Outline

1. Intro+kinematics
2. Dynamics+Dimensional Analysis
3. Low Reynolds number flow/ Stokes drag
4. Stokes drag-sedimentation-reversibility-Hele Shaw-Pipe Flows
5. Lubrication
6. Unsteady flows
7. Boundary layers

D'Alembert's paradox

Inviscid flow theory does not provide a good prediction of the drag on a flat plate

$$\frac{\partial u}{\partial x} + \frac{\partial \upsilon}{\partial y} = 0$$

$$\left(u\frac{\partial}{\partial x} + \upsilon\frac{\partial}{\partial y} \right) u = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\cancel{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}} \right) u$$

$$\left(u\frac{\partial}{\partial x} + \upsilon\frac{\partial}{\partial y} \right) \upsilon = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\cancel{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}} \right) \upsilon$$

Singular limit as $\mathrm{Re} \to \infty$

$$\overline{\mathrm{div}}\,\underline{\overline{U}} = 0$$

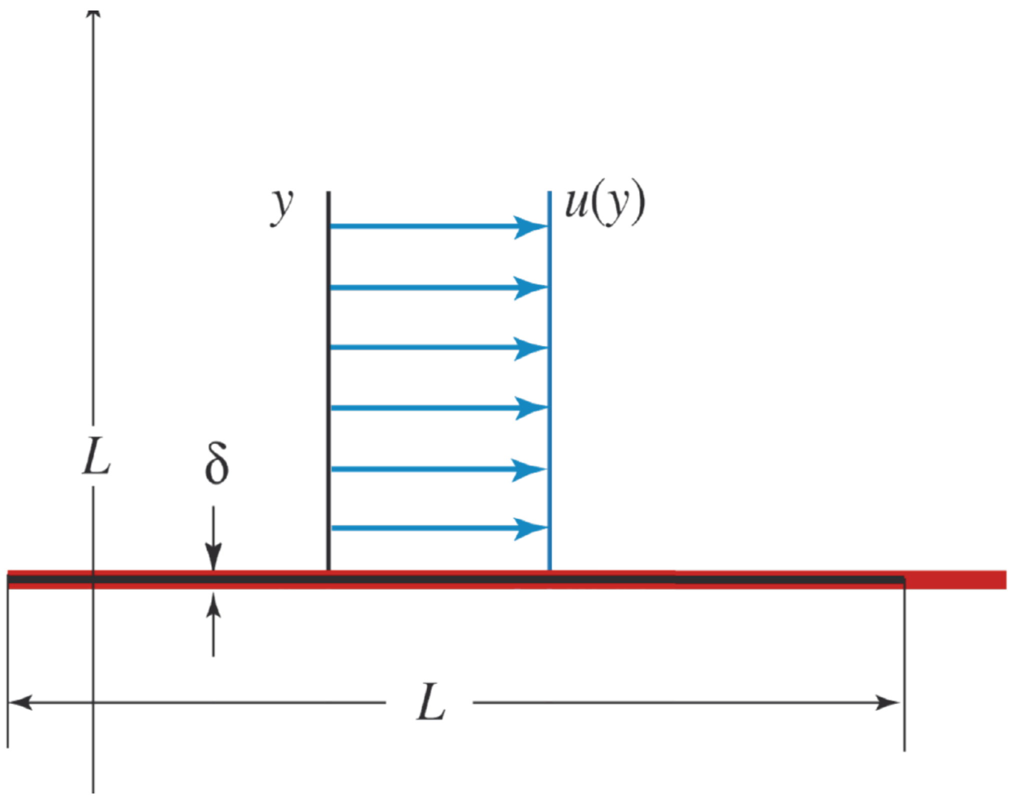
$$\underline{\underline{\overline{\mathrm{grad}}\,\overline{U}}}\,.\,\underline{\overline{U}} = -\underline{\overline{\mathrm{grad}}}\,\overline{p} + \cancel{\frac{1}{\mathrm{Re}}\overline{\Delta}\,\underline{\overline{U}}}$$

$$\underline{\overline{U}}\,.\,\underline{n} = 0$$

$$\cancel{\underline{\overline{U}}_T} = 0$$

D'Alembert's paradox

Inviscid flow theory does not provide a good prediction of the drag on a flat plate

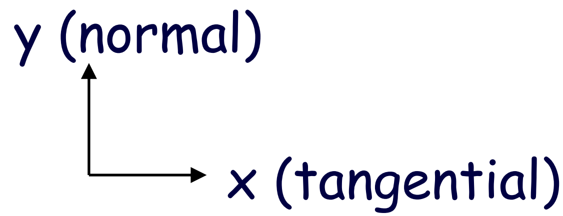


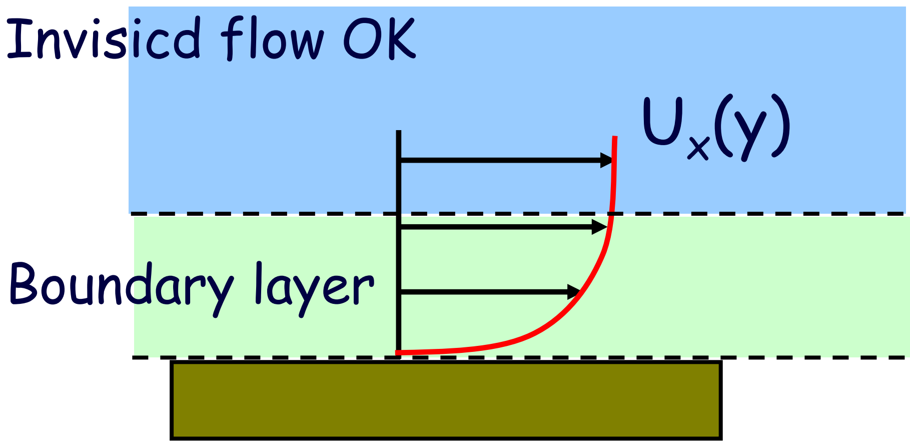
D'Alembert's paradox

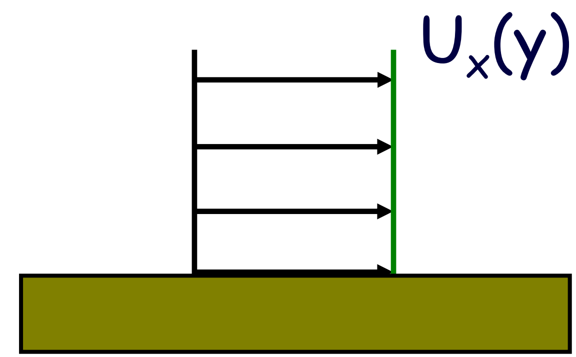
Obviously, a moving body experiments a drag force in real life.

In order to resolve this paradox, we need to account for the viscous friction, even at high Re, that should not be neglected in a boundary layer located close to the obstacle. Inviscid flow theory can still be applied far from the obstacle

Viscous friction



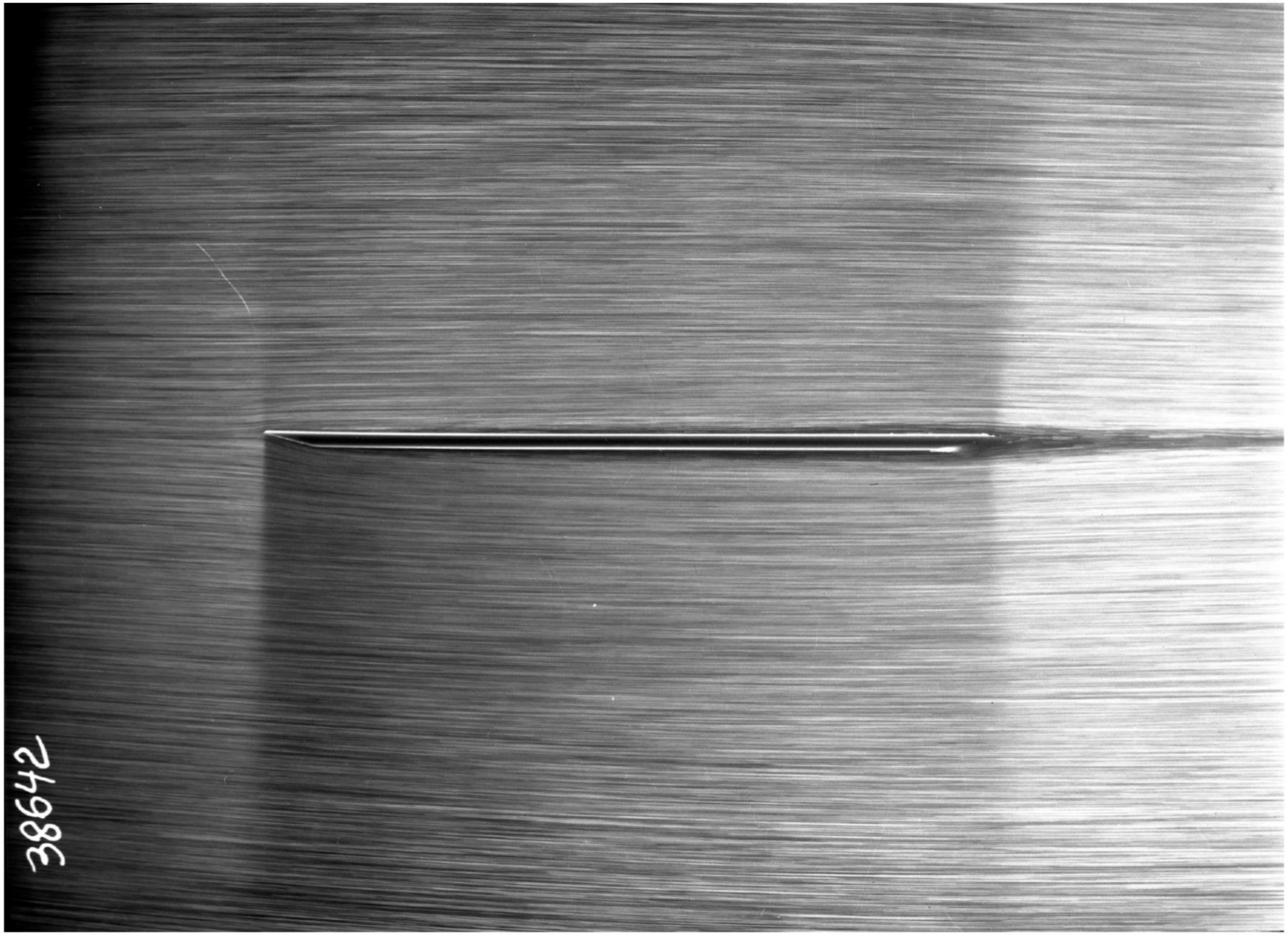




Real flow: the viscous friction decelerates the fluid close to the wall (no slip)

Inviscid flow: the fluid slips on the wall

Écoulement le long d'une plaque plane
$Re_L = 10^4$



Werlé (1974)

The boundary layer

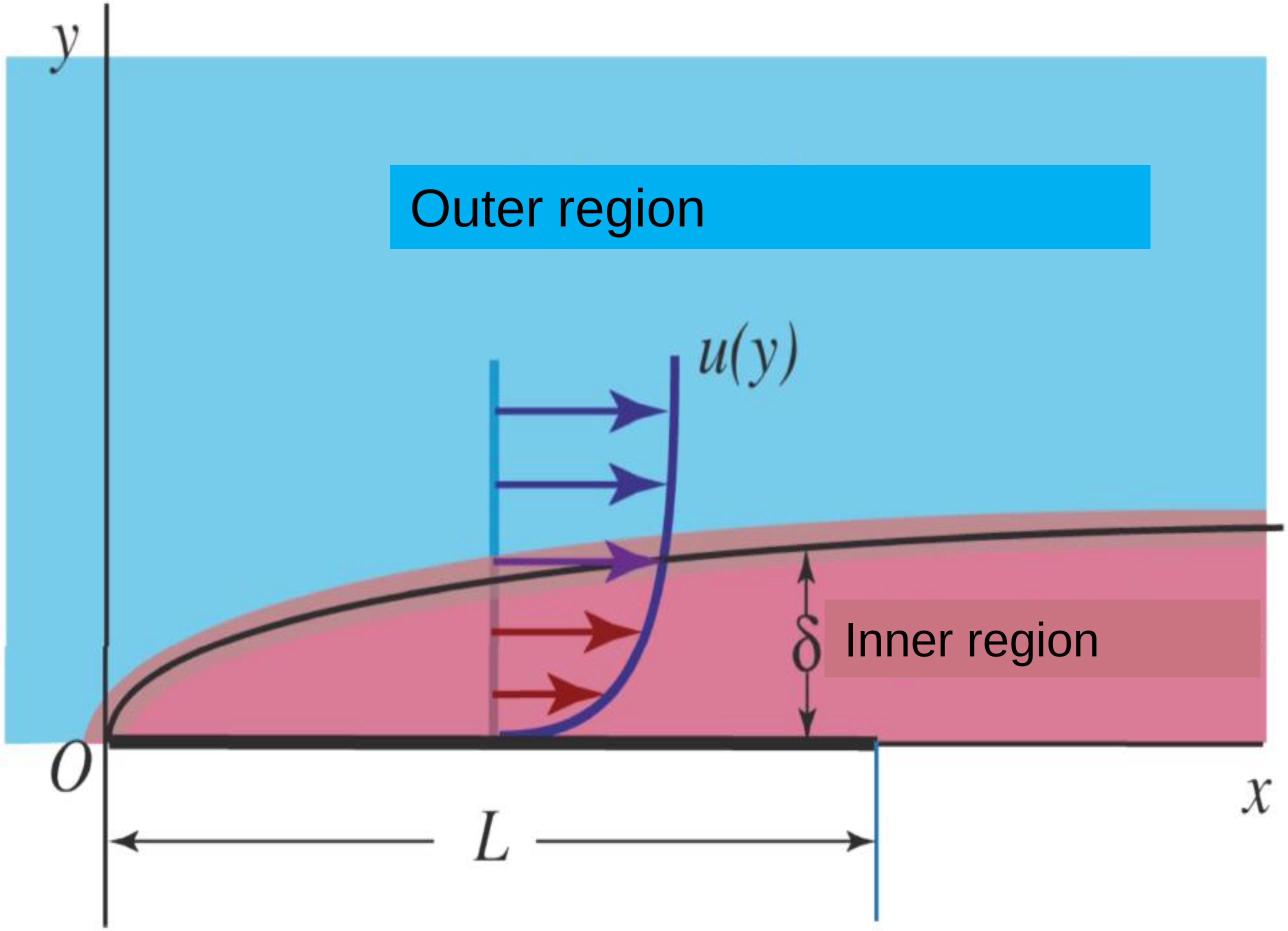
The boundary layer is a region where the velocity (streamwise and normal) is reduced due to no slip condition and friction.

It is a region where the streamwise velocity component is much larger than the normal $U_x >> U_y$

It is a region of large shear, where the normal gradient of the streamwise velocity is large.

$\dfrac{\partial U_x}{\partial y}$ is not negligible even if $U_x << 1$

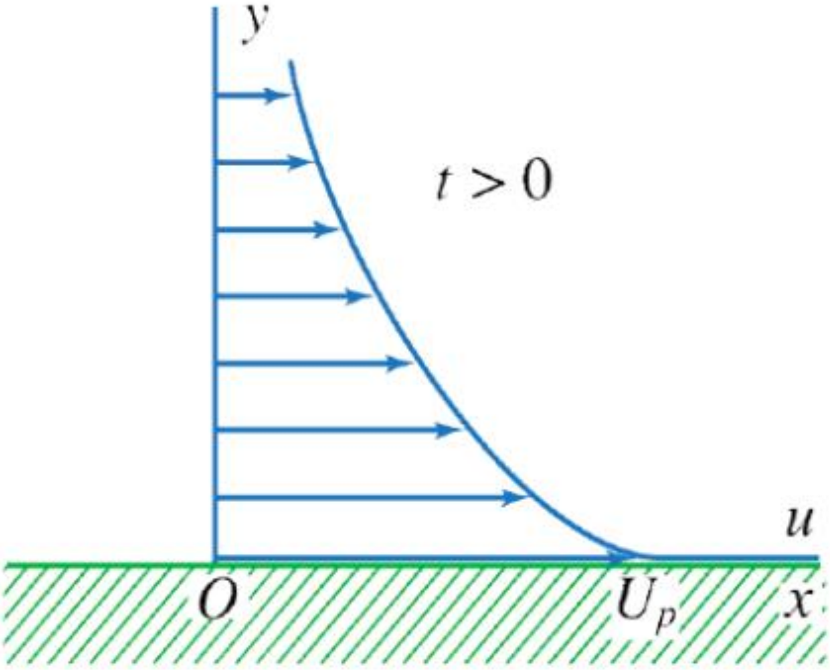
Boundary layer



Fundamental question:

What is the thickness of the boundary layer?

Have a look at a similar experiment : an impulsively started plate (1st Stokes problem)



$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$$

$$u(0,t) = U_p \qquad t > 0$$

$$u(y,t) \to 0 \qquad y \to +\infty$$

With a space-time analogy, dimensional analysis suggests to replace t by x/U in δ

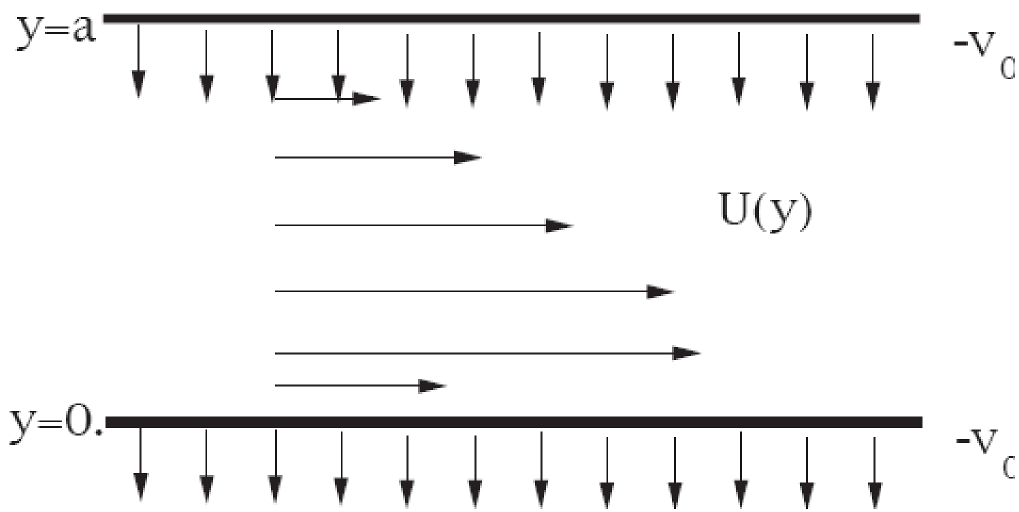
$$\delta(t) \sim (\nu t)^{1/2}$$

$$\Rightarrow \delta(x) \sim (\nu x/U)^{1/2}$$

$$\delta(x)/x \sim Re_x^{-1/2}$$

The method of matched asymptotic expansion

Escher



A 2D channel flow from left to right due to an imposed pressure gradient with suction at the lower wall and blowing at the upper.

We look for a streamwise independent solution

$$\vec{u}(x,y,z) = U(y)\,\vec{e}_x + V(y)\,\vec{e}_y$$

Continuity ensures that $V=V_0$

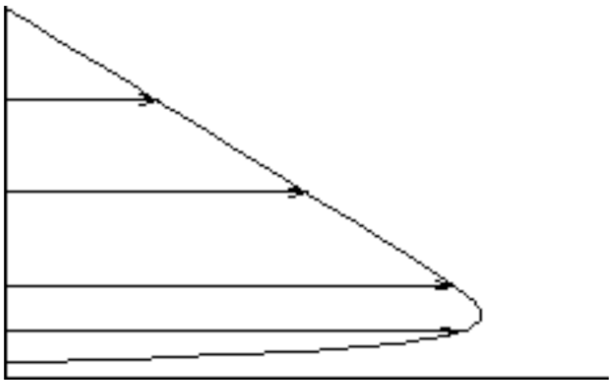
$$0 = -\frac{\partial p}{\partial y} \;\; \text{so} \;\; p(x) = -kx + P_0$$

$$-\,v_0\frac{\partial U}{\partial y} = \frac{k}{\rho} + \nu\frac{\partial^2 U}{\partial y^2}$$

$U(0)=U(a)=0$

$$U_{exact}(y) = \frac{ka}{\rho v_0}\left(-\frac{y}{a} + \left(\frac{1 - exp(-v_0 y/\nu)}{1 - exp(-v_0 a/\nu)}\right)\right)$$

Weak aspiration



strong aspiration

Nondimensionalization

$$y = a\bar{y} \qquad u = U_0\bar{u}$$

$$-(v_0 a/\nu)\frac{\partial \bar{U}}{\partial \bar{y}} = ka^2/(\rho\nu U_0) + \frac{\partial^2 \bar{U}}{\partial \bar{y}^2}$$

$$U_0 = ka^2/\rho/\nu$$

$$\boxed{-(v_0 a/\nu)\frac{\partial \bar{U}}{\partial \bar{y}} = 1 + \frac{\partial^2 \bar{U}}{\partial \bar{y}^2}}$$

Reynolds

$$\bar{U}(0) = \bar{U}(1) = 0$$

Light blowing/suction, Re<<1 ε=Re

$$\bar{U}'' + \varepsilon \bar{U}' + 1 = 0, \text{ and } \bar{U}(0) = \bar{U}(1) = 0$$

$U = u_0 + \varepsilon u_1$

$u_0 = \frac{\bar{y}}{2} - \frac{\bar{y}^2}{2}$

$$\bar{u}_1'' + 1/2 - \bar{y} = 0 \text{ and } \bar{u}_1(0) = \bar{u}_1(1) = 0$$

$$\bar{u}_1 = \frac{\bar{y}^3}{6} - \frac{\bar{y}^2}{4} + \frac{\bar{y}}{12}$$

Validation

$$\bar{U}_{exact} = -\frac{-e^{\varepsilon}\bar{y} + \bar{y} + e^{\varepsilon} - e^{\varepsilon - \varepsilon\bar{y}}}{\varepsilon - \varepsilon e^{\varepsilon}}$$

$$= \left(\frac{\bar{y}}{2} - \frac{\bar{y}^2}{2}\right) + \left(\frac{\bar{y}^3}{6} - \frac{\bar{y}^2}{4} + \frac{\bar{y}}{12}\right)\varepsilon \cdot$$

Large blowing/suction, Re>>1 ε=1/Re

$$\varepsilon \bar{U}'' + \bar{U}' + 1 = 0; \text{ with } \bar{U}(0) = \bar{U}(1) = 0,$$

$$\bar{U} = 1 - \bar{y}$$

$$\bar{U}(0) = 1,$$

Contradiction in y=0!

Singular perturbation!

Large blowing/suction, Re>>1 ε=1/Re

$$\varepsilon \bar{U}'' + \bar{U}' + 1 = 0; \text{ with } \bar{U}(0) = \bar{U}(1) = 0.$$

Rescaling close to y=0

$$U_0 = ka/\rho/v_0$$

$$\tilde{U} = \bar{U} \text{ and } \bar{y} = \delta\tilde{y}, \text{ with } \delta << 1$$

$$\varepsilon \frac{1}{\delta^2} \frac{d^2\tilde{U}}{d\tilde{y}^2} + \frac{1}{\delta} \frac{d\tilde{U}}{d\tilde{y}} + 1 = 0$$

Dominant balance

$$\varepsilon \frac{1}{\delta^2} = \frac{1}{\delta} \qquad \delta = \varepsilon$$

$$\varepsilon \frac{1}{\delta^2}\frac{d^2\tilde{U}}{d\tilde{y}^2} + \frac{1}{\delta}\frac{d\tilde{U}}{d\tilde{y}} + \cancel{1} = 0$$

$$\frac{d^2\tilde{U}}{d\tilde{y}^2} + \frac{d\tilde{U}}{d\tilde{y}} = 0 \qquad \tilde{U}(0) = 0$$

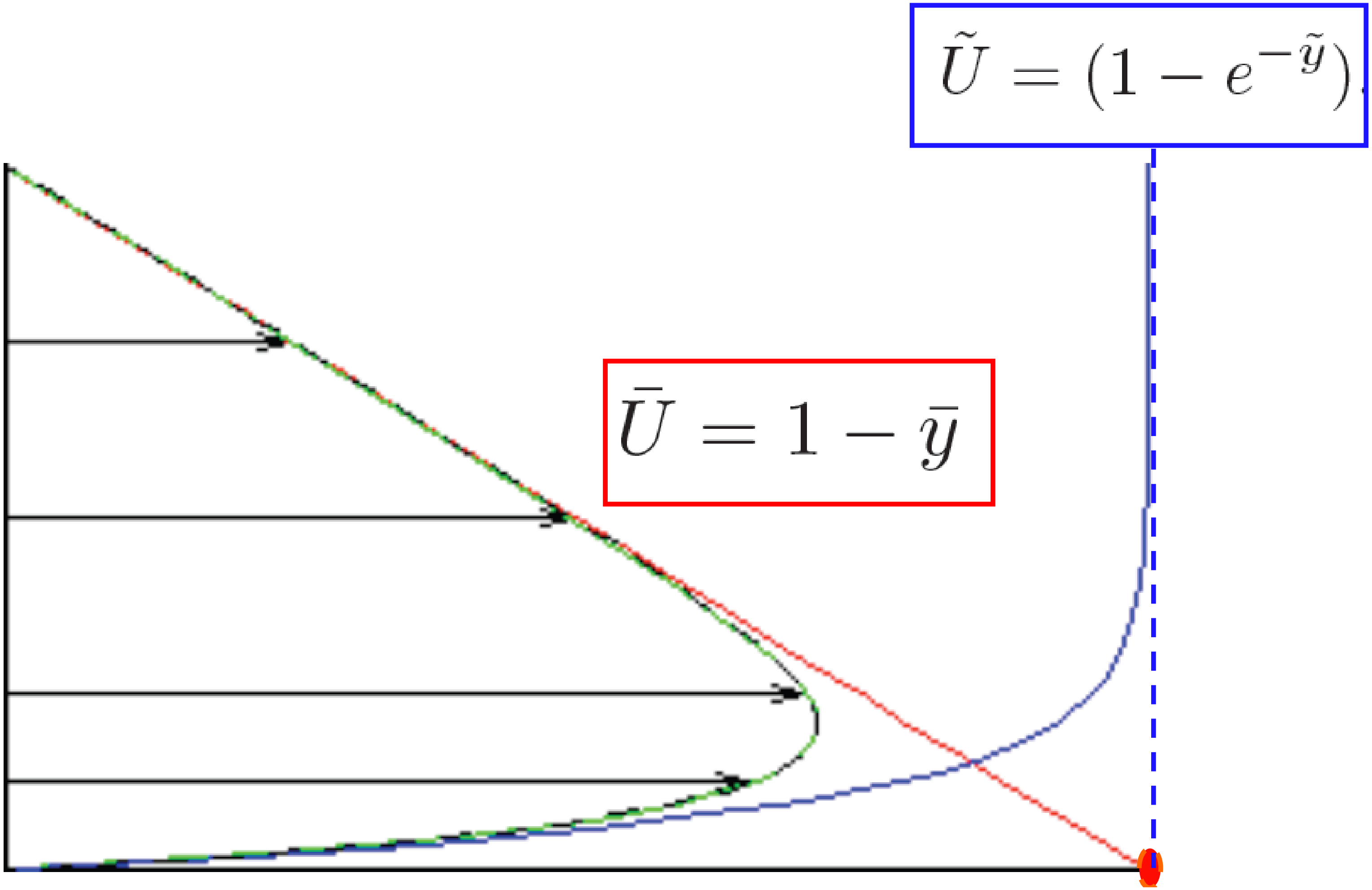
$$\tilde{U} = A(1 - e^{\tilde{y}})$$

Matching

$$\lim_{\bar{y}\to 0} \bar{U}(\bar{y}) = \lim_{\tilde{y}\to\infty} \tilde{U}(\tilde{y})$$

$$\bar{U}(0) = 1 \qquad \tilde{U}(\infty) = A$$

$$A = 1$$

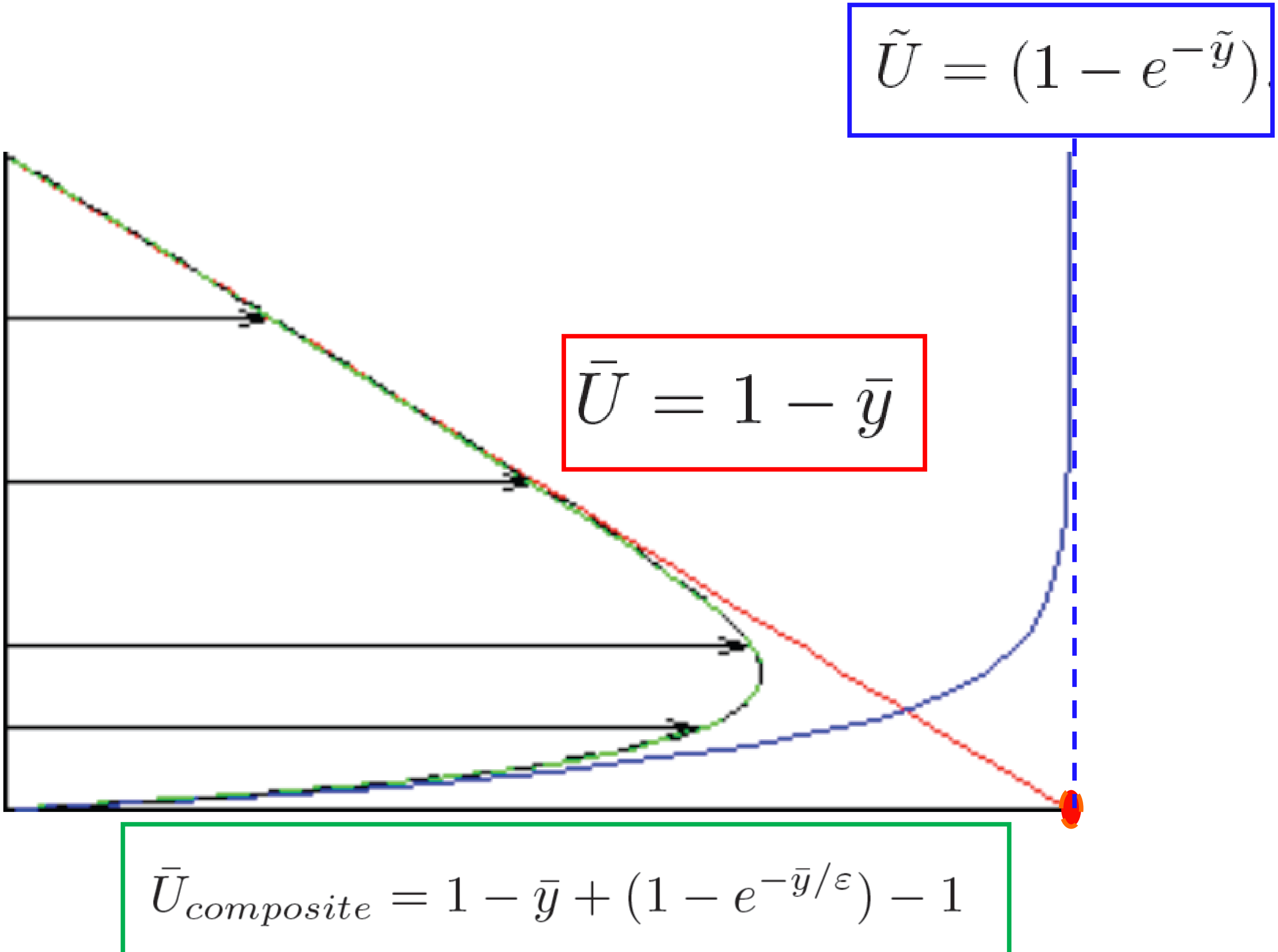


$$\bar{U}_{exact} \simeq -\varepsilon(\bar{y}/\varepsilon) + \frac{(1 - e^{\bar{y}/\varepsilon})}{(1 - 0)} \simeq (1 - e^{-\bar{y}/\varepsilon})$$

Composite expansion

$$\bar{U}_{composite} = \bar{U}(\bar{y}) + \tilde{U}(\bar{y}/\varepsilon) - \bar{U}(0)$$

$$\bar{U}_{composite} = 1 - \bar{y} + (1 - e^{-\bar{y}/\varepsilon}) - 1$$


$\tilde{U} = (1 - e^{-\tilde{y}})$
$\bar{U} = 1 - \bar{y}$
$\bar{U}_{composite} = 1 - \bar{y} + (1 - e^{-\bar{y}/\varepsilon}) - 1$

Matching condition

$\lim_{\tilde{x}\to+\infty}$ DI = $\lim_{x\to 0}$ Dext

2D Flow

$$\frac{\partial u}{\partial x}+\frac{\partial \upsilon}{\partial y}=0$$

$$\left(u\frac{\partial}{\partial x}+\upsilon\frac{\partial}{\partial y}\right)u=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)u$$

$$\left(u\frac{\partial}{\partial x}+\upsilon\frac{\partial}{\partial y}\right)\upsilon=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\upsilon$$

Scales

$$\tilde{x} = \frac{x}{L} \qquad \tilde{y} = \frac{y}{\delta}$$

$$\tilde{u} = \frac{u}{U_\infty} \qquad \tilde{\upsilon} = \frac{\upsilon}{V} \qquad \tilde{p} = \frac{p}{\rho U_\infty^2}$$

$$\frac{\delta}{L} << 1 \qquad \frac{V}{U_\infty} << 1$$

Dominant balance

$$\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{L}{\delta}\frac{V}{U_\infty}\frac{\partial \tilde{v}}{\partial \tilde{y}} = 0$$

$$\tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{L}{\delta}\frac{V}{U_\infty}\tilde{v}\frac{\partial \tilde{u}}{\partial \tilde{y}} = -\frac{\partial \tilde{p}}{\partial \tilde{x}}$$

$$+\frac{1}{\mathrm{Re}}\frac{L^2}{\delta^2}\left(\frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2} + \frac{\delta^2}{L^2}\frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2}\right)$$

$$\frac{V}{U_\infty}\tilde{u}\frac{\partial \tilde{v}}{\partial \tilde{x}} + \left(\frac{V}{U_\infty}\right)^2\frac{L}{\delta}\tilde{v}\frac{\partial \tilde{v}}{\partial \tilde{y}} = -\frac{L}{\delta}\frac{\partial \tilde{p}}{\partial \tilde{y}}$$

$$+\frac{1}{\mathrm{Re}}\frac{L^2}{\delta^2}\frac{V}{U_\infty}\left(\frac{\partial^2 \tilde{v}}{\partial \tilde{y}^2} + \frac{\delta^2}{L^2}\frac{\partial^2 \tilde{v}}{\partial \tilde{x}^2}\right)$$

Dominant balance

$$\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{L}{\delta}\frac{V}{U_\infty}\frac{\partial \tilde{\upsilon}}{\partial \tilde{y}} = 0$$

$$\tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{L}{\delta}\frac{V}{U_\infty}\tilde{\upsilon}\frac{\partial \tilde{u}}{\partial \tilde{y}} = -\frac{\partial \tilde{p}}{\partial \tilde{x}}$$

$$+\frac{1}{\mathrm{Re}}\frac{L^2}{\delta^2}\left(\frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2} + \frac{\delta^2}{L^2}\frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2}\right)$$

$$\frac{V}{U_\infty}\tilde{u}\frac{\partial \tilde{\upsilon}}{\partial \tilde{x}} + \left(\frac{V}{U_\infty}\right)^2\frac{L}{\delta}\tilde{\upsilon}\frac{\partial \tilde{\upsilon}}{\partial \tilde{y}} = -\frac{L}{\delta}\frac{\partial \tilde{p}}{\partial \tilde{y}}$$

$$+\frac{1}{\mathrm{Re}}\frac{L^2}{\delta^2}\frac{V}{U_\infty}\left(\frac{\partial^2 \tilde{\upsilon}}{\partial \tilde{y}^2} + \frac{\delta^2}{L^2}\frac{\partial^2 \tilde{\upsilon}}{\partial \tilde{x}^2}\right)$$

Dominant balance

$$\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{L}{\delta}\frac{V}{U_\infty}\frac{\partial \tilde{\upsilon}}{\partial \tilde{y}} = 0$$

$$\tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{L}{\delta}\frac{V}{U_\infty}\tilde{\upsilon}\frac{\partial \tilde{u}}{\partial \tilde{y}} = -\frac{\partial \tilde{p}}{\partial \tilde{x}}$$

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$$\frac{V}{U_\infty}\tilde{u}\frac{\partial \tilde{\upsilon}}{\partial \tilde{x}} + \left(\frac{V}{U_\infty}\right)^2 \frac{L}{\delta}\tilde{\upsilon}\frac{\partial \tilde{\upsilon}}{\partial \tilde{y}} = -\frac{L}{\delta}\frac{\partial \tilde{p}}{\partial \tilde{y}}$$

$$+\frac{1}{\mathrm{Re}}\frac{L^2}{\delta^2}\frac{V}{U_\infty}\left(\frac{\partial^2 \tilde{\upsilon}}{\partial \tilde{y}^2} + \frac{\delta^2}{L^2}\frac{\partial^2 \tilde{\upsilon}}{\partial \tilde{x}^2}\right)$$

Dominant balance

$$\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{L}{\delta}\frac{V}{U_\infty}\frac{\partial \tilde{v}}{\partial \tilde{y}} = 0$$

$$\tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{L}{\delta}\frac{V}{U_\infty}\tilde{v}\frac{\partial \tilde{u}}{\partial \tilde{y}} = -\frac{\partial \tilde{p}}{\partial \tilde{x}}$$

$$+\frac{1}{\mathrm{Re}}\frac{L^2}{\delta^2}\left(\frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2} + \frac{\delta^2}{L^2}\frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2}\right)$$

$$\frac{V}{U_\infty}\tilde{u}\frac{\partial \tilde{v}}{\partial \tilde{x}} + \left(\frac{V}{U_\infty}\right)^2\frac{L}{\delta}\tilde{v}\frac{\partial \tilde{v}}{\partial \tilde{y}} = -\frac{L}{\delta}\frac{\partial \tilde{p}}{\partial \tilde{y}}$$

$$+\frac{1}{\mathrm{Re}}\frac{L^2}{\delta^2}\frac{V}{U_\infty}\left(\frac{\partial^2 \tilde{v}}{\partial \tilde{y}^2} + \frac{\delta^2}{L^2}\frac{\partial^2 \tilde{v}}{\partial \tilde{x}^2}\right)$$

Dominant balance

$$\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{L}{\delta}\frac{V}{U_\infty}\frac{\partial \tilde{\upsilon}}{\partial \tilde{y}} = 0$$

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$$+\frac{1}{\mathrm{Re}}\frac{L^2}{\delta^2}\left(\frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2} + \frac{\delta^2}{L^2}\frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2}\right)$$

$$\frac{V}{U_\infty}\tilde{u}\frac{\partial \tilde{\upsilon}}{\partial \tilde{x}} + \left(\frac{V}{U_\infty}\right)^2\frac{L}{\delta}\tilde{\upsilon}\frac{\partial \tilde{\upsilon}}{\partial \tilde{y}} = -\frac{L}{\delta}\frac{\partial \tilde{p}}{\partial \tilde{y}}$$

$$+\frac{1}{\mathrm{Re}}\frac{L^2}{\delta^2}\frac{V}{U_\infty}\left(\frac{\partial^2 \tilde{\upsilon}}{\partial \tilde{y}^2} + \frac{\delta^2}{L^2}\frac{\partial^2 \tilde{\upsilon}}{\partial \tilde{x}^2}\right)$$

Dominant balance
Leading order terms

$$\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{L}{\delta}\frac{V}{U_\infty}\frac{\partial \tilde{v}}{\partial \tilde{y}} = 0$$

$$\tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{L}{\delta}\frac{V}{U_\infty}\tilde{v}\frac{\partial \tilde{u}}{\partial \tilde{y}} = -\frac{\partial \tilde{p}}{\partial \tilde{x}}$$

$$+\frac{1}{\mathrm{Re}}\frac{L^2}{\delta^2}\left(\frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2} + \frac{\delta^2}{L^2}\frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2}\right)$$

$$\frac{V}{U_\infty}\tilde{u}\frac{\partial \tilde{v}}{\partial \tilde{x}} + \left(\frac{V}{U_\infty}\right)^2\frac{L}{\delta}\tilde{v}\frac{\partial \tilde{v}}{\partial \tilde{y}} = -\frac{L}{\delta}\frac{\partial \tilde{p}}{\partial \tilde{y}}$$

$$+\frac{1}{\mathrm{Re}}\frac{L^2}{\delta^2}\frac{V}{U_\infty}\left(\frac{\partial^2 \tilde{v}}{\partial \tilde{y}^2} + \frac{\delta^2}{L^2}\frac{\partial^2 \tilde{v}}{\partial \tilde{x}^2}\right)$$

Scales

$$\frac{L}{\delta}\frac{V}{U_\infty} = 1$$

$$\frac{L^2}{\delta^2}\frac{1}{\mathrm{Re}} = 1$$

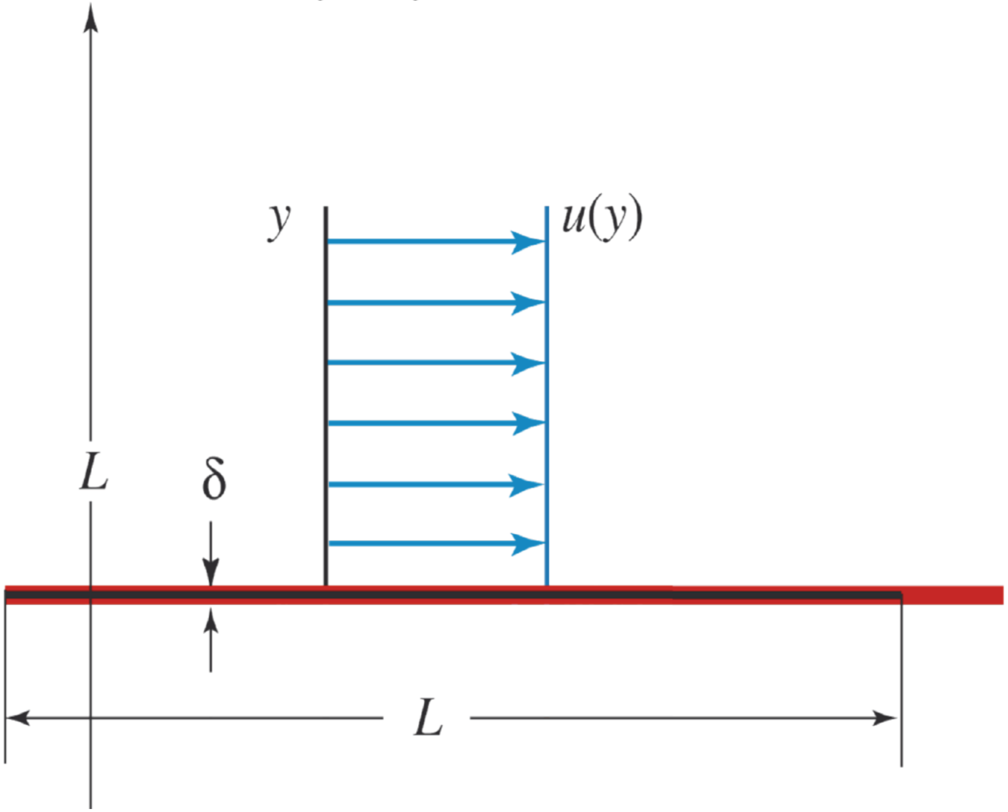
$$\cdot\delta = \frac{L}{\mathrm{Re}^{1/2}} \qquad V = \frac{U_\infty}{\mathrm{Re}^{1/2}}$$

Small parameter

$$\left(\frac{1}{\mathrm{Re}}\right)^{1/2} = \frac{\delta}{L} << 1$$

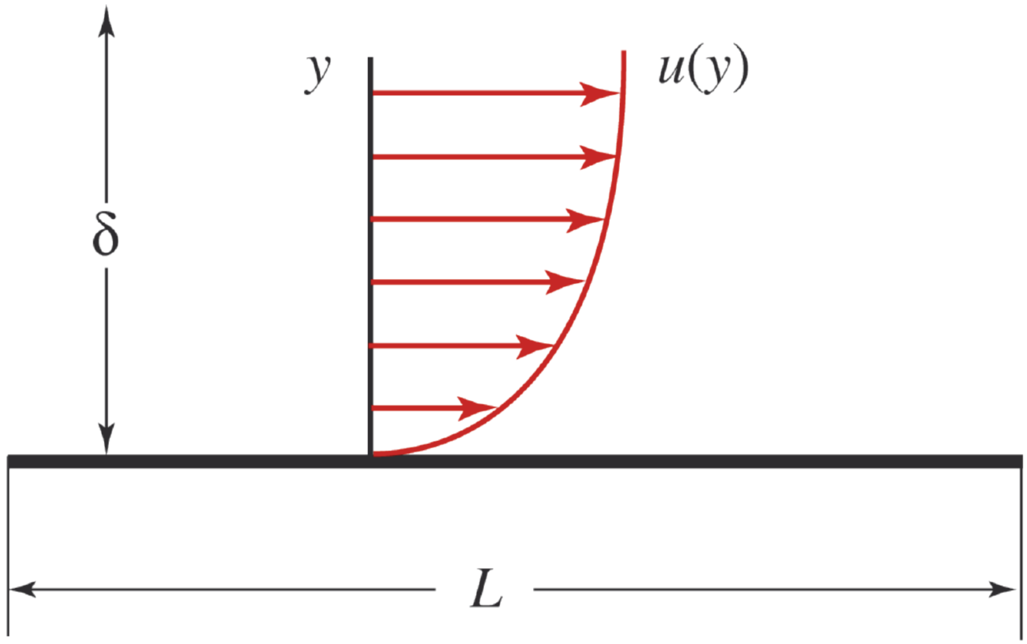
Outer approximation
Inviscid fluid

$$y = \bar{y}\,L = \mathcal{O}(L)$$



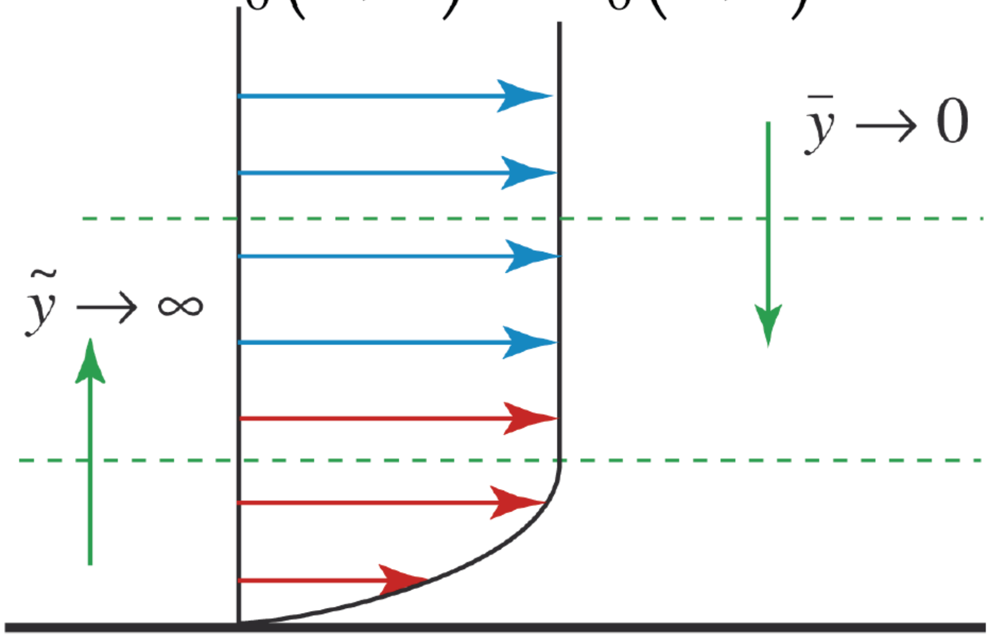
Inner approximation
Boundary layer equations

$$y = \tilde{y}\,\delta = \mathcal{O}(\delta)$$



Matching condition

$$\tilde{\mathcal{B}}_0(\overline{x},\infty) = \overline{\mathcal{B}}_0(\overline{x},0)$$

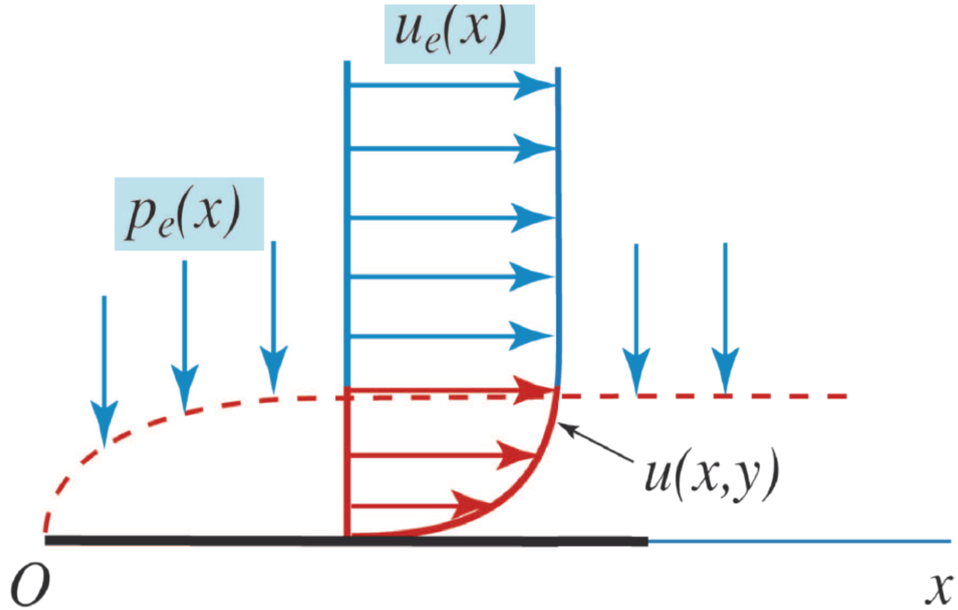


$$\tilde{u}_0(\overline{x},\infty) = \overline{u}_0(\overline{x},0) \equiv \overline{u}_e(\overline{x})$$

$$0 = \overline{v}_0(\overline{x},0)$$

$$\tilde{p}_0(\overline{x},\infty) = \overline{p}_0(\overline{x},0) \equiv \overline{p}_e(\overline{x})$$

Boundary layer equations



$$\frac{\partial u}{\partial x} + \frac{\partial \upsilon}{\partial y} = 0$$

$$\left(u\frac{\partial}{\partial x} + \upsilon\frac{\partial}{\partial y} \right) u = -\frac{1}{\rho}\frac{dp_e}{dx} + \nu\frac{\partial^2 u}{\partial y^2}$$

$$p(x,y) = p_e(x)$$

$$u(x,0) = \upsilon(x,0) = 0$$

$$u(x,\infty) = u_e(x)$$

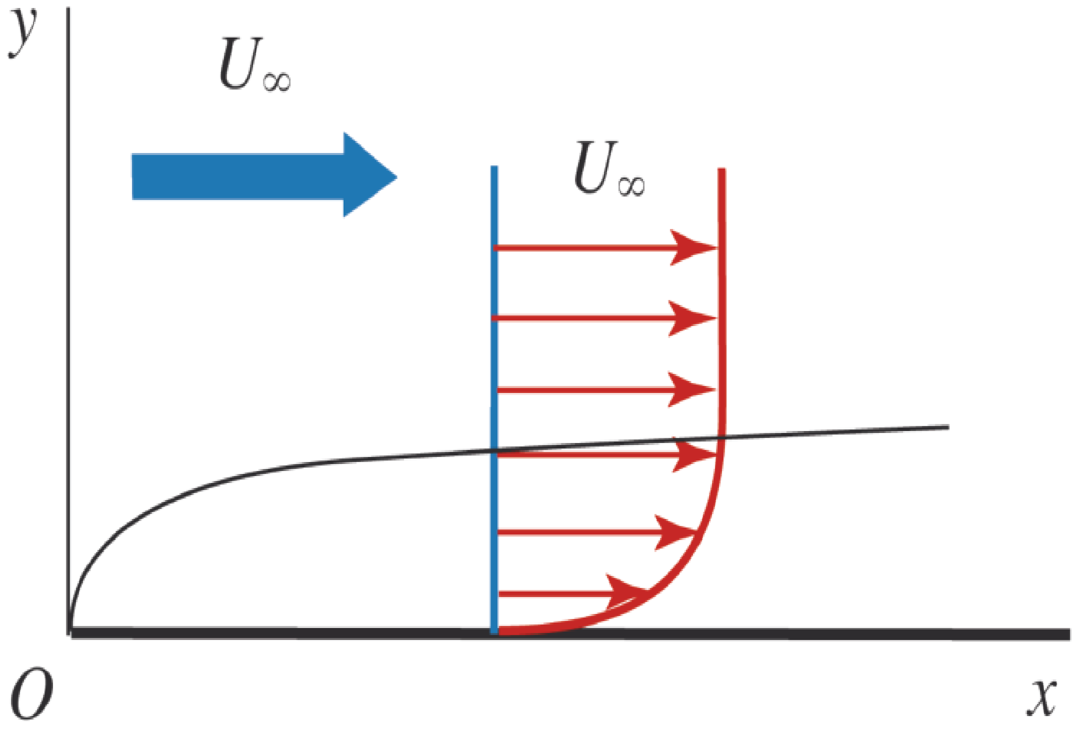
Boundary layer equations in streamfunction formulation

$$\left(\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\right)\frac{\partial\psi}{\partial y}=-\frac{1}{\rho}\frac{dp_e}{dx}+\nu\frac{\partial^3\psi}{\partial y^3}$$

$$\psi(x,0)=\frac{\partial\psi}{\partial y}(x,0)=0$$

$$\psi(x,y)\sim u_e(x)y \qquad , \qquad y\to\infty$$

Boundary layer on a semi-infinite plate



Self-similar solution?

$$\psi = A\psi' \qquad x = Bx' \qquad y = Cy'$$

$$v = Dv' \qquad U_\infty = EU_\infty'$$

$$\frac{AC}{BD}\left(\frac{\partial\psi'}{\partial y'}\frac{\partial}{\partial x'} - \frac{\partial\psi'}{\partial x'}\frac{\partial}{\partial y'}\right)\frac{\partial\psi'}{\partial y'} = v'\frac{\partial^3\psi'}{\partial y'^3}$$

$$\psi'(x',0) = \frac{\partial\psi'}{\partial y'}(x',0) = 0$$

$$\frac{A}{CE}\psi'(x',y') \sim U_\infty' y' \quad , \quad y' \to \infty$$

$$\boxed{\frac{AC}{BD} = 1 \qquad \frac{A}{CE} = 1}$$

Self-similar solution

$$\frac{\psi y}{\nu x} \qquad \frac{\psi}{U_\infty y}$$

$$\frac{\psi^2}{U_\infty \nu x} \qquad \frac{y^2}{\nu x / U_\infty}$$

$$\underbrace{\frac{\psi}{(U_\infty \nu x)^{1/2}}}_{U_\infty \delta(x)} \qquad \underbrace{\frac{y}{(\nu x / U_\infty)^{1/2}}}_{\delta(x)}$$

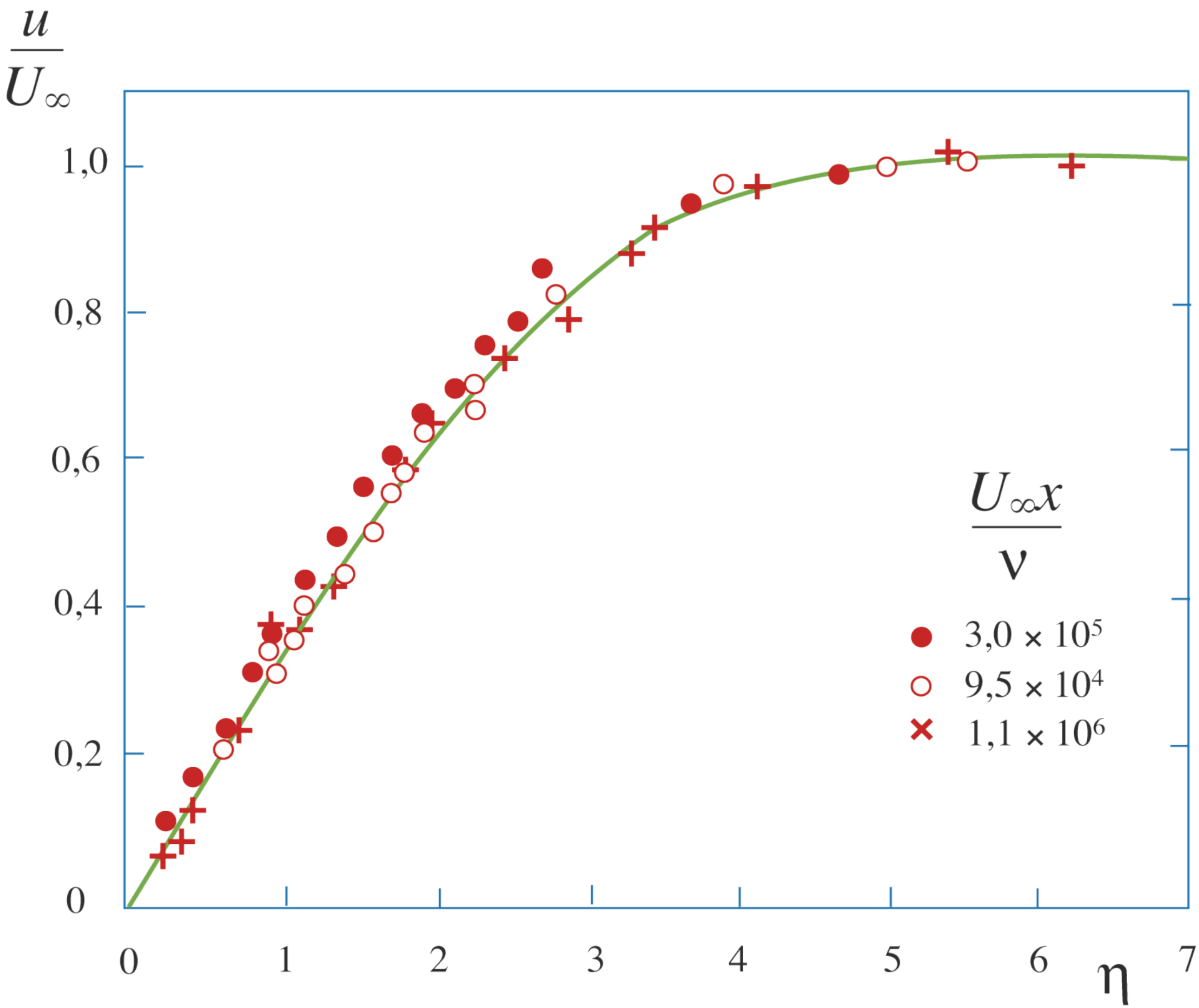
Boundary layer on a semi-infinite plate
Blasius solution

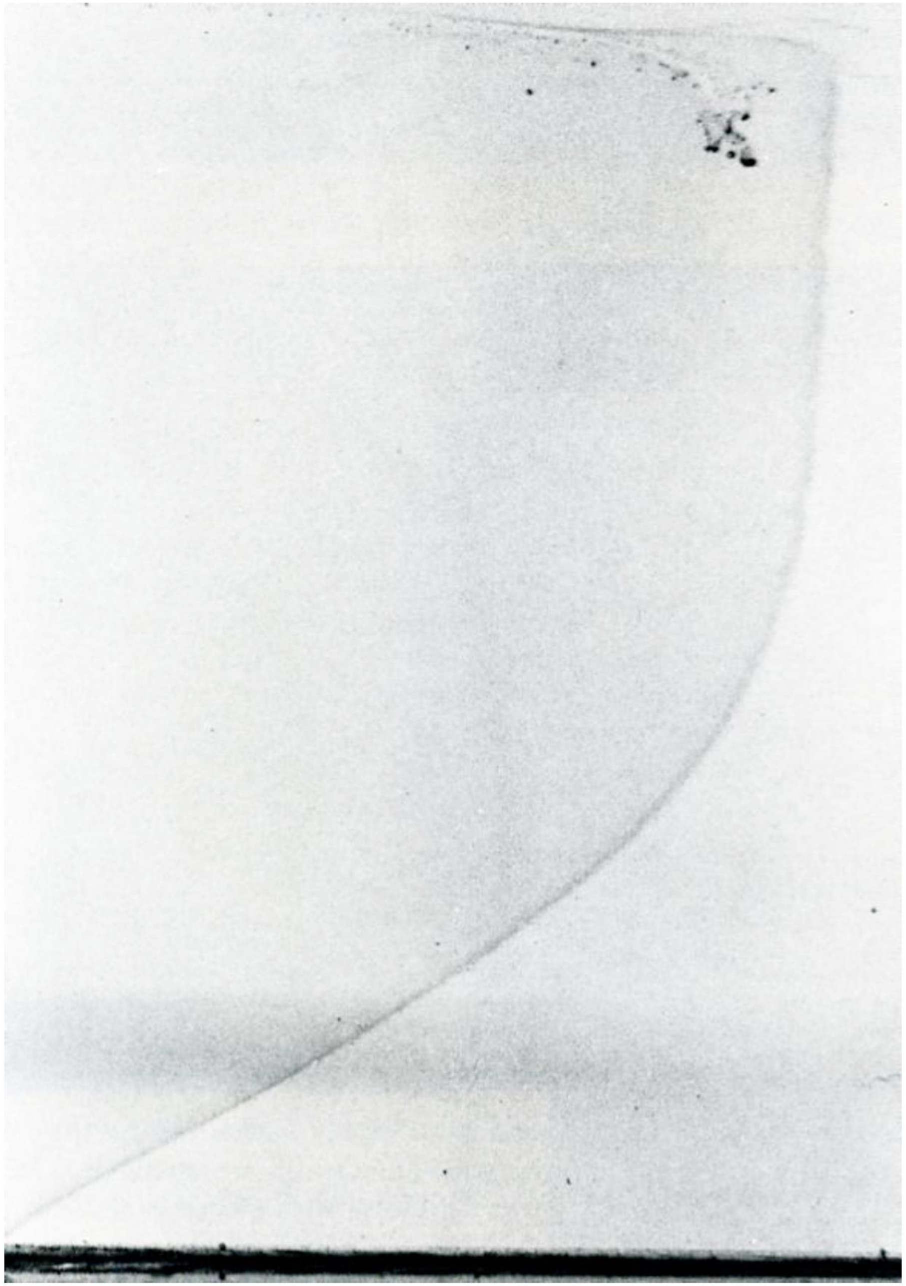
$$\frac{\psi(x,y)}{U_\infty \delta(x)} = f(\eta)$$

$$\eta = \frac{y}{\delta(x)} = \frac{y}{\left(\nu x / U_\infty\right)^{1/2}}$$

$$\boxed{\begin{gathered} 2f''' + ff'' = 0 \\ f(0) = f'(0) = 0 \\ f'(\infty) = 1 \end{gathered}}$$

Velocity profiles in BL

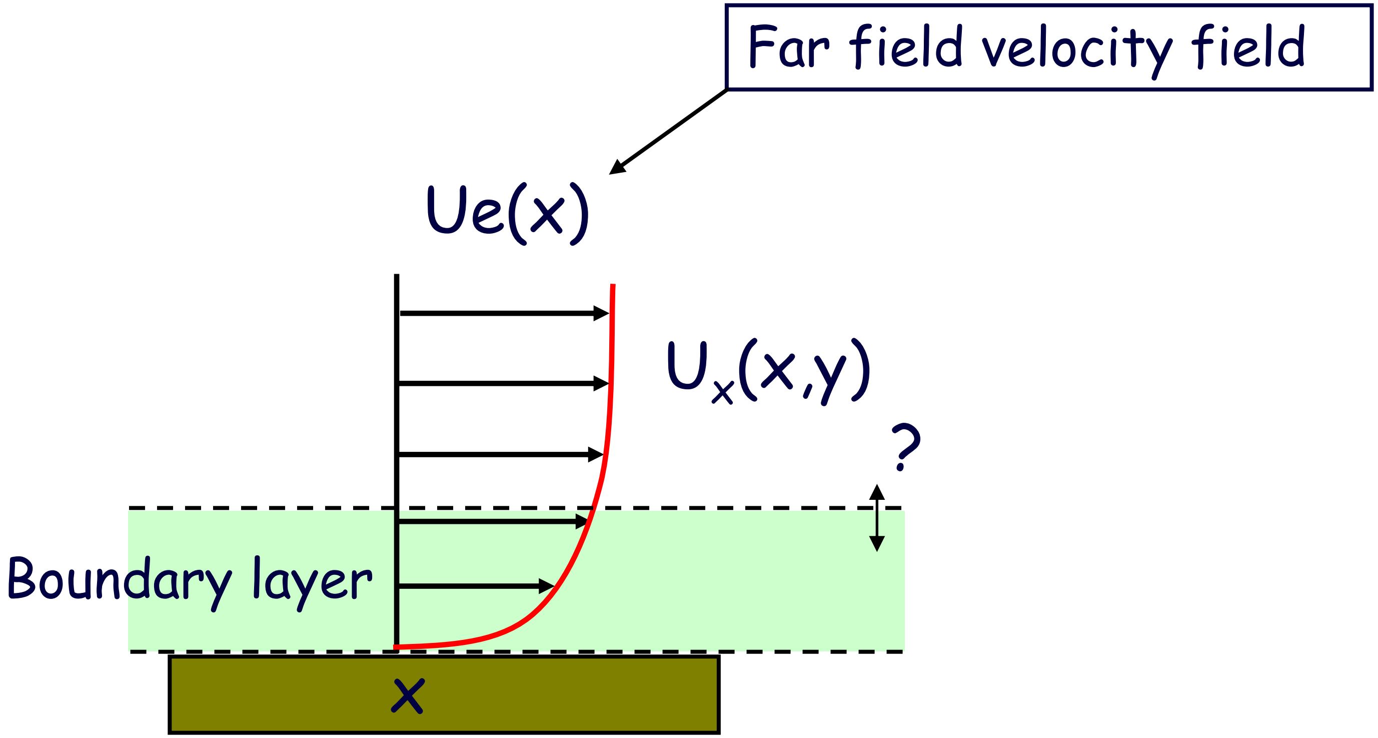




Profil de vitesse expérimental dans la couche limite de Blasius à Re_x = 500

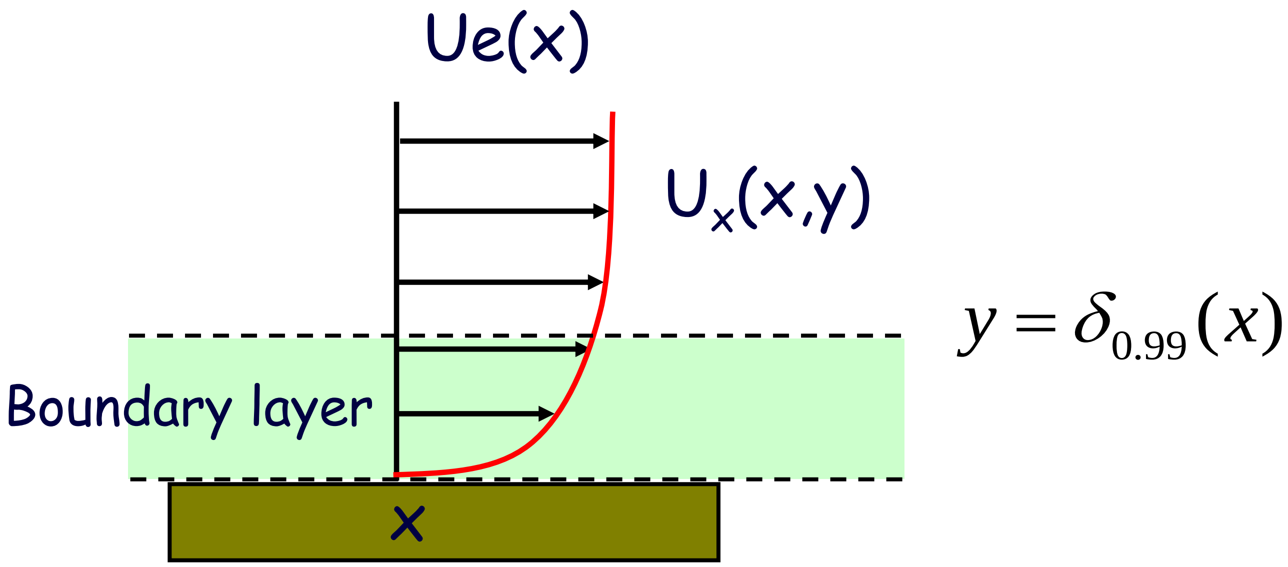
Wortmann (1982)

Boundary layer thickness



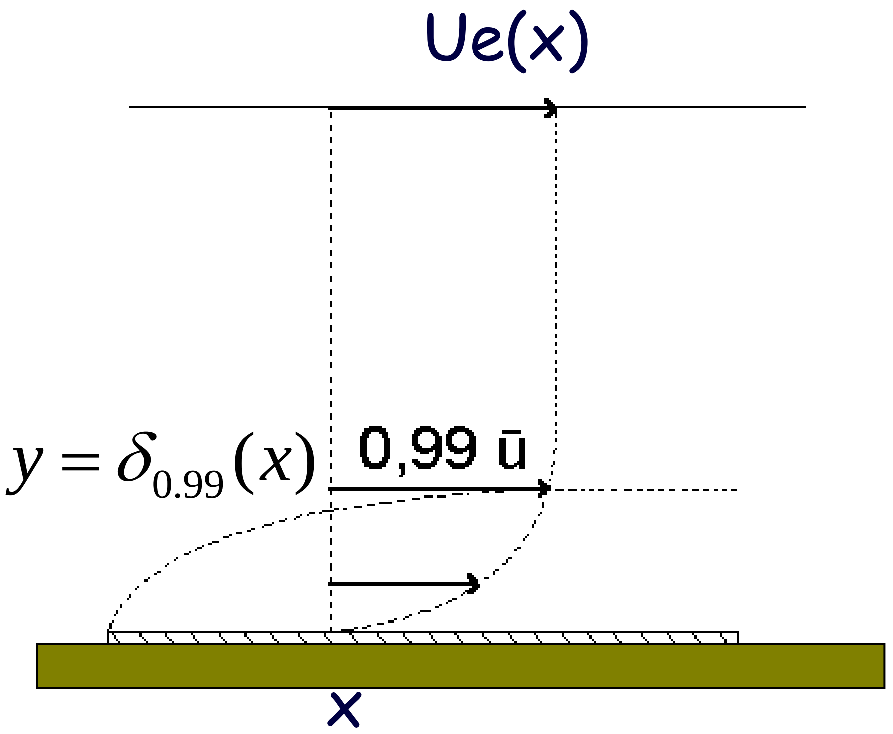
How to characterize or define the boundary layer thickness?

$\delta_{0.99}$ thickness



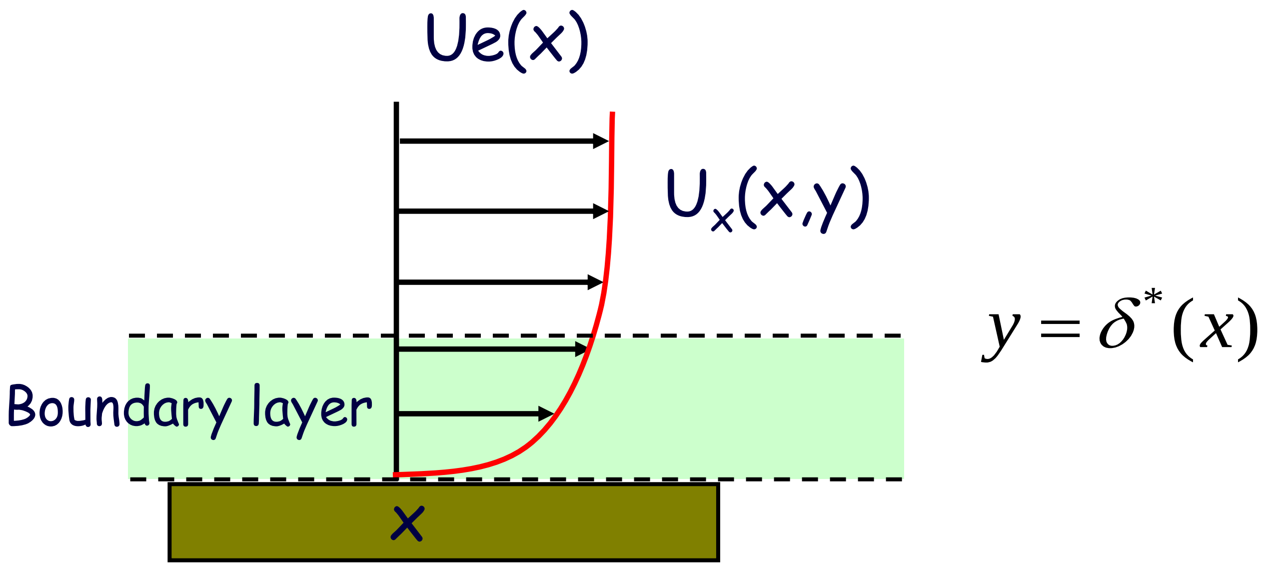
Thickness such that $$U_x(x, \delta_{0.99}) = 0.99 U_e(x)$$

Thickness $\delta_{0.99}$



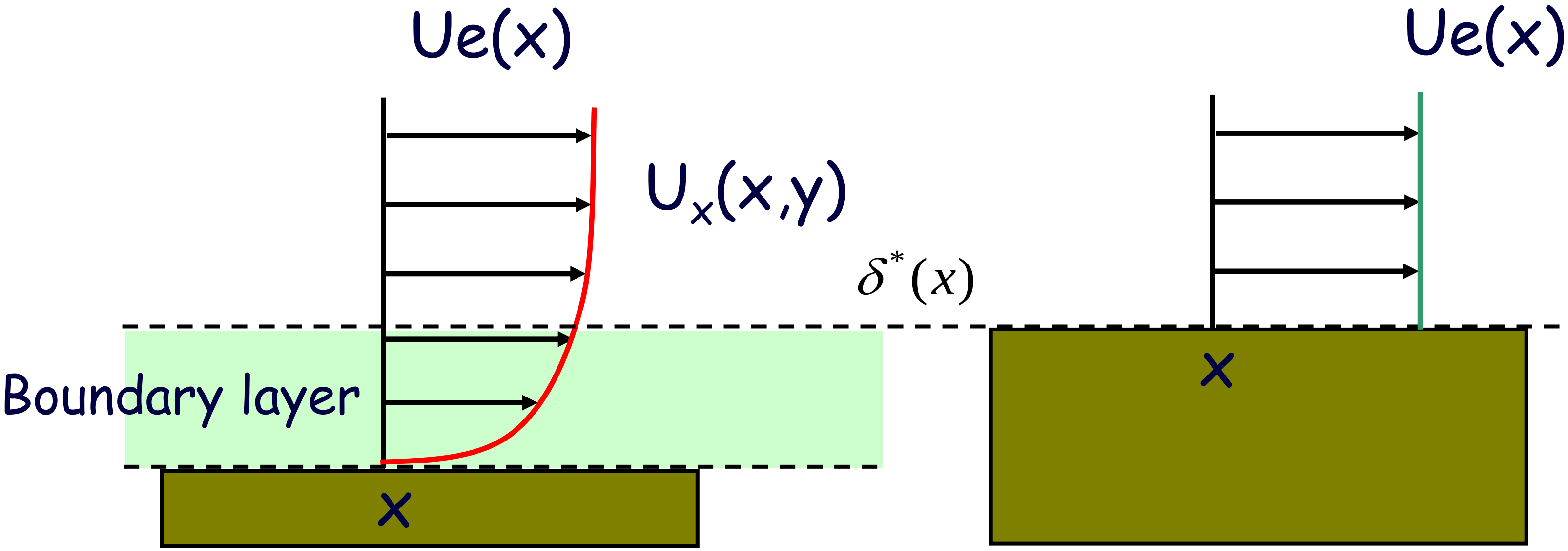
→The boundary layer thickness is local!

Displacement thickness δ*



Thickness defined such that $\delta^*(x) = \int_{y=0}^{\infty} \left(1 - \frac{U_x(x,y)}{U_e(x)}\right) dy$

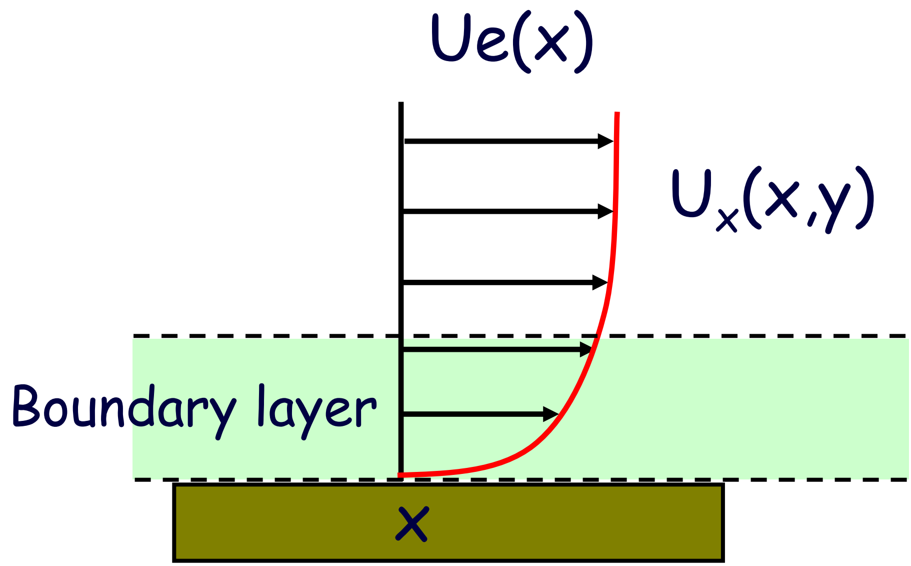
Physical meaning δ^*



Flow rate $Q_1 = \int_{y=0}^{\infty} U_x(x,y)dy$

$$Q_2 = \int_{y=\delta^*(x)}^{\infty} U_e(x)dy = Q_1$$

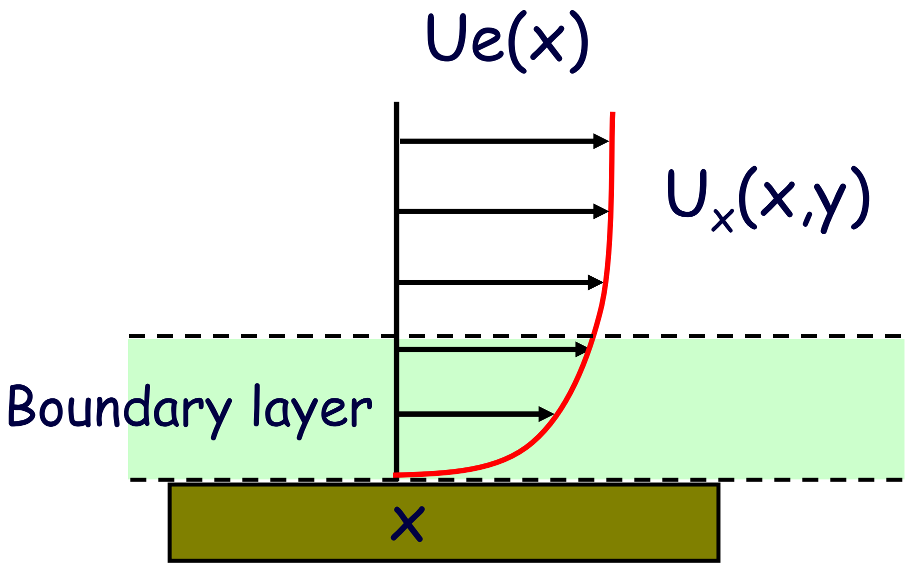
Momentum thickness Θ



$$y = \Theta(x)$$

Defined as $$\Theta(x) = \int_{y=0}^{\infty} \frac{U_x(x,y)}{U_e(x)} \left(1 - \frac{U_x(x,y)}{U_e(x)} \right) dy$$

Physical meaning of Θ



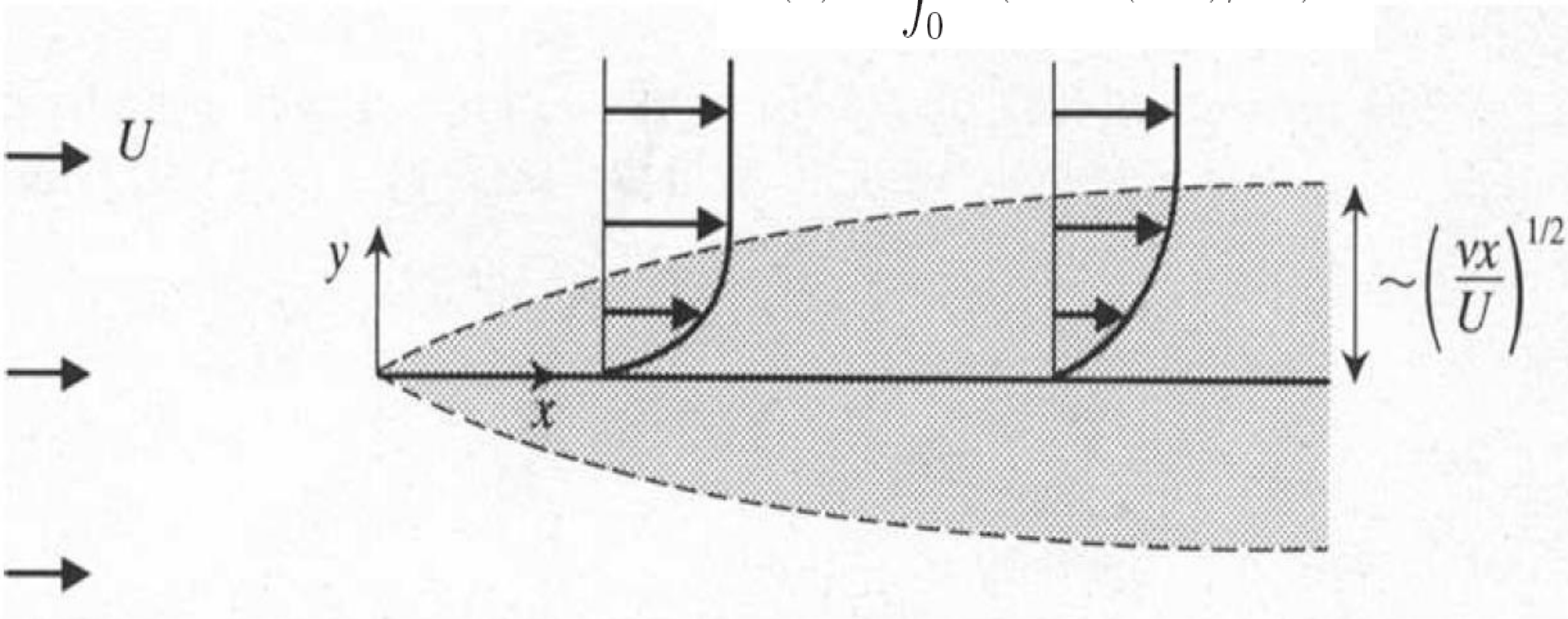
$$y = \Theta(x)$$

It is the thickness by which the wall should be increased to yield the same momentum for a inviscid flow.

Epaisseurs de couche limite

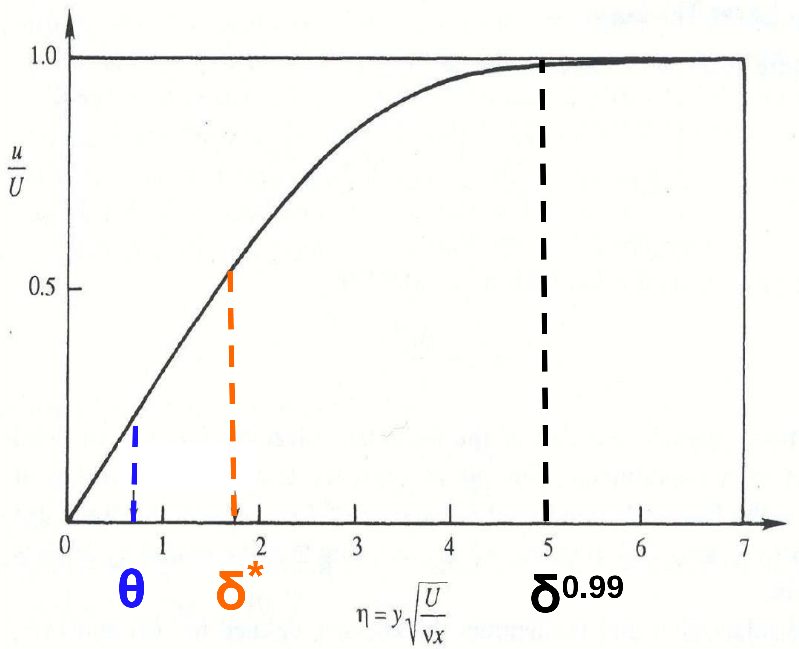
épaisseur à 99% $u\left[x,\delta_{0,99}(x)\right] = 0,99\,U_e$ **~ 5 δ**

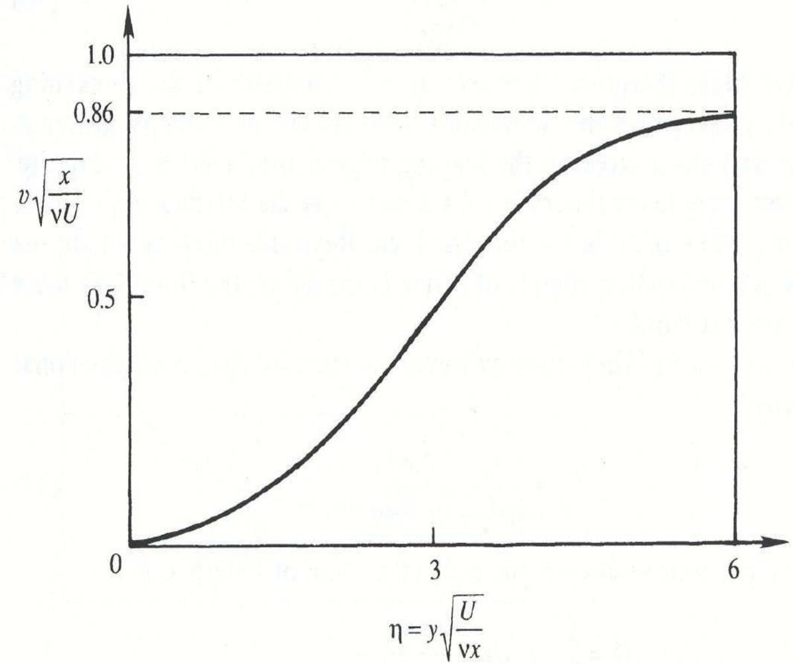
épaisseur de déplacement $\delta^*(x) = \int_0^\infty \left(1 - u(x,y)/U_e\right) dy$ **~ 1.73 δ**



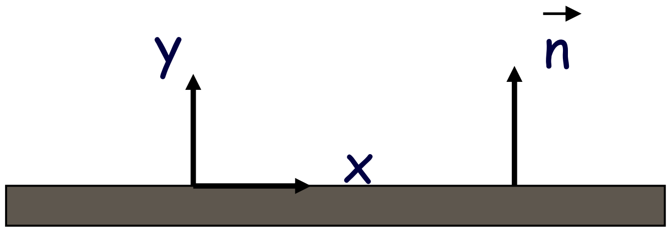
épaisseur de quantité de mouvement **θ** $= \int_0^\infty \frac{u}{U_e}(1 - u/U_e) dy$ **~.66 δ**

Profils de couche limite





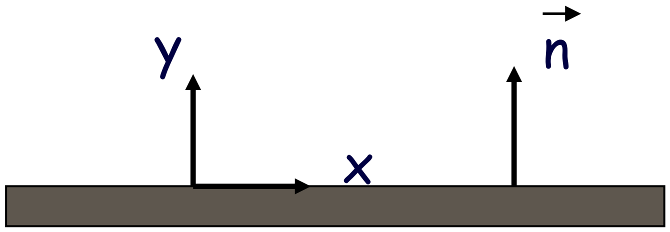
Viscous friction



$$\underbrace{\begin{pmatrix} 2\mu \dfrac{\partial U_x}{\partial x} & \mu\left(\dfrac{\partial U_x}{\partial y} + \dfrac{\partial U_y}{\partial x}\right) \\ \mu\left(\dfrac{\partial U_x}{\partial y} + \dfrac{\partial U_y}{\partial x}\right) & 2\mu \dfrac{\partial U_y}{\partial y} \end{pmatrix}}_{\vec{\vec{\tau}}} \cdot \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\vec{n}} = \begin{pmatrix} \mu \dfrac{\partial U_x}{\partial y} + \mu \dfrac{\partial U_y}{\partial x} \\ 2\mu \dfrac{\partial U_y}{\partial y} \end{pmatrix}$$

A priori both drag and lift can be caused by viscous friction

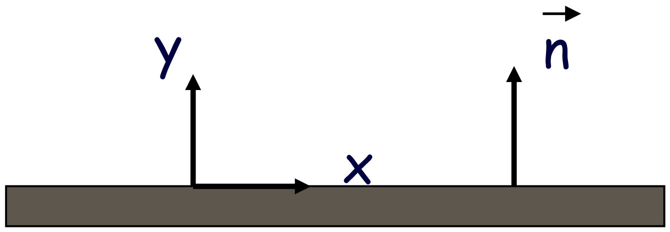
Viscous friction



$$\begin{pmatrix} 2\mu\dfrac{\partial U_x}{\partial x} & \mu\left(\dfrac{\partial U_x}{\partial y}+\dfrac{\partial U_y}{\partial x}\right) \\ \mu\left(\dfrac{\partial U_x}{\partial y}+\dfrac{\partial U_y}{\partial x}\right) & 2\mu\dfrac{\partial U_y}{\partial y} \end{pmatrix}\cdot\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \mu\dfrac{\partial U_x}{\partial y}+\mu\dfrac{\partial U_y}{\partial x} \\ 2\mu\dfrac{\partial U_y}{\partial y} \end{pmatrix}$$

Note however $U_x >> U_y$ in the boundary layer

Viscous friction

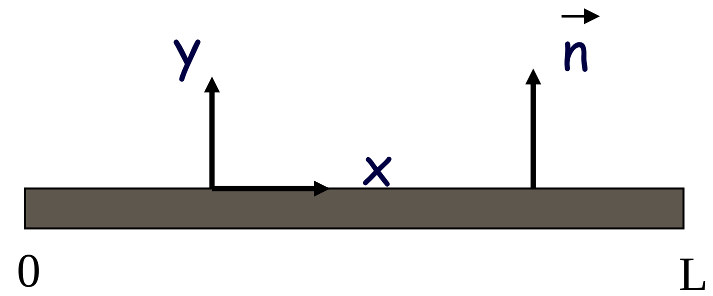


$$\begin{pmatrix} 2\mu\dfrac{\partial U_x}{\partial x} & \mu\left(\dfrac{\partial U_x}{\partial y}+\dfrac{\partial U_y}{\partial x}\right) \\ \mu\left(\dfrac{\partial U_x}{\partial y}+\dfrac{\partial U_y}{\partial x}\right) & 2\mu\dfrac{\partial U_y}{\partial y} \end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}\mu\dfrac{\partial U_x}{\partial y}+\cancel{\mu\dfrac{\partial U_y}{\partial x}} \\ \cancel{2\mu\dfrac{\partial U_y}{\partial y}}\end{pmatrix}\approx\begin{pmatrix}\mu\dfrac{\partial U_x}{\partial y}\\ 0\end{pmatrix}$$

At leading order, only shear stress

Wall friction shear coefficient

Flat plate of length L



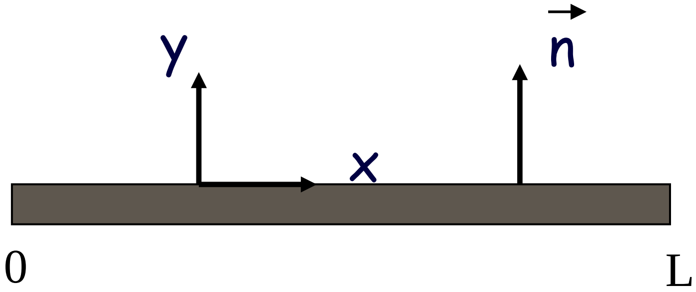
Linear density of friction coefficient

dimensional : $F_X = \int_0^L \tau_p(x)dx = \int_0^L \mu \frac{\partial U_x}{\partial y}(x)dx$ $\rightarrow$ N/m ! (2D)

nondimensional : $C_X = \frac{F_X}{\frac{1}{2}\rho_\infty U_\infty{}^2 L}$

Wall friction shear coefficient

Flat plate of length L



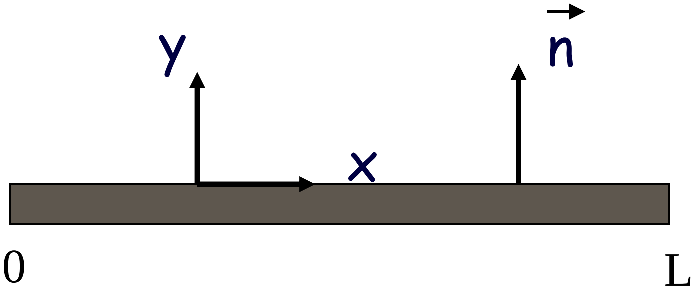
Linear density of friction coefficient

dimensional : $$\tau_p(x) = \mu \frac{\partial U_x}{\partial y}(x) \quad \rightarrow N/m^2$$

nondimensional : $$\boxed{Cf(x) = \frac{\tau_p(x)}{\frac{1}{2}\rho_\infty U_\infty^{\ 2}} = \frac{0.664}{\mathrm{Re}_x^{\ 1/2}}}$$

Wall friction shear coefficient Cf

Plaque de longueur L



Densité de contrainte pariétale (i.e. qui s'exerce à la paroi)

Dimensionné : $\tau_p(x) = \mu \dfrac{\partial U_x}{\partial y}(x) \quad \rightarrow N/m^2$

$$Cf(x) = \frac{\tau_p(x)}{\frac{1}{2}\rho_\infty U_\infty^{\ 2}} \approx \frac{\mu U_\infty}{\delta} \frac{1}{\frac{1}{2}\rho_\infty U_\infty^{\ 2}}$$

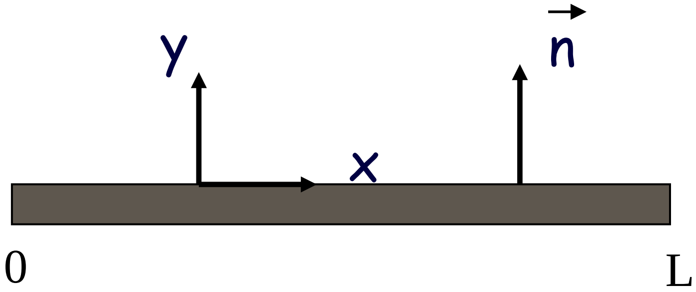
Adimensionné :

$$\approx \frac{\mu U_\infty}{\sqrt{\dfrac{\mu x}{\rho_\infty U_\infty}}} \frac{1}{\frac{1}{2}\rho_\infty U_\infty^{\ 2}}$$

$$\approx \sqrt{\frac{\rho_\infty \mu U_\infty^{\ 3}}{x}} \frac{1}{\frac{1}{2}\rho_\infty U_\infty^{\ 2}}$$

Drag coefficient Cx

Plaque de longueur L

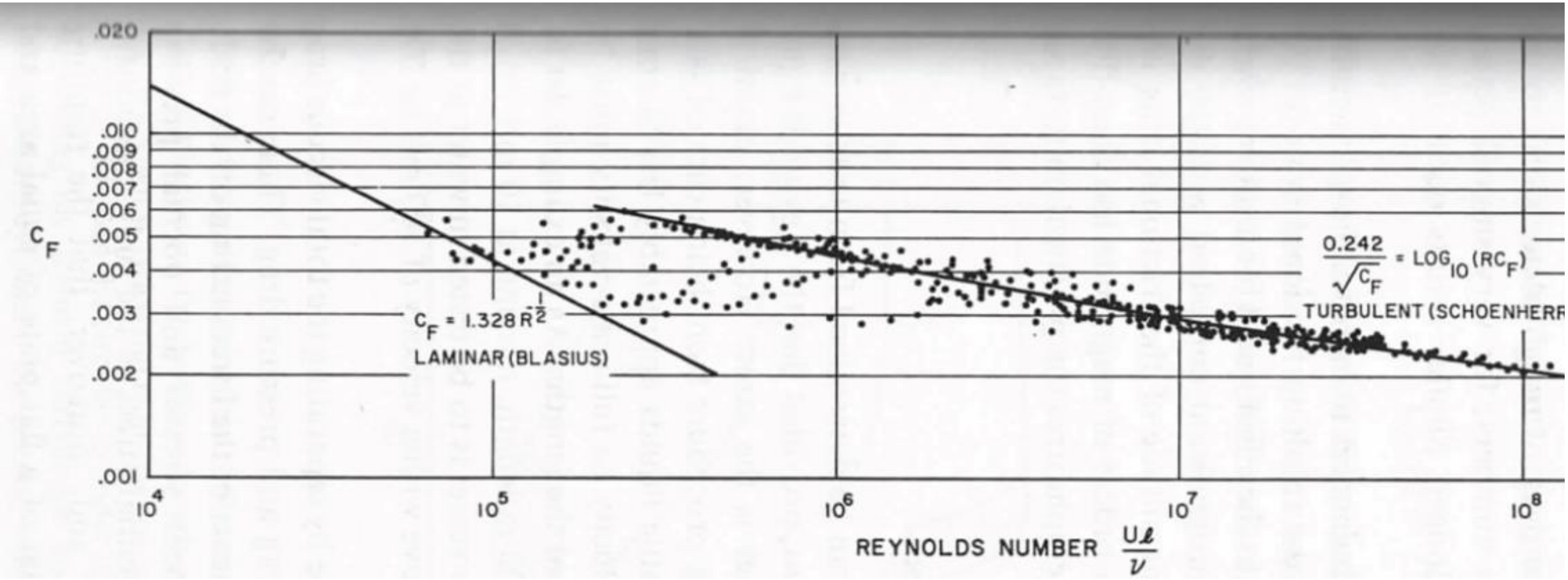


Force de trainée s'exerçant sur la paroi

Dimensionné : $F_X = \int_0^L \tau_p(x)dx = \int_0^L \mu \frac{\partial U_x}{\partial y}(x)dx$ $\rightarrow$ N/m ! (2D)

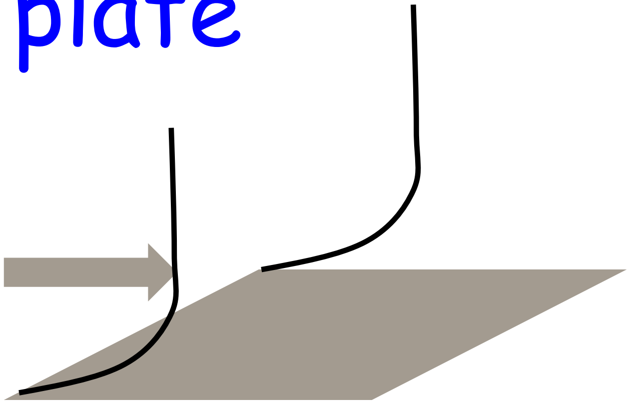
Adimensionné : $C_X = \dfrac{F_X}{\frac{1}{2}\rho_\infty U_\infty{}^2 L} \approx \dfrac{\sqrt{\rho_\infty \mu L U_\infty{}^3}}{\frac{1}{2}\rho_\infty U_\infty{}^2 L} \approx \sqrt{\dfrac{\mu}{\rho_\infty U_\infty L}} \approx \mathrm{Re}^{-1/2}$

Drag of a plate



Drag of a plate

Plate of length L, width W



$$F_D \approx W\sqrt{\rho \mu L U_\infty^{\ 3}}$$