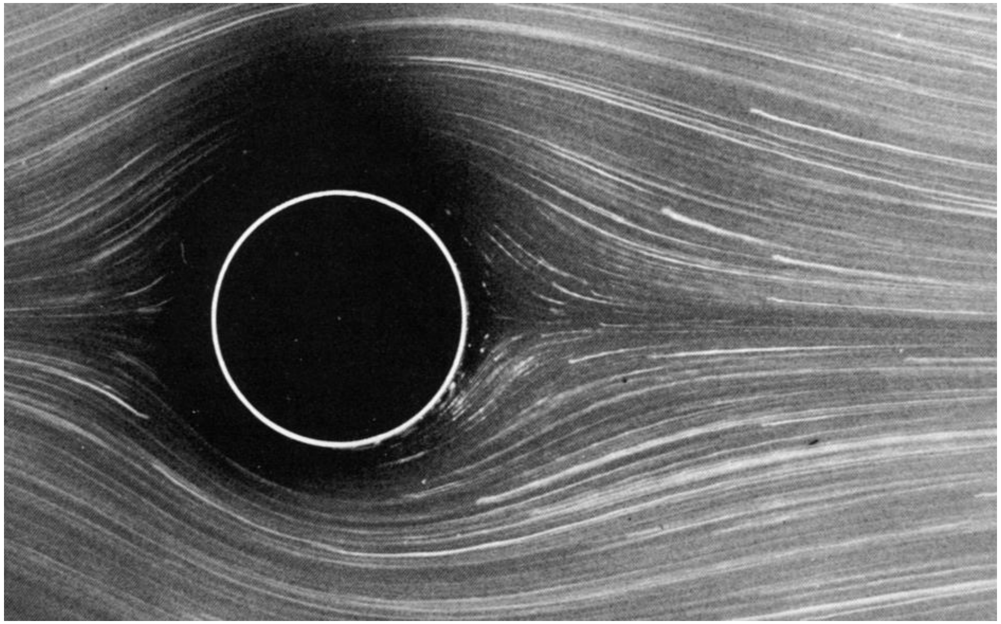
Chapter 6:
Lubrication



Outline

1. Hele-Shaw cells
2. Pipe flows (parallel flows)
3. Lubrication

A strange potential limit of Stokes equations

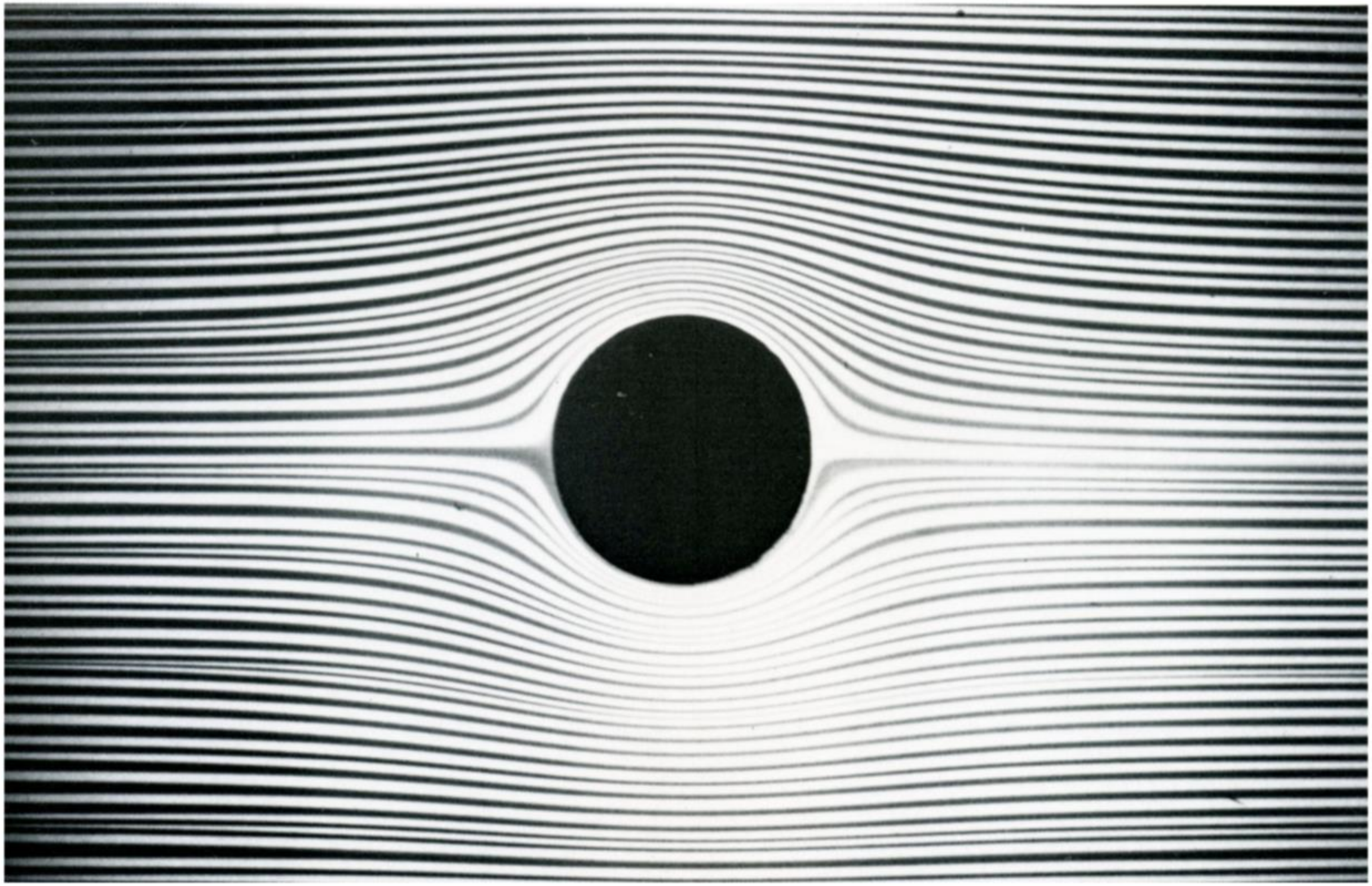


Figure 20: Hele-Shaw flow, Peregrine (1982)

Hele-Shaw flow (flow in Hele-Shaw cell)

$$Re = \frac{U_\infty h}{\nu} \qquad \epsilon = \frac{h}{L} \ll 1$$

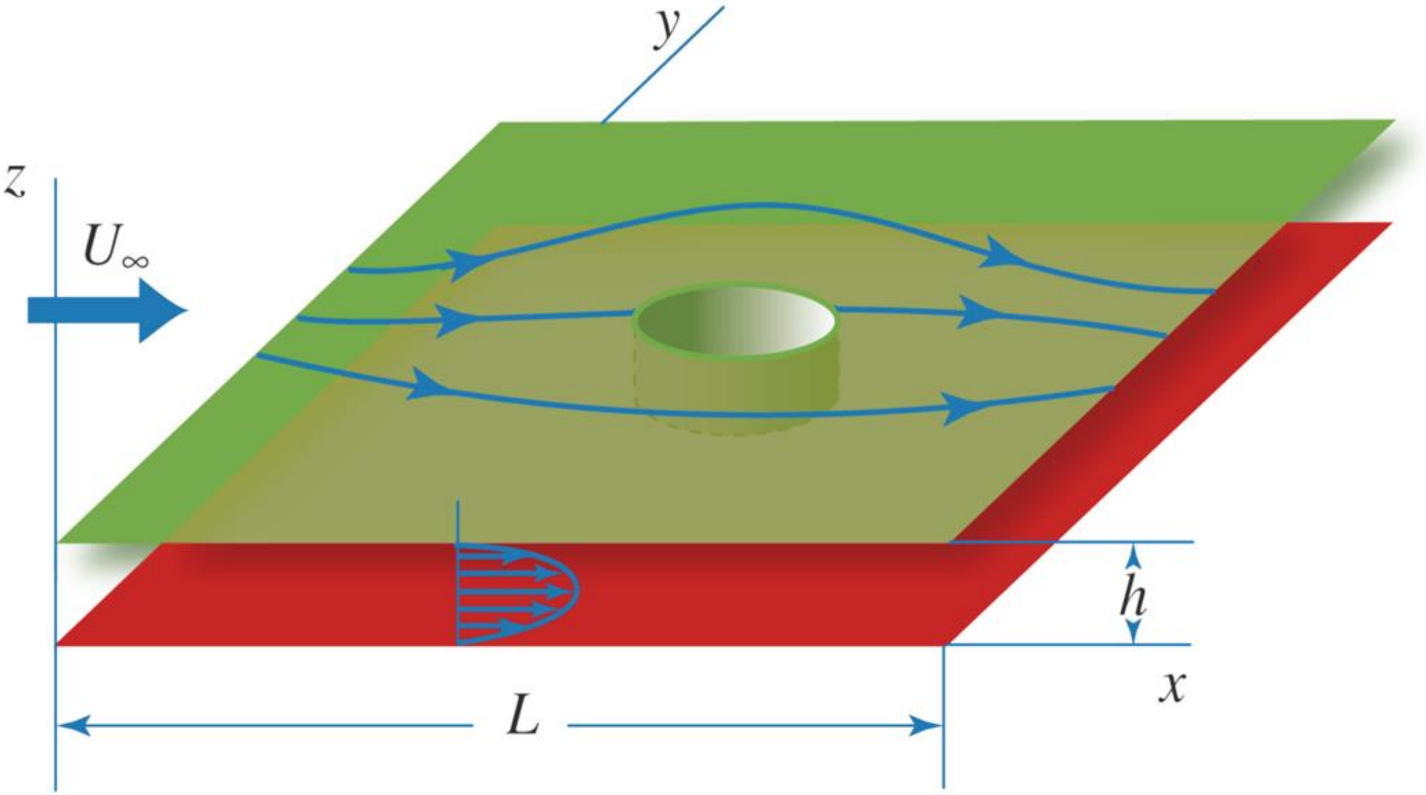


Figure 21: schematic representation of Hele-Shaw flow

Movie (A. Garcia)

Hele-Shaw flow

- Non-dimensional variables

$$(\bar{x}, \bar{y}) = \frac{(x, y)}{L} \ , \ \bar{z} = \frac{z}{h}$$

$$(\bar{u}, \bar{v}) = \frac{(u, v)}{U_\infty} \ , \ \bar{w} = \frac{w}{W} \ , \ \bar{p}\frac{p}{P}$$

- Non-dimensional continuity equation: **dominant balance**

$$\boxed{\frac{\partial \bar{u}}{\partial \bar{x}}} + \boxed{\frac{\partial \bar{v}}{\partial \bar{y}}} + \boxed{\frac{W}{U_\infty}\frac{L}{h}\frac{\partial \bar{w}}{\partial \bar{z}}} = 0$$

- Non-dimensional momentum equation (along $\mathbf{e}_x$) : **dominant balance**

$$Re \cdot \epsilon \left(\bar{u}\frac{\partial}{\partial \bar{x}} + \bar{v}\frac{\partial}{\partial \bar{y}} + \frac{W}{U_\infty}\frac{L}{h}\bar{w}\frac{\partial}{\partial \bar{z}} \right) \bar{u}$$
$$= \boxed{-\frac{Ph^2}{\mu U_\infty L}\frac{\partial \bar{p}}{\partial \bar{x}}} + \left(\boxed{\frac{\partial^2}{\partial \bar{z}^2}} + \epsilon^2 \frac{\partial^2}{\partial \bar{x}^2} + \epsilon^2 \frac{\partial^2}{\partial \bar{y}^2} \right) \bar{u}$$

Hele-Shaw flow: Fundamental equations

With a similar dominant balance reasoning along $\mathbf{e_y}$ and $\mathbf{e_z}$, one finds the fundamental equations of the Hele-Shaw flow

- Continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

- Momentum equations

$\mathbf{e_x}$) $$\mu \frac{\partial^2 u}{\partial z^2} = \frac{\partial p}{\partial x}$$

$\mathbf{e_y}$) $$\mu \frac{\partial^2 v}{\partial z^2} = \frac{\partial p}{\partial y}$$

$\mathbf{e_z}$) $$\frac{\partial p}{\partial z} = 0$$

- With

$$\epsilon \ll 1 \quad , \quad Re \cdot \epsilon \ll 1$$

Hele-Shaw flow: flow-field velocity and vorticity

- Solving the equations, one finds

$$u(x,y,z) = -\frac{1}{2\mu}\frac{\partial p}{\partial x}z(h-z)$$

$$v(x,y,z) = -\frac{1}{2\mu}\frac{\partial p}{\partial y}z(h-z)$$

$$\boxed{w(x,y,z) = 0}$$!

- Vorticity

$$\boxed{\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0}$$

Potential flow in the (x-y) plane!

Hele-Shaw flow: Flow rate

- Integrating the velocities, one finds

$$
\begin{aligned}
Q &= \int_0^h \int_{-w/2}^{w/2} -\frac{1}{2\mu}\frac{dp}{dx} z(h-z)\,dz\,dy \\
&= \int_0^h \int_{-w/2}^{w/2} \frac{1}{2\mu}\frac{\Delta p}{L} z(h-z)\,dz\,dy \\
&= \frac{wh^3}{12\mu}\frac{\Delta p}{L}
\end{aligned}
$$

Towards Navier-Stokes equations

- In strictly parallel flows, the inertial terms disappear as the consequence of geometry

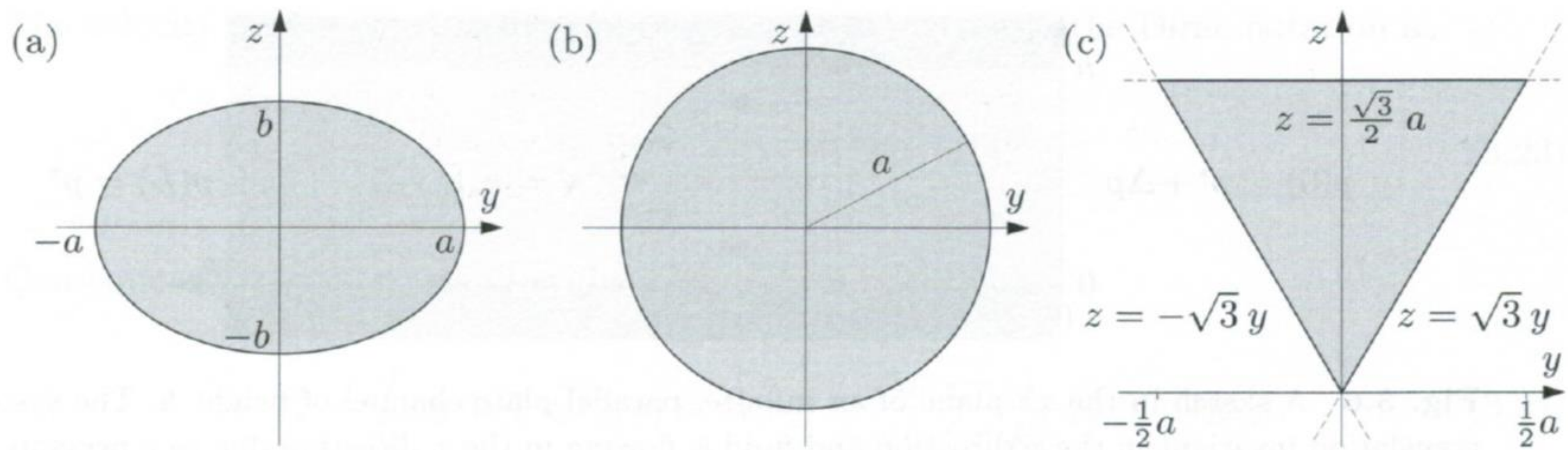


Figure 22: Parallel flows in a a) elliptical duct b) cylindrical duct c) triangular duct

Outline

1. Hele-Shaw cells
2. Pipe flows (parallel flows)
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Flow in a rectangular duct

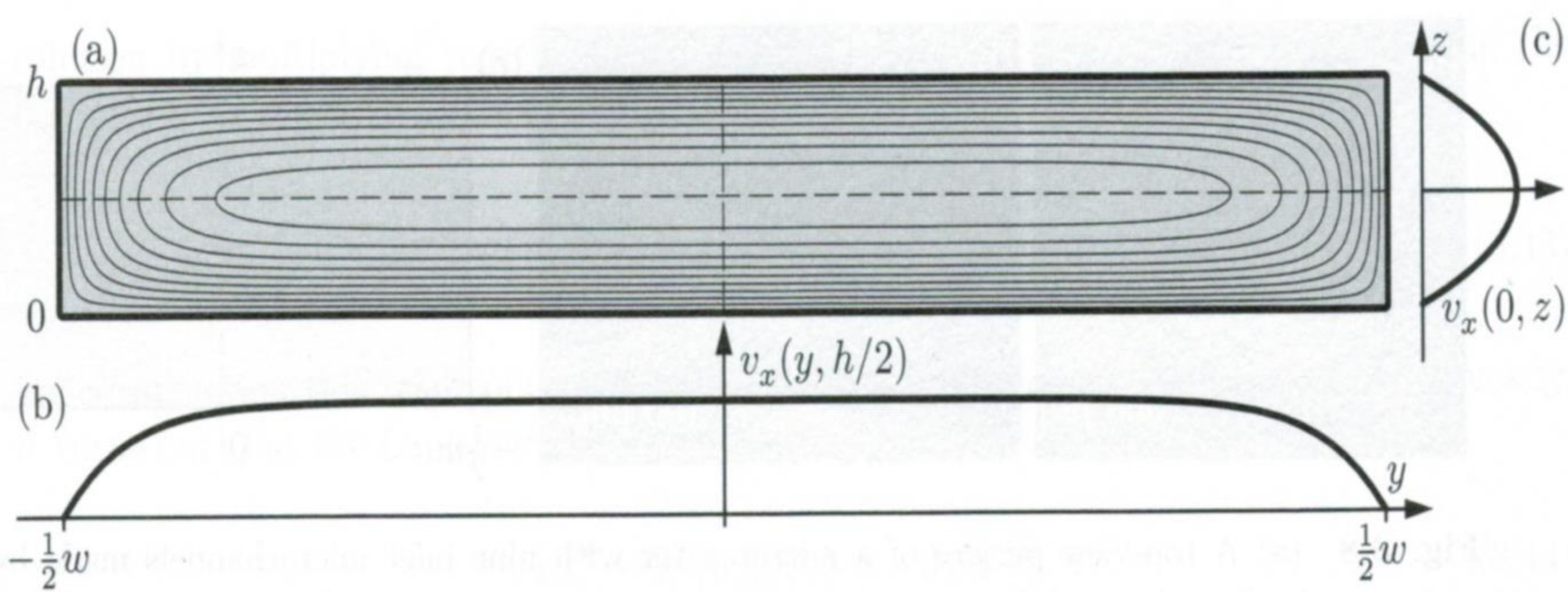


Figure 23: Flow in a rectangular duct

$$v_x(y,z) = \frac{4h^2\Delta p}{\pi^3 \mu L} \sum_{n,\text{odd}}^{\infty} \frac{1}{n^3} \left[1 - \frac{\cosh\left(n\pi \frac{y}{h}\right)}{\cosh\left(n\pi \frac{w}{2h}\right)}\right] \sin\left(n\pi \frac{z}{h}\right)$$

Flow in a rectangular duct

- Velocity

$$v_x(y,z) = \frac{4h^2\Delta p}{\pi^3 \mu L} \sum_{n,\mathrm{odd}}^{\infty} \frac{1}{n^3} \left[1 - \frac{\cosh\left(n\pi\frac{y}{h}\right)}{\cosh\left(n\pi\frac{w}{2h}\right)}\right] \sin\left(n\pi\frac{z}{h}\right)$$

- Integrating to find flow rate

$$\begin{aligned} Q &= 2\int_0^{\frac{1}{2}\omega} dy \int_0^h dz\ v_x(y,z) \\ &= \frac{4h^2\Delta p}{\pi^3 \mu L} \sum_{n,\mathrm{odd}}^{\infty} \frac{1}{n^3}\frac{2h}{n\pi} \left[w - \frac{2h}{n\pi}\tanh\left(n\pi\frac{w}{2h}\right)\right] \\ &= \frac{8h^3 w\Delta p}{\pi^4 \mu L} \sum_{n,\mathrm{odd}}^{\infty} \left[\frac{1}{n^4} - \frac{2h}{\pi w}\frac{1}{n^5}\tanh\left(n\pi\frac{w}{2h}\right)\right] \\ &= \frac{h^3 w\Delta p}{12\mu L}\left[1 - \sum_{n,\mathrm{odd}}^{\infty} \frac{1}{n^5}\frac{192}{\pi^5}\frac{h}{w}\tanh\left(n\pi\frac{w}{2h}\right)\right] \end{aligned}$$

Flow in a rectangular duct

With $\sum_{n,\text{odd}}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{96}$, one finds

$$Q \approx \frac{h^3 w \Delta p}{12\mu L}\left[1 - 0.630\frac{h}{w}\right], \ \text{for } h < w$$

13% error for square, 0.2% for aspect ratio 2

Flow in a triangular duct

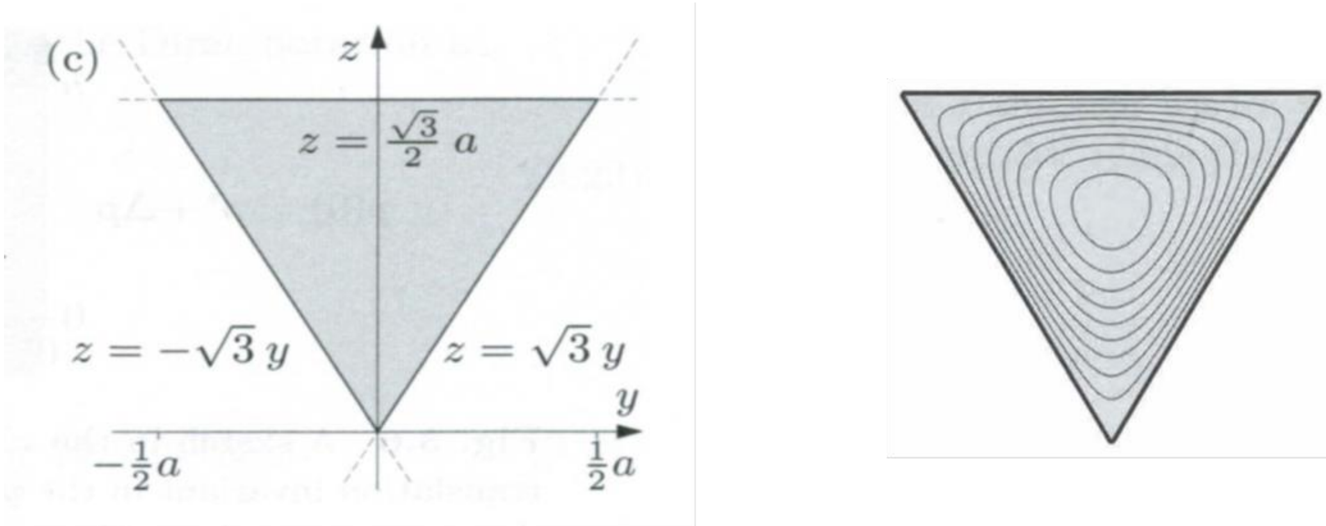


Figure 24: Flow in a triangular duct

$$v_x(y,z) = \frac{v_0}{a^3}\left(\frac{\sqrt{3}}{2}a - z\right)\left(z - \sqrt{3}y\right)\left(z + \sqrt{3}y\right) = \frac{v_0}{a^3}\left(\frac{\sqrt{3}}{2}a - z\right)(z^2 - 3y^2)$$

$$v_0 = \frac{1}{2\sqrt{3}}\frac{\Delta p}{\mu L}a^2$$

Flow in an elliptical duct

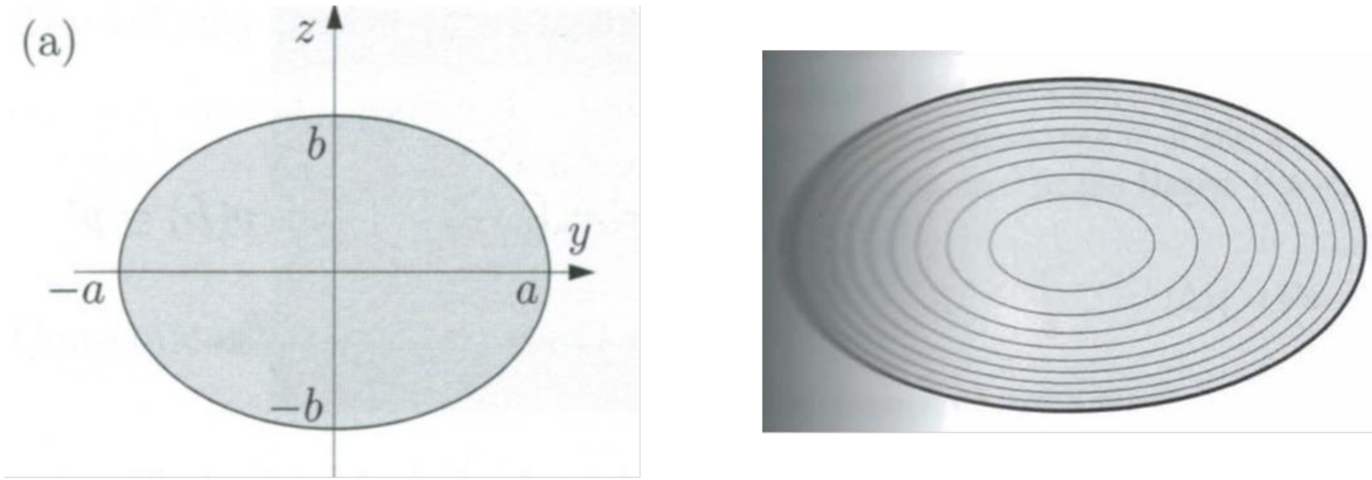
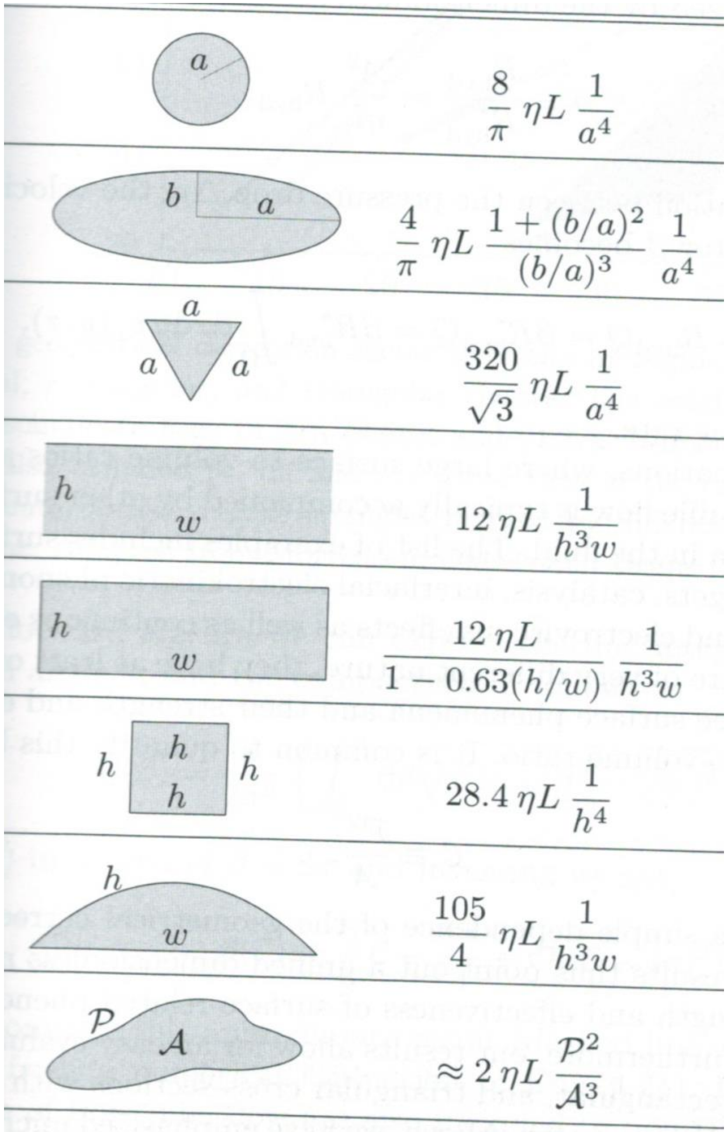


Figure 24: Flow in an elliptical duct

$$v_x(y,z) = v_0\left(1 - \frac{y^2}{a^2} - \frac{z^2}{b^2}\right)$$

$$v_0 = \frac{\Delta p}{2\mu L}\frac{a^2 b^2}{a^2 + b^2}$$

Hydraulic resistance



a	$\frac{8}{\pi}\,\eta L\,\frac{1}{a^4}$
b, a	$\frac{4}{\pi}\,\eta L\,\frac{1+(b/a)^2}{(b/a)^3}\,\frac{1}{a^4}$
a, a, a	$\frac{320}{\sqrt{3}}\,\eta L\,\frac{1}{a^4}$
h, w	$12\,\eta L\,\frac{1}{h^3 w}$
h, w	$\frac{12\,\eta L}{1-0.63(h/w)}\,\frac{1}{h^3 w}$
h, h, h, h	$28.4\,\eta L\,\frac{1}{h^4}$
h, w	$\frac{105}{4}\,\eta L\,\frac{1}{h^3 w}$
$\mathcal{P}$, $\mathcal{A}$	$\approx 2\,\eta L\,\frac{\mathcal{P}^2}{\mathcal{A}^3}$

$$R_{\mathrm{hyd.}} \cdot Q = \Delta p$$

Figure 25: Hydraulic resistance for various duct configuration Bruus (2008)

How does tape glue?

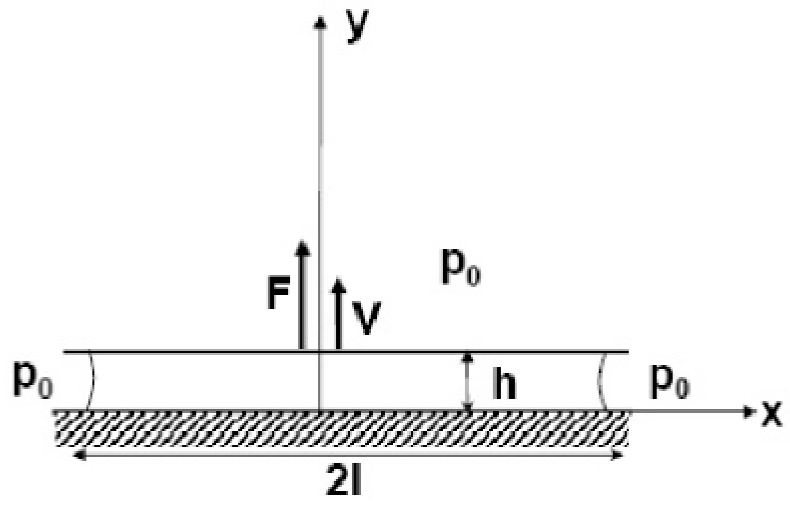


Figure1: Duck tape with glue layer on a plane surface, force F and pulling velocity V

What is the intensity of F that should be applied to pull at velocity V?

Outline

1. Hele-Shaw cells
2. Pipe flows (parallel flows)
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Stokes equations

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial p}{\partial x} - \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = 0$$

$$\frac{\partial p}{\partial y} - \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) = 0$$

Dimensional analysis … and more

Introduce following scales

Variables : x, y, u, v, p

Dimensionless variables : $\tilde{x}, \tilde{y}, \tilde{u}, \tilde{v}, \tilde{p}$

Gauges (scales) : l, h, U, V, P

$$x = l\tilde{\boldsymbol{x}}$$
$$y = h\tilde{\boldsymbol{y}}$$
$$u = U\tilde{\boldsymbol{u}}$$
$$v = V\tilde{\boldsymbol{v}}$$
$$p = P\tilde{\boldsymbol{p}}$$

V, *P* are unknown at this stage!

Scaling analysis

Continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \rightarrow \frac{U}{l}\frac{\partial \widetilde{u}}{\partial \widetilde{x}} + \frac{V}{h}\frac{\partial \widetilde{v}}{\partial \widetilde{y}} = 0$$

Dominant balance=
keep terms when possible + throw them away if necessary

$$V = \frac{Uh}{l}, \qquad h \ll l$$

$$\Rightarrow \qquad V \ll U$$

$$\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{v}}{\partial \tilde{y}} = 0$$

Scaling analysis

Momentum equation along Ox

$$\frac{\partial p}{\partial x} - \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = 0$$

$$\frac{P}{l}\frac{\partial \tilde{p}}{\partial \tilde{x}} - \mu \left(\frac{U}{l^2}\frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} + \frac{U}{h^2}\frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2} \right) = 0$$

$$\frac{P}{l} \qquad \mu\frac{U}{l^2} \ll \mu\frac{U}{h^2} \quad (h \ll l)$$

P is still free, let us chose it according to dominant balance.

$$\rightarrow P = \frac{\mu U l}{h^2} \rightarrow \frac{\partial \tilde{p}}{\partial \tilde{x}} - \frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}$$

Scaling analysis

Momentum balance along Oy

$$\frac{\partial p}{\partial y} - \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) = 0$$

$$\frac{P}{h}\frac{\partial \tilde{p}}{\partial \tilde{y}} - \mu \left(\frac{V}{l^2}\frac{\partial^2 \tilde{v}}{\partial \tilde{x}^2} + \frac{V}{h^2}\frac{\partial^2 \tilde{v}}{\partial \tilde{y}^2} \right) = 0$$

$$\frac{P}{h} \qquad \mu\frac{V}{l^2} \ll \mu\frac{V}{h^2} \qquad (h \ll l)$$

$$\frac{\mu U}{lh}\frac{l^2}{h^2} = \frac{\mu U l}{h^3} \quad >> \quad \frac{\mu U}{lh}$$

$\frac{l^2}{h^2}$ >>1

$$P = \frac{\mu U l}{h^2}$$

$$V = \frac{U h}{l}$$

$$\frac{\partial \tilde{p}}{\partial \tilde{y}} = 0$$

Lubrication equations

Hypothesis h<<l

starting from Stokes

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial p}{\partial x} - \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = 0$$

$$\frac{\partial p}{\partial y} - \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) = 0$$

$$\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{v}}{\partial \tilde{y}} = 0$$

$$\frac{\partial \tilde{p}}{\partial \tilde{x}} - \frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}$$

$$\frac{\partial \tilde{p}}{\partial \tilde{y}} = 0$$

Equations de lubrification:

←(without dimensions)

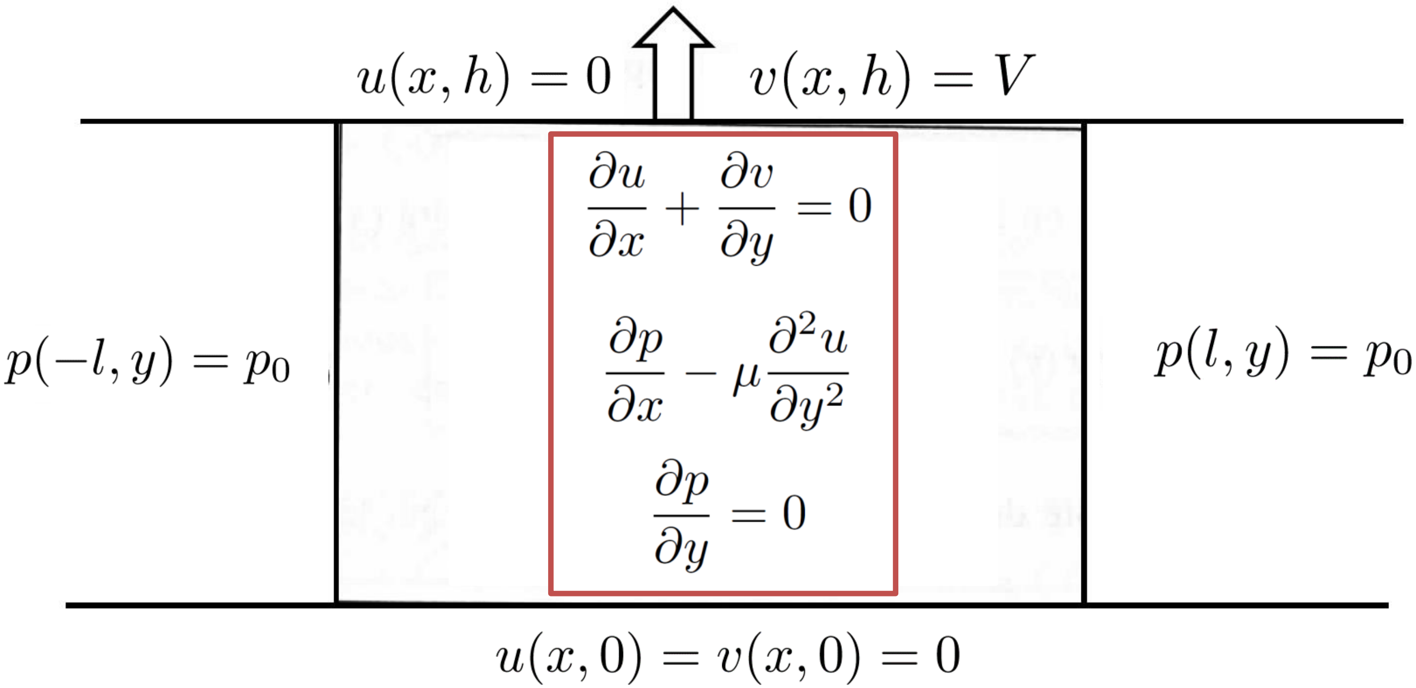
(with dimensions)→

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial p}{\partial x} - \mu \frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial p}{\partial y} = 0$$

Boundary conditions



Resolution

- u profile

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu}\frac{dp}{dx}(x), \qquad u(x=0) = u(x,h) = 0$$

$$\boxed{u(x,y) = \frac{1}{2\mu}\frac{dp}{dx}y(y-h)}$$

- find v

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$v(x,h) \quad = \quad -\int_0^h \frac{\partial u}{\partial x} dy \qquad \text{car v(x,0)=0}$$

$$\boxed{v(x,h) \quad = \quad \frac{h^3}{12\mu}\frac{d^2p}{dx^2}}$$

Resolution

$$v(x,h) = \frac{h^3}{12\mu}\frac{d^2p}{dx^2}$$

but $v(x,h) = V$

therefore $\frac{d^2p}{dx^2} = \frac{12\mu V}{h^3}$

$$p(x) = p_0 - \frac{6\mu V}{h^3}(l^2 - x^2)$$

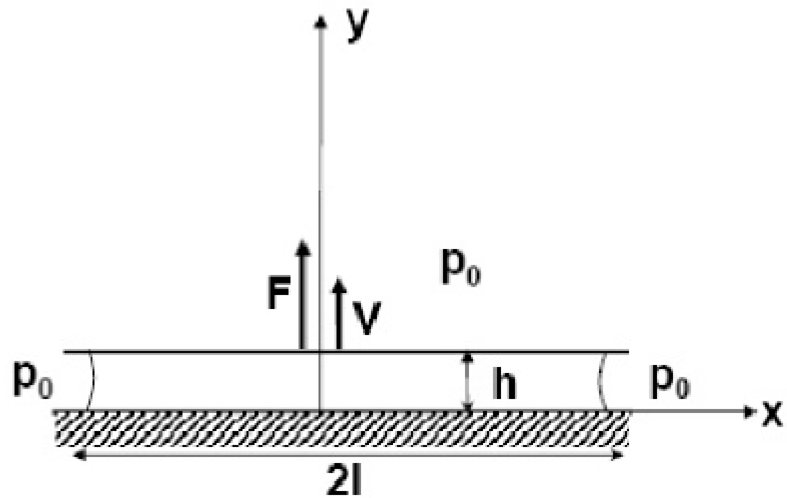
$$v(x,y) = -\int_0^y \frac{\partial u}{\partial x} dy \quad \text{car v(x,0)=0}$$

$$v(y) = V\left(\frac{y}{h}\right)^2\left[3 - 2\left(\frac{y}{h}\right)\right]$$

Note that $\frac{dv}{dy}|_h = 0$

Stress on upper plate

$$\bar{\bar{\sigma}} = -p\mathbf{I} + 2\mu\mathbf{D}$$



$$\sigma_{yy}(x,h) = -p(x,h) + 2\mu\frac{dv}{dy}(x,h)$$

$$(=0)$$

Force on plate

$$\mathbf{f_1} = \int_{-l}^{l} [p(x) - p_0]\, dx \; \mathbf{e}_y$$

$$\longrightarrow \quad \mathbf{f_1} = -\frac{8\mu V l^3}{h^3}\mathbf{e}_y$$

$$\boxed{\mathbf{f} = \frac{8\mu V l^3}{h^3}\mathbf{e}_y}$$

Squeezing of an oil film

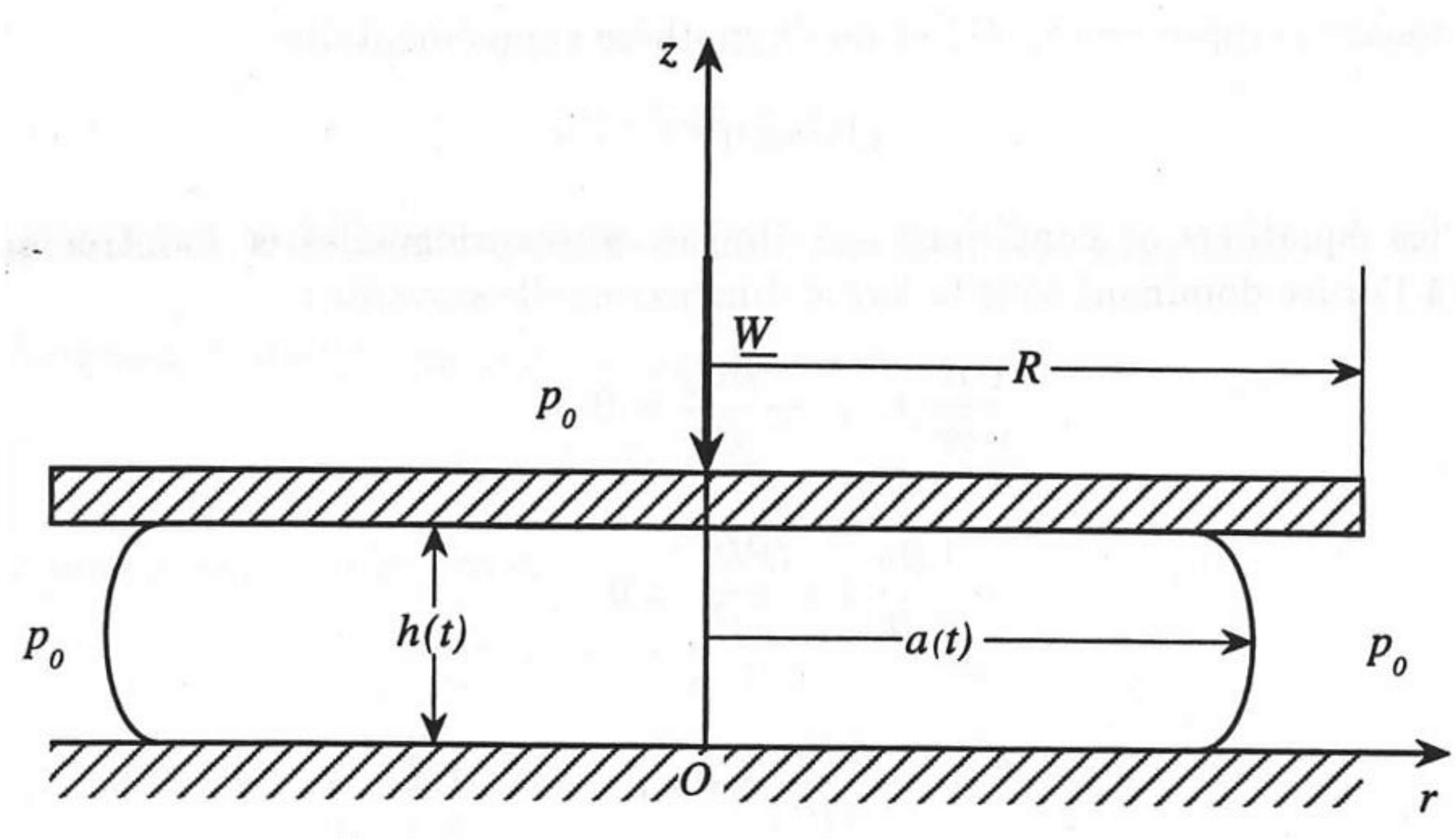


Figure 29: Squeezing of an oil film

Squeezing of an oil film

- Axisymmetric lubrication equations and boundary conditions

$$u_r(r, h(t), t) = 0 \qquad u_z(r, h(t), t) = \frac{dh}{dt}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{\partial u_z}{\partial z} = 0$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu \frac{\partial^2 u_r}{\partial z^2} = 0$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial z} = 0$$

$$p(a(t), z, t) = p_0$$

$$u_r(r, 0, t) = u_z(r, 0, t) = 0$$

Squeezing of an oil film

- Integrating the radial momentum equation twice and using BC:

$$u_r(r,z,t) = \frac{1}{2\mu}\frac{\partial p}{\partial r} z(z-h)$$

- Using continuity equation, one finds

$$u_z(r,h,t) = -\frac{1}{r}\int_0^h \frac{\partial}{\partial r}(r u_r) dz$$

$$\boxed{u_z(r,h,t) = \frac{h^3}{12\mu}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right)}$$

- Using BC in z=h(t) to find the pressure

$$\frac{h^3}{12\mu}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) = \frac{dh}{dt}$$

$$\boxed{p(r,t) = p_0 + \frac{3\mu}{h^3}\frac{dh}{dt}(r^2 - a^2)}$$

Squeezing of an oil film

- Recalling

$$u_r(r,z,t) = \frac{1}{2\mu}\frac{\partial p}{\partial r} z(z-h)$$

- And going back to continuity equation

$$u_z(r,z,t) = -\frac{1}{2\mu}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right)\int_0^z z(z-h)dz$$

- With the previous result for pressure, one finds

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) = \frac{12\mu}{h^3}\frac{dh}{dt}$$

$$\boxed{u_z(r,z,t) = \frac{dh}{dt}\left(\frac{z}{h}\right)^2\left[3-2\left(\frac{z}{h}\right)\right]}$$

- Since the viscous stress vanishes

$$\mathbf{f} = \int_0^a [p(r,t)-p_0]2\pi r dr\ \mathbf{e_x} = -\frac{3\pi\mu}{2h^3}a^4\frac{dh}{dt}\ \mathbf{e_x}$$

Squeezing of an oil film

- Lubrication force

$$\mathbf{f} = \int_0^a [p(r,t) - p_0] 2\pi r dr \ \mathbf{e_x} = -\frac{3\pi\mu}{2h^3} a^4 \frac{dh}{dt} \ \mathbf{e_x}$$

- Using conservation of volume

$$\pi a^2(t) h(t) = \pi a_0^2 h_0$$

$$a^2(t) = a_0^2 \frac{h_0}{h(t)}$$

$$\boxed{\mathbf{f} = -\frac{3\pi\mu a_0^4 h_0^2}{2h^5} \frac{dh}{dt} \ \mathbf{e_x}}$$

Squeezing of an oil film

- The lubrication force equals the weight W of the wall squeezing the film

$$-\frac{3\pi\mu a_0^4 h_0^2}{2h^5}\frac{dh}{dt} = W$$

- Integration yields

$$\frac{1}{h^4} - \frac{1}{h_0^4} = \frac{8Wt}{3\pi\mu a_0^4 h_0^2}$$

$$h(t) = \frac{h_0}{\left[1 + \frac{8Wh_0^2}{3\pi\mu a_0^4}t\right]^{1/4}}$$

$$a(t) = a_0\left[1 + \frac{8Wh_0^2}{3\pi\mu a_0^4}t\right]^{1/8}$$

Squeezing of an oil film

- How can you transpose this solution to the case of adhesive tape?

- What time is required for the adhesive tape glued on the ceiling to fall under its own weight?

Almost unidirectional flow

- Hele-Shaw
- Lubrication

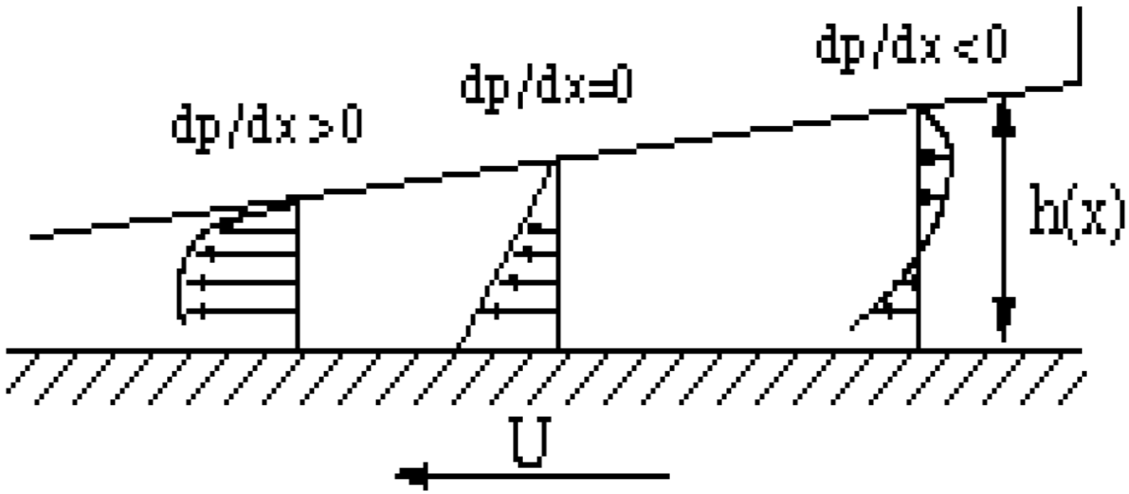


Figure 26: Lubrication flow

Almost unidirectional flow

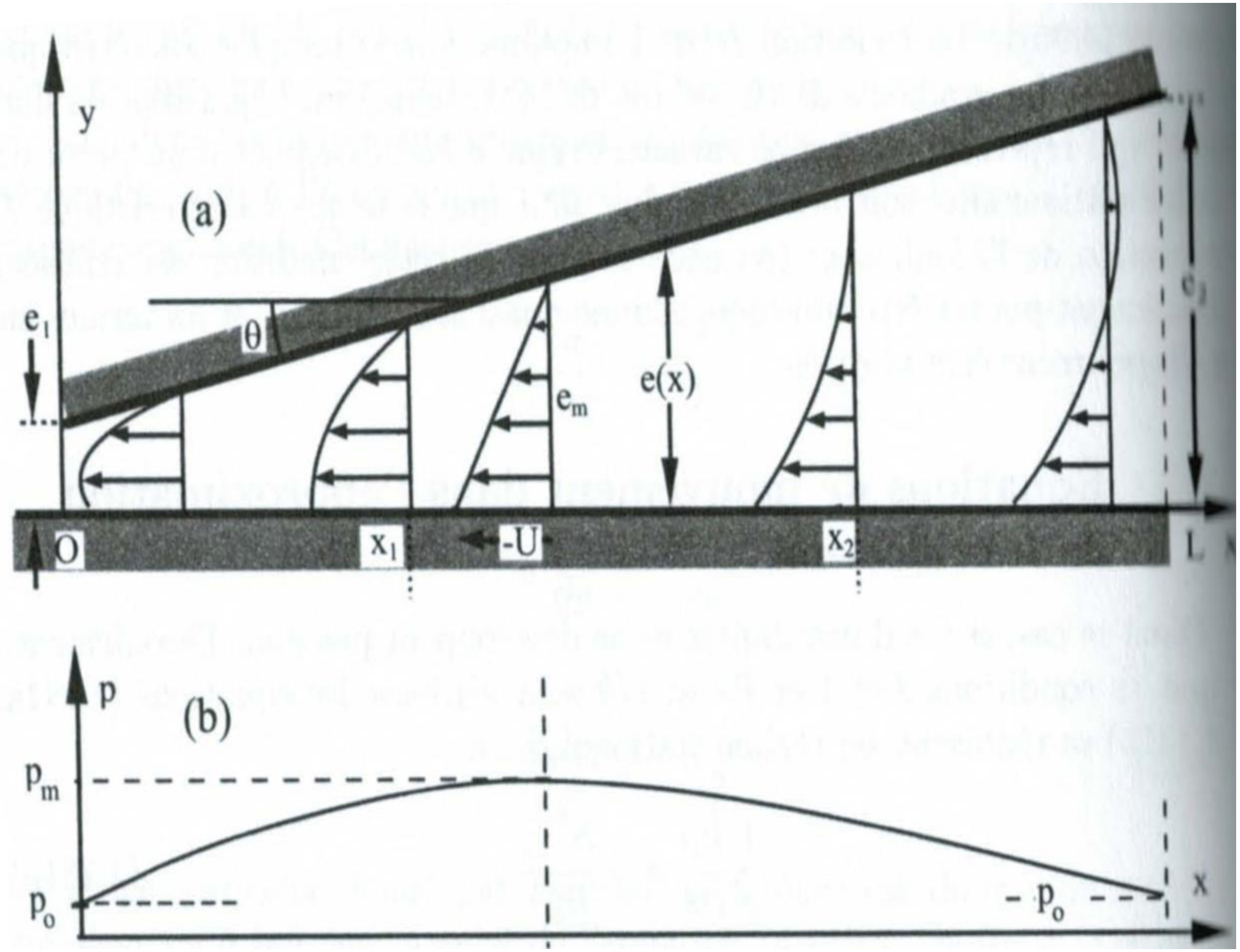


Figure 27: Lubrication flow, pressure distribution

Almost unidirectional flow

- Velocity profile

$$v(x,y) = -\frac{1}{2\mu}\frac{dp}{dx}y[e(x)-y] - U\frac{e(x)-y}{e(x)}$$

- Pressure gradient

$$\frac{dp}{dx} = \theta\frac{dp}{de} = -\frac{12\mu Q}{e(x)^3} - \frac{6\mu U}{e(x)^2}$$

- Integrating, one finds the pressure

$$p(x) = p_0 + \frac{6\mu Q}{\theta}\left[\frac{1}{e(x)^2} - \frac{1}{e_1^2}\right] + \frac{6\mu U}{\theta}\left[\frac{1}{e(x)} - \frac{1}{e_1}\right]$$

Almost unidirectional flow

- Tangent viscous forces

$$F_T = \frac{2\mu U}{\theta}\left[-\ln\frac{e_2}{e_1} - \frac{3(e_2 - e_1)}{e_2 + e_1}\right]$$

- Normal pressure forces

$$F_N = \int_0^L (p - p_0)dx = \frac{6\mu U}{\theta^2}\left[\ln\frac{e_2}{e_1} - \frac{2(e_2 - e_1)}{e_2 + e_1}\right]$$

$$\boxed{F_N > F_T}$$

Lubrication

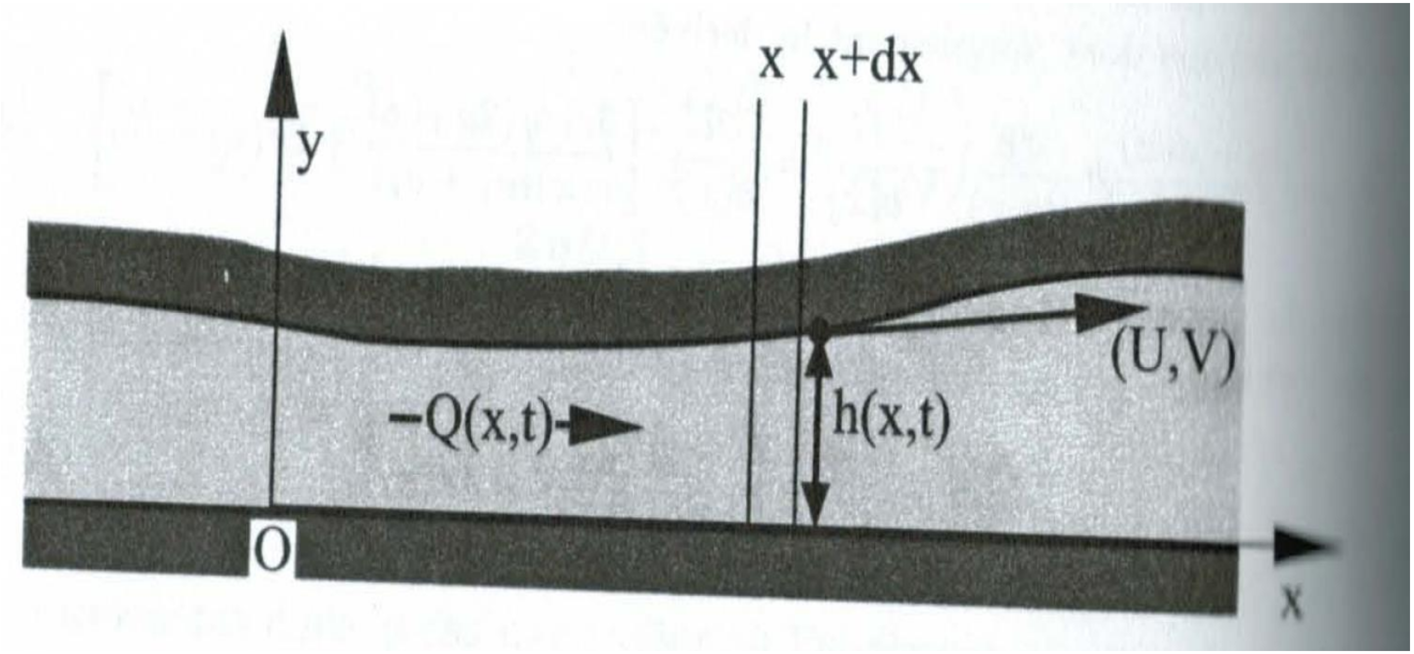


Figure 28: Lubrication flow

$$\frac{\partial h}{\partial t} = \frac{1}{12\mu}\frac{\partial}{\partial x}\left[h^3\frac{\partial p}{\partial x}\right] - \frac{1}{2}\left(h\frac{\partial U}{\partial x} + U\frac{\partial h}{\partial x}\right)$$