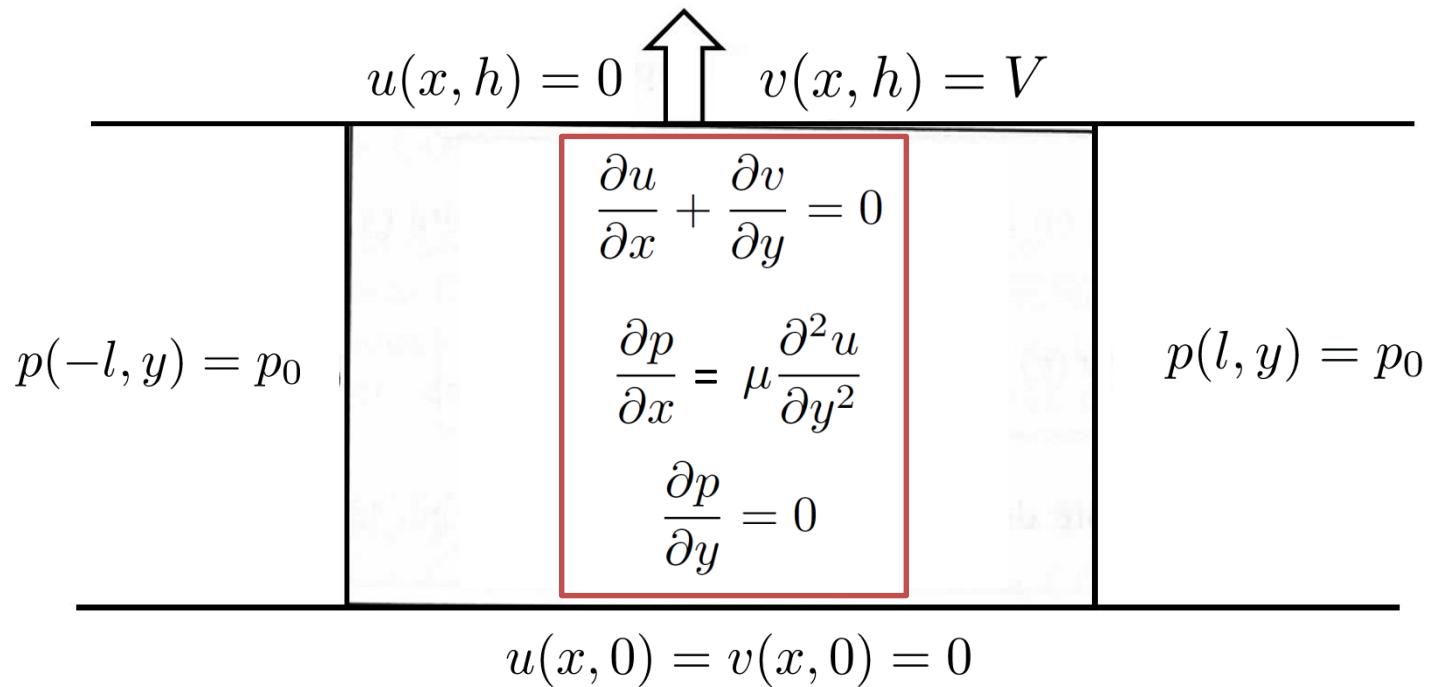
(with dimensions)→

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial p}{\partial x} = \mu \frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial p}{\partial y} = 0$$

# Boundary conditions



# Resolution

- $u$  profile

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu} \frac{dp}{dx}(x), \quad u(x = 0) = u(x, h) = 0$$

- find  $v$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

# Resolution

- u profile

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu} \frac{dp}{dx}(x), \quad u(x=0) = u(x, h) = 0$$

$$u(x, y) = \frac{1}{2\mu} \frac{dp}{dx} y(y - h)$$

- find v

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$v(x, h) = - \int_0^h \frac{\partial u}{\partial x} dy \quad \text{since} \quad v(x, 0) = 0$$

$$v(x, h) = \frac{h^3}{12\mu} \frac{d^2 p}{dx^2}$$

# Resolution

$$v(x, h) = \frac{h^3}{12\mu} \frac{d^2 p}{dx^2}$$

but  $v(x, h) = V$   
therefore  $\frac{d^2 p}{dx^2} = \frac{12\mu V}{h^3}$

$$v(x, y) = - \int_0^y \frac{\partial u}{\partial x} dy \text{ since } v(x, 0) = 0$$

$$u(x, y) = \frac{1}{2\mu} \frac{dp}{dx} y(y - h)$$

# Resolution

$$v(x, h) = \frac{h^3}{12\mu} \frac{d^2 p}{dx^2}$$

but  $v(x, h) = V$   
therefore  $\frac{d^2 p}{dx^2} = \frac{12\mu V}{h^3}$

$$p(x) = p_0 - \frac{6\mu V}{h^3} (l^2 - x^2)$$

$$v(x, y) = - \int_0^y \frac{\partial u}{\partial x} dy \text{ since } v(x, 0) = 0$$

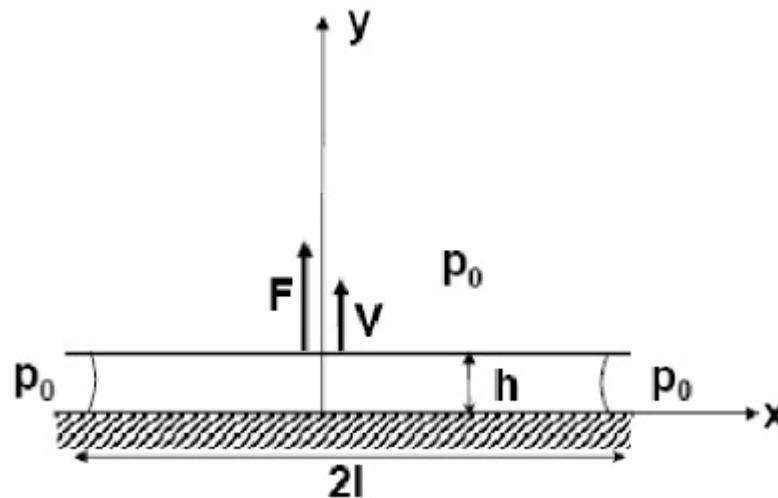
$$u(x, y) = \frac{1}{2\mu} \frac{dp}{dx} y(y - h)$$

$$v(y) = V \left( \frac{y}{h} \right)^2 \left[ 3 - 2 \left( \frac{y}{h} \right) \right]$$

Note that  $\frac{dv}{dy} \Big|_{h=0} = 0$

# Stress on upper plate

$$\bar{\bar{\sigma}} = -p\mathbf{I} + 2\mu\mathbf{D}$$



$$\sigma_{yy}(x, h) = -p(x, h) + 2\mu \frac{dv}{dy}(x, h) \\ (= 0)$$

# Force on plate

$$\mathbf{f}_1 = \int_{-l}^l [p(x) - p_0] dx \mathbf{e}_y$$

$$p(x) = p_0 - \frac{6\mu V}{h^3} (l^2 - x^2)$$



# Force on plate

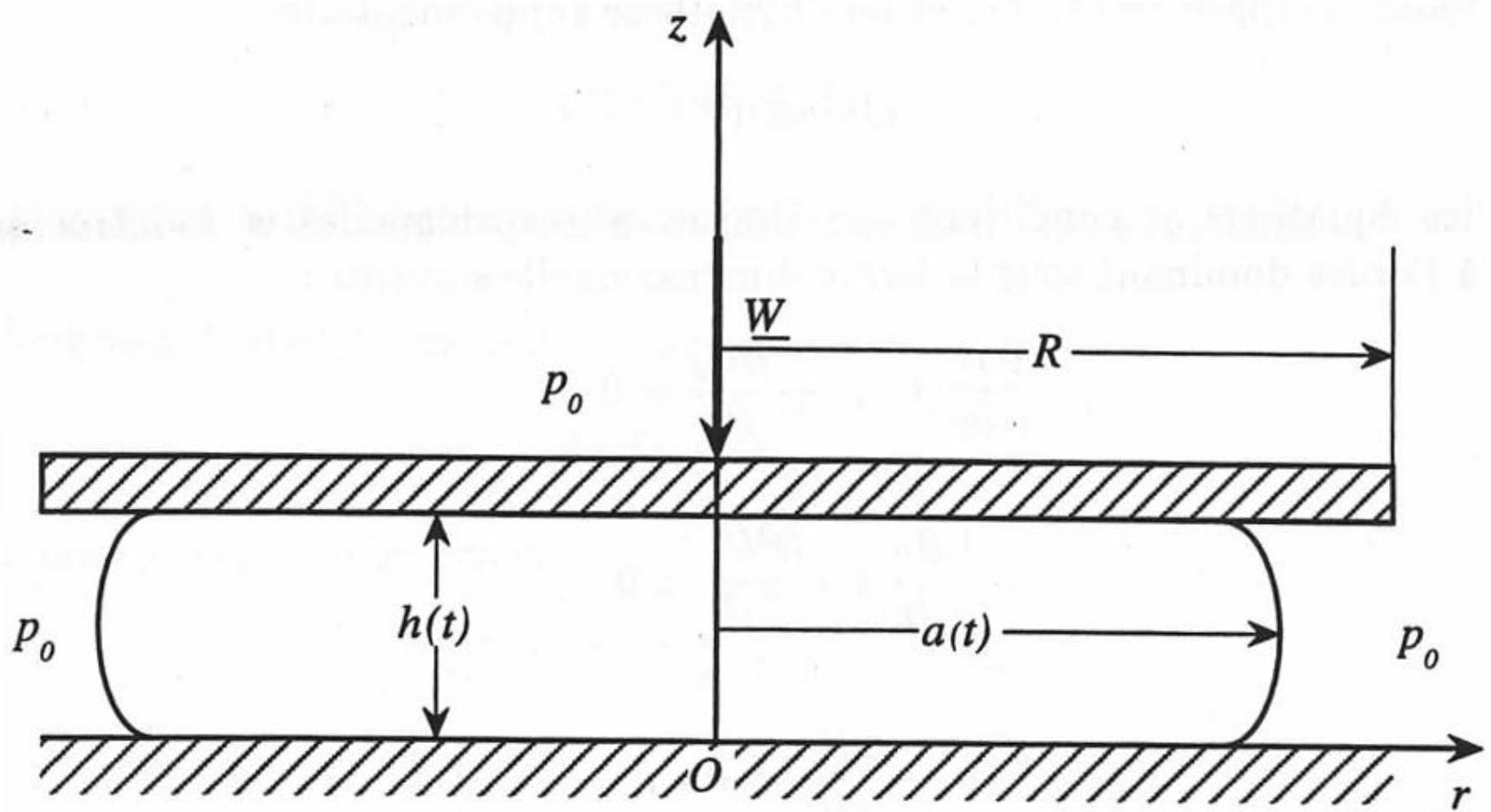
$$\mathbf{f}_1 = \int_{-l}^l [p(x) - p_0] dx \mathbf{e}_y$$

$$p(x) = p_0 - \frac{6\mu V}{h^3} (l^2 - x^2)$$

$$\rightarrow \mathbf{f}_1 = -\frac{8\mu V l^3}{h^3} \mathbf{e}_y$$

$$\mathbf{f} = \frac{8\mu V l^3}{h^3} \mathbf{e}_y$$

# Squeezing of an oil film



Squeezing of an oil film

# Squeezing of an oil film

- Axisymmetric lubrication equations and boundary conditions

$$u_r(r, h(t), t) = 0 \quad u_z(r, h(t), t) = \frac{dh}{dt}$$

$$\frac{1}{r} \frac{\partial}{\partial r} (r u_r) + \frac{\partial u_z}{\partial z} = 0$$

$$- \frac{1}{\rho} \frac{\partial p}{\partial r} + \nu \frac{\partial^2 u_r}{\partial z^2} = 0$$

$$- \frac{1}{\rho} \frac{\partial p}{\partial z} = 0$$

$$p(a(t), z, t) = p_0$$

$$u_r(r, 0, t) = u_z(r, 0, t) = 0$$

# Squeezing of an oil film

- Integrating the radial momentum equation twice and using BC:

$$u_r(r, z, t) = \frac{1}{2\mu} \frac{\partial p}{\partial r} z(z - h)$$

- Using continuity equation, one finds

$$u_z(r, h, t) = -\frac{1}{r} \int_0^h \frac{\partial}{\partial r} (r u_r) dz$$

$$u_z(r, h, t) = \frac{h^3}{12\mu} \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial p}{\partial r} \right)$$

- Using BC in  $z=h(t)$  to find the pressure

$$\frac{h^3}{12\mu} \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial p}{\partial r} \right) = \frac{dh}{dt}$$

$$p(r, t) = p_0 + \frac{3\mu}{h^3} \frac{dh}{dt} (r^2 - a^2)$$

# Squeezing of an oil film

- Recalling

$$u_r(r, z, t) = \frac{1}{2\mu} \frac{\partial p}{\partial r} z(z - h)$$

- And going back to continuity equation

$$u_z(r, z, t) = -\frac{1}{2\mu} \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial p}{\partial r} \right) \int_0^z z(z - h) dz$$

- With the previous result for pressure, one finds

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial p}{\partial r} \right) = \frac{12\mu}{h^3} \frac{dh}{dt}$$

$$u_z(r, z, t) = \frac{dh}{dt} \left( \frac{z}{h} \right)^2 \left[ 3 - 2 \left( \frac{z}{h} \right) \right]$$

- Since the viscous stress vanishes

$$\mathbf{f} = \int_0^a [p(r, t) - p_0] 2\pi r dr \ \mathbf{e}_x = -\frac{3\pi\mu}{2h^3} a^4 \frac{dh}{dt} \ \mathbf{e}_x$$

# Squeezing of an oil film

- Lubrication force

$$\mathbf{f} = \int_0^a [p(r, t) - p_0] 2\pi r dr \ \mathbf{e}_x = -\frac{3\pi\mu}{2h^3} a^4 \frac{dh}{dt} \ \mathbf{e}_x$$

- Using conservation of volume

$$\pi a^2(t) h(t) = \pi a_0^2 h_0$$

$$a^2(t) = a_0^2 \frac{h_0}{h(t)}$$

$$\mathbf{f} = -\frac{3\pi\mu a_0^4 h_0^2}{2h^5} \frac{dh}{dt} \ \mathbf{e}_x$$

# Squeezing of an oil film

- The lubrication force equals the weight  $W$  of the wall squeezing the film

$$-\frac{3\pi\mu a_0^4 h_0^2}{2h^5} \frac{dh}{dt} = W$$

- Integration yields

# Squeezing of an oil film

- The lubrication force equals the weight  $W$  of the wall squeezing the film

$$-\frac{3\pi\mu a_0^4 h_0^2}{2h^5} \frac{dh}{dt} = W$$

- Integration yields

$$\frac{1}{h^4} - \frac{1}{h_0^4} = \frac{8Wt}{3\pi\mu a_0^4 h_0^2}$$

$$h(t) = \frac{h_0}{\left[1 + \frac{8Wh_0^2}{3\pi\mu a_0^4} t\right]^{1/4}}$$

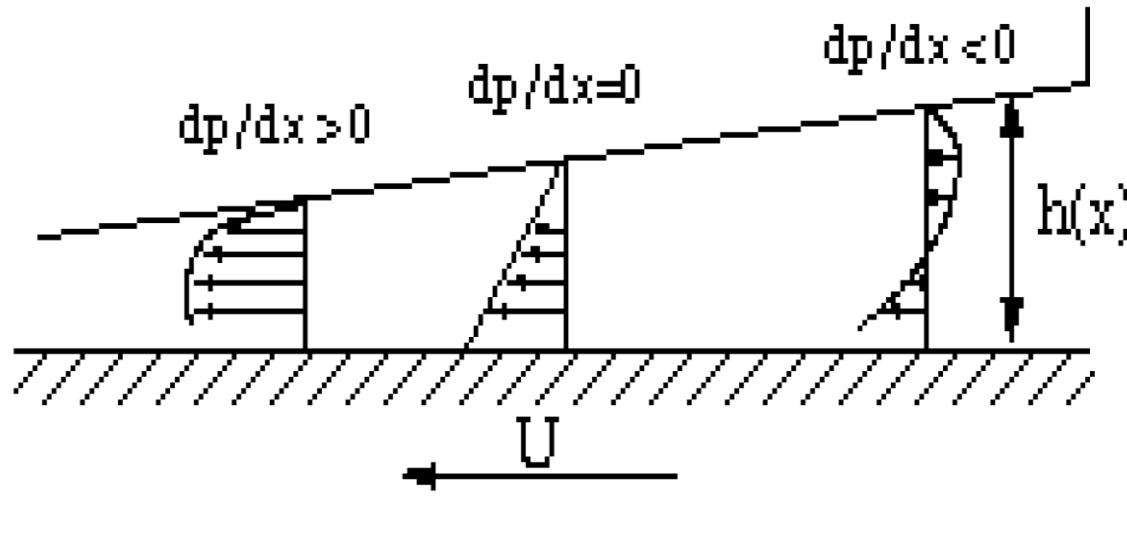
$$a(t) = a_0 \left[1 + \frac{8Wh_0^2}{3\pi\mu a_0^4} t\right]^{1/8}$$

# Squeezing of an oil film

- How can you transpose this solution to the case of adhesive tape?
- What time is required for the adhesive tape glued on the ceiling to fall under its own weight?

# Almost unidirectional flow

- Hele-Shaw
- Lubrication



Lubrication flow

# Almost unidirectional flow

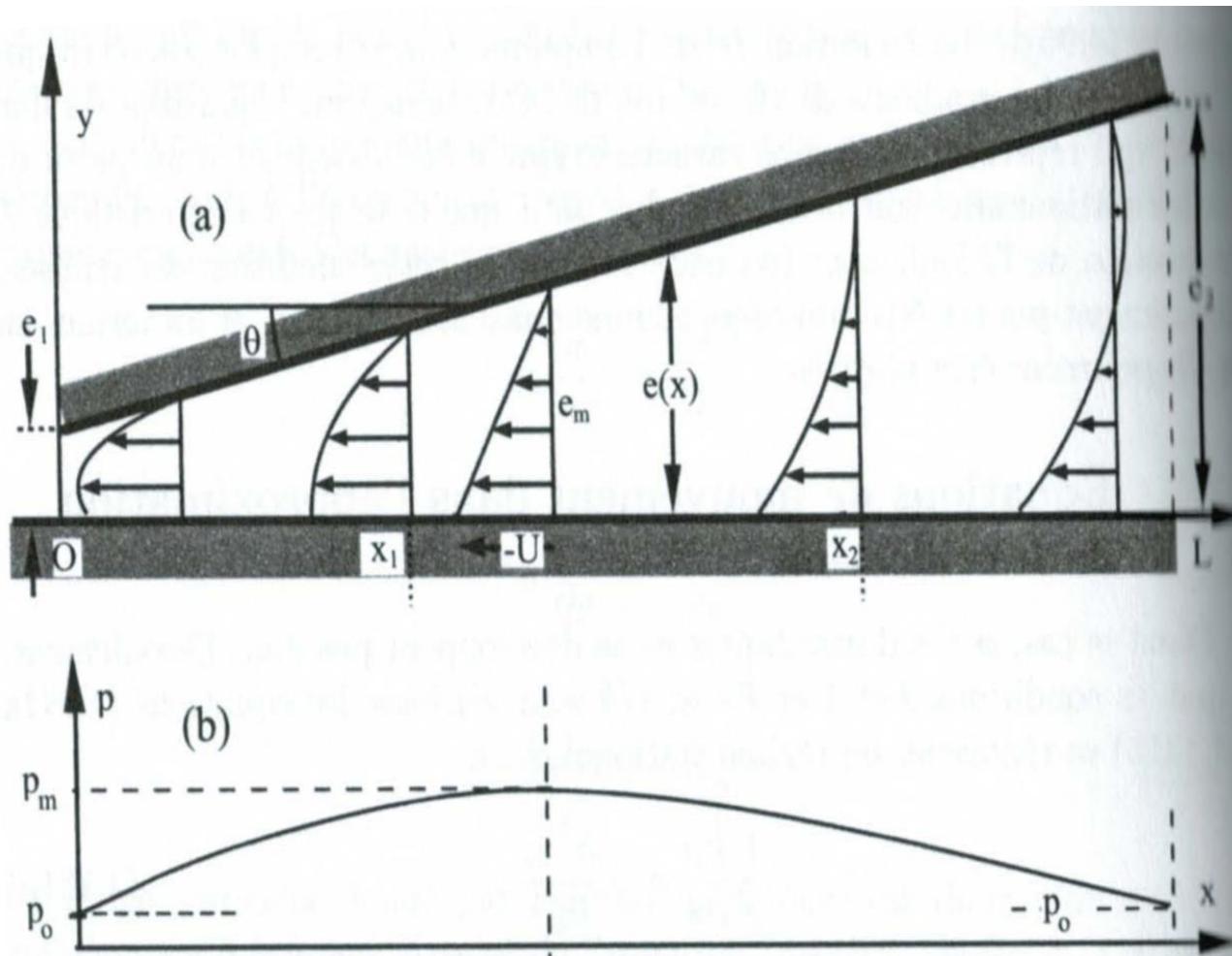


Figure 27: Lubrication flow, pressure distribution

# Almost unidirectional flow

- Velocity profile

$$v(x, y) = -\frac{1}{2\mu} \frac{dp}{dx} y [e(x) - y] - U \frac{e(x) - y}{e(x)}$$

- Pressure gradient

$$\frac{dp}{dx} = \theta \frac{dp}{de} = -\frac{12\mu Q}{e(x)^3} - \frac{6\mu U}{e(x)^2}$$

- Integrating, one finds the pressure

$$p(x) = p_0 + \frac{6\mu Q}{\theta} \left[ \frac{1}{e(x)^2} - \frac{1}{e_1^2} \right] + \frac{6\mu U}{\theta} \left[ \frac{1}{e(x)} - \frac{1}{e_1} \right]$$

# Almost unidirectional flow

- Tangent viscous forces

$$F_T = \frac{2\mu U}{\theta} \left[ -\ln \frac{e_2}{e_1} - \frac{3(e_2 - e_1)}{e_2 + e_1} \right]$$

- Normal pressure forces

$$F_N = \int_0^L (p - p_0) dx = \frac{6\mu U}{\theta^2} \left[ \ln \frac{e_2}{e_1} - \frac{2(e_2 - e_1)}{e_2 + e_1} \right]$$

$$F_N > F_T$$

# Lubrication

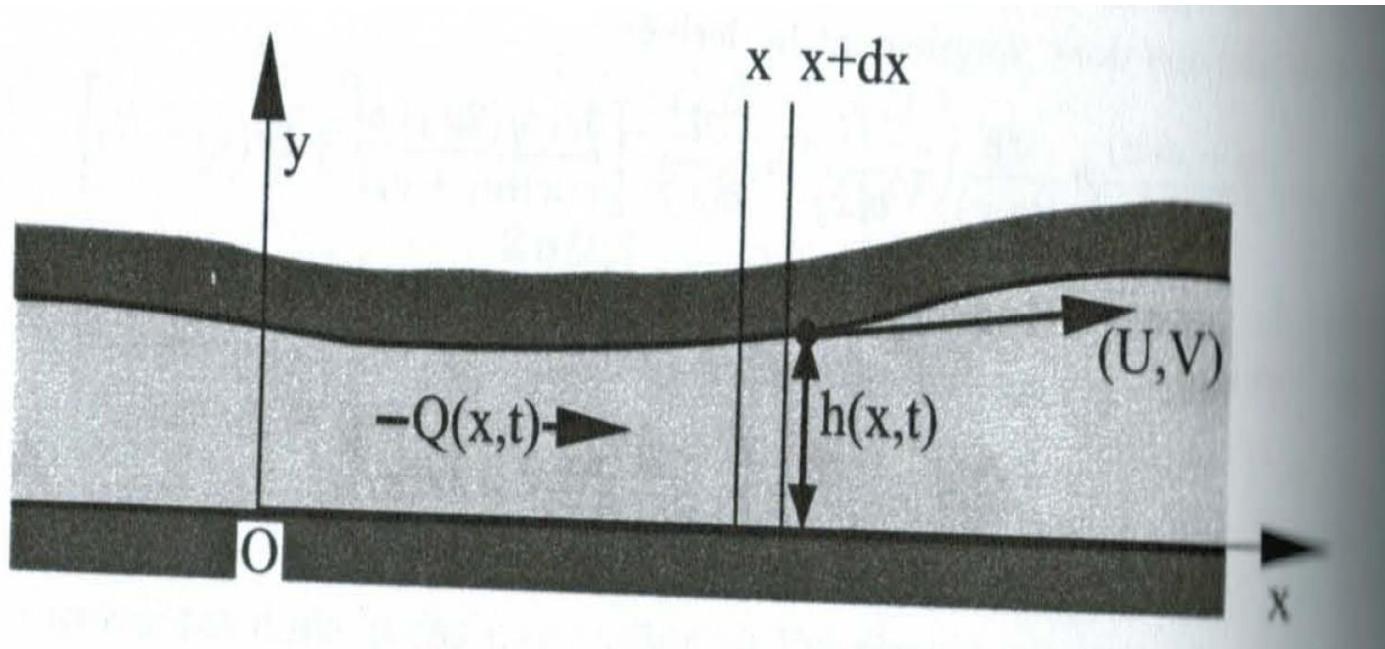


Figure 28: Lubrication flow

$$\frac{\partial h}{\partial t} = \frac{1}{12\mu} \frac{\partial}{\partial x} \left[ h^3 \frac{\partial p}{\partial x} \right] - \frac{1}{2} \left( h \frac{\partial U}{\partial x} + U \frac{\partial h}{\partial x} \right)$$