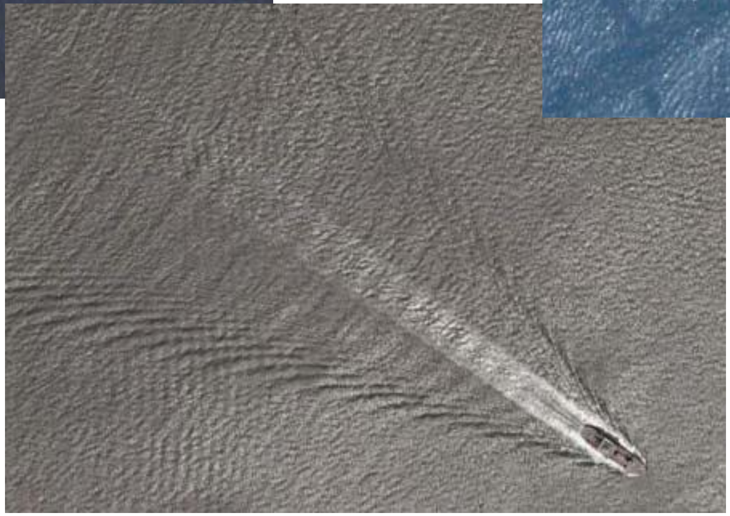
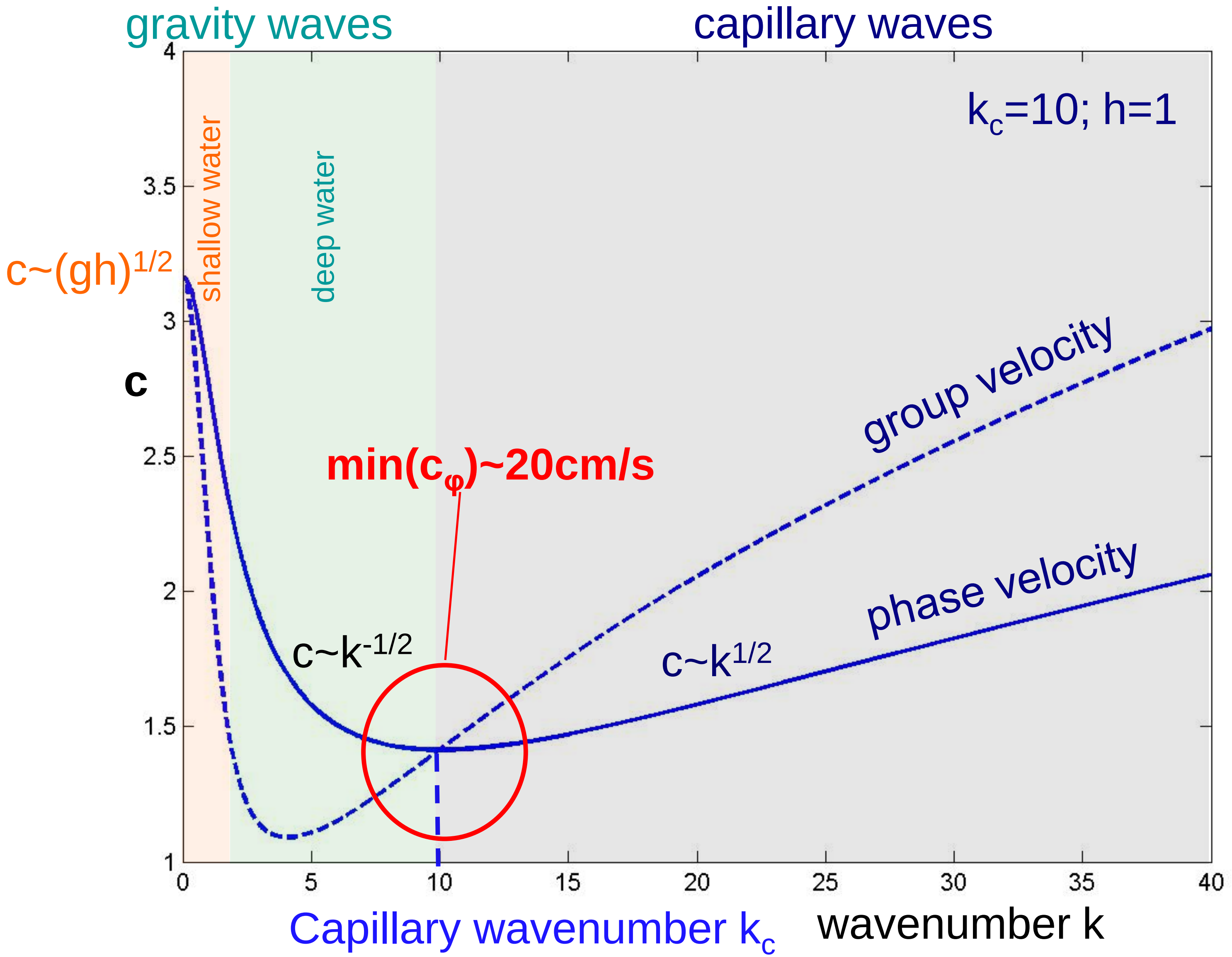
Hydrodynamics 14
Waves



Dispersion relation



Conditions for wave pattern formation?

$$V_{duck} \lesseqgtr c_{min} \quad ?$$

Dispersion relation

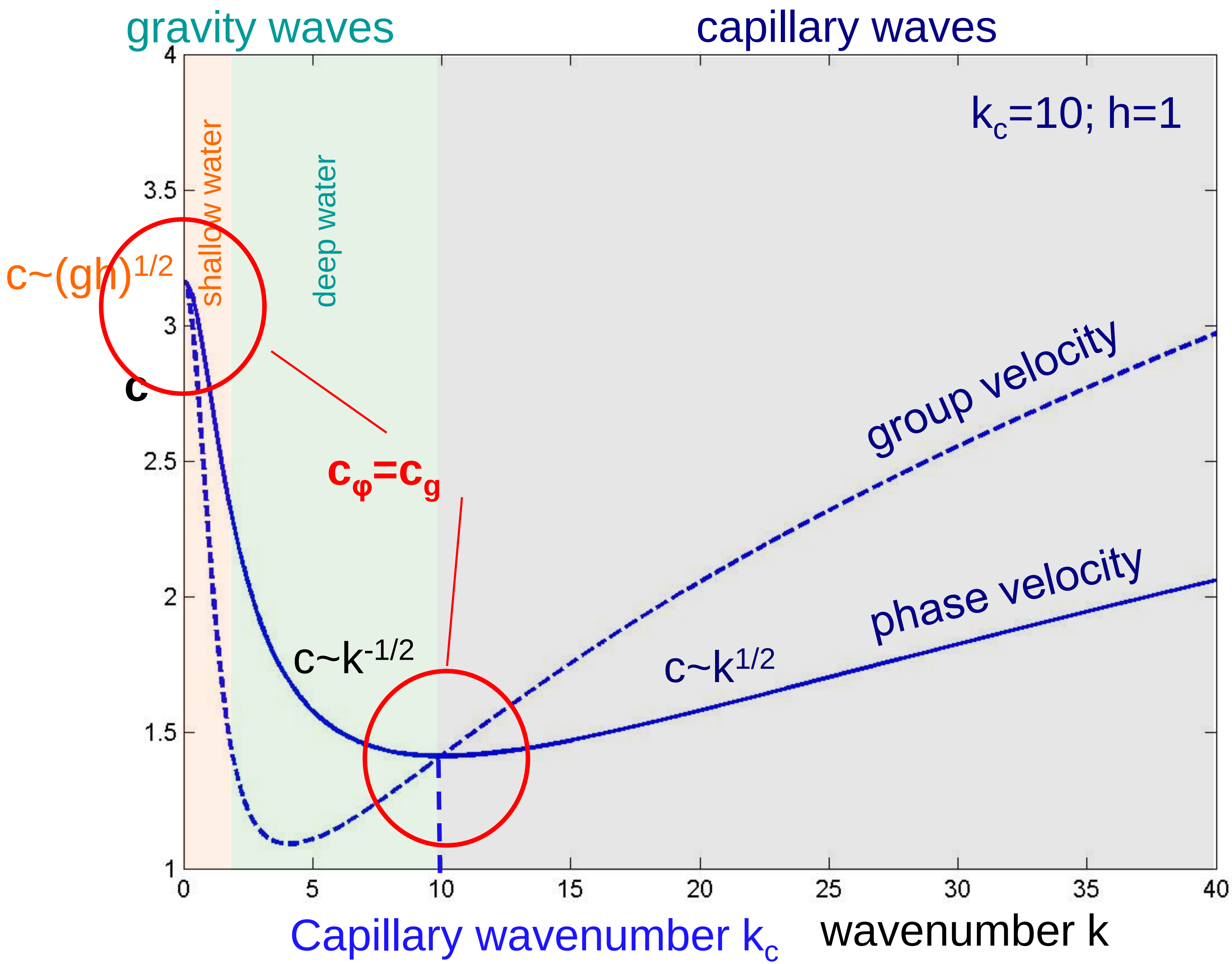
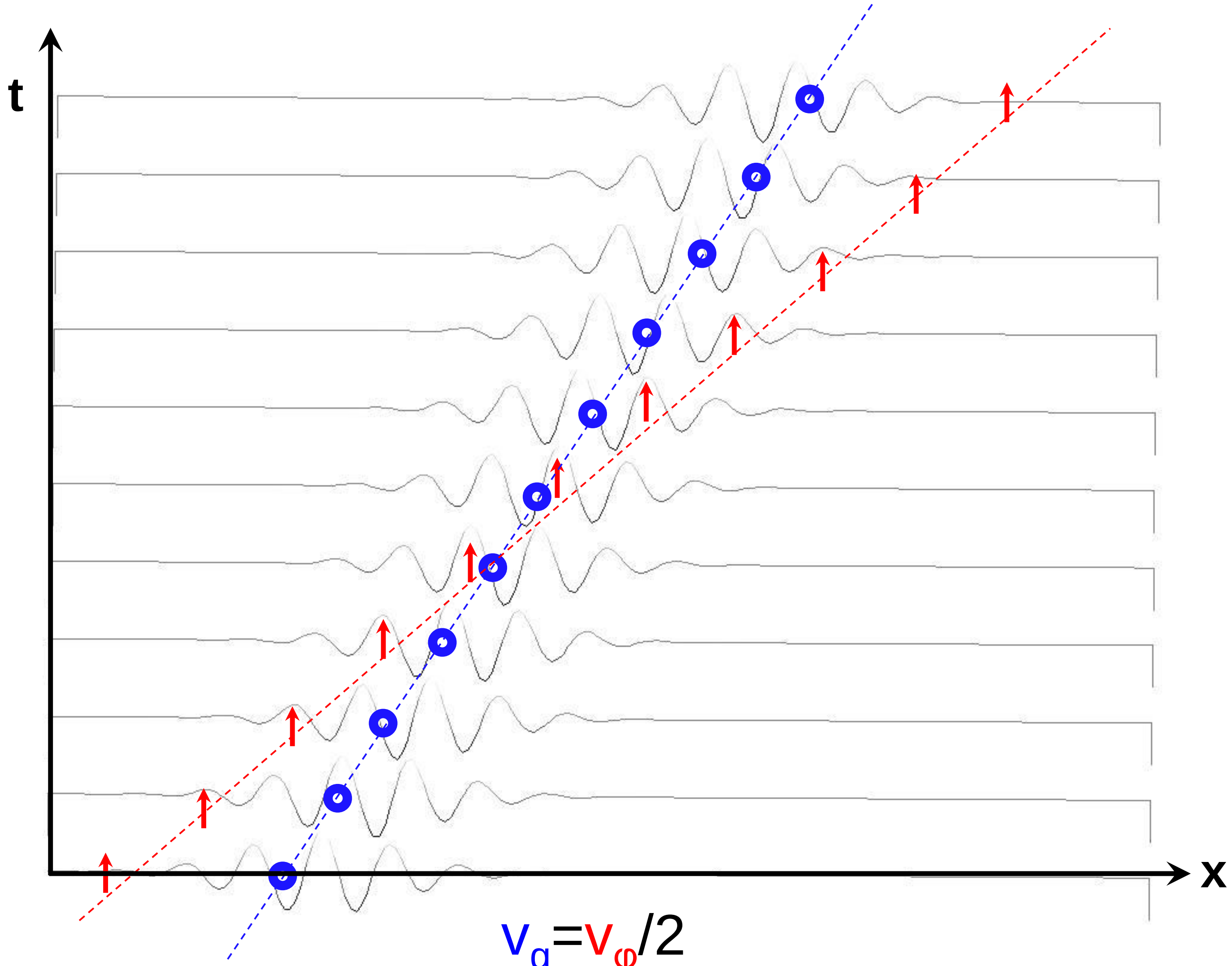


Diagramme spatio-temporel



$v_g = v_\varphi / 2$

Diagramme spatio-temporel

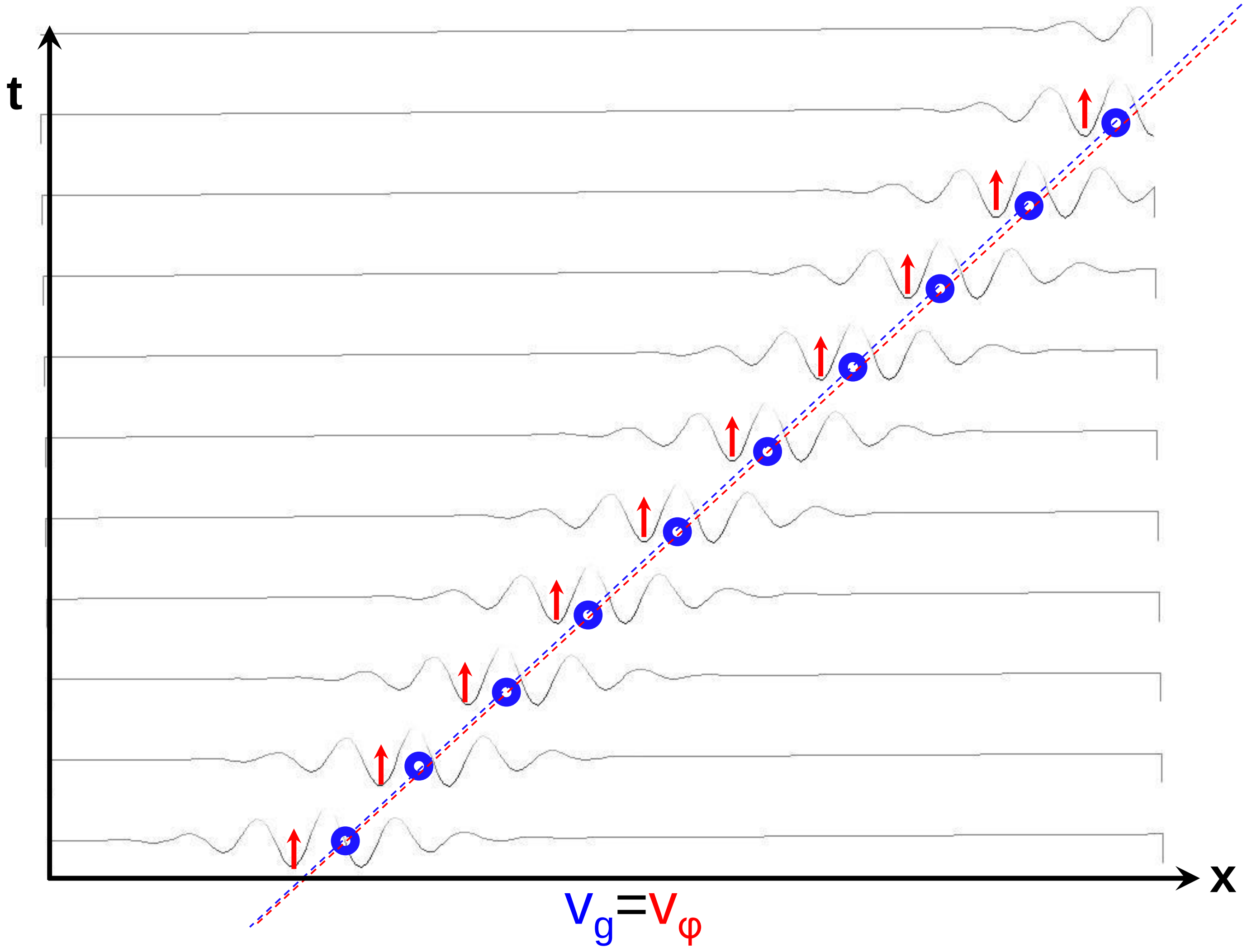
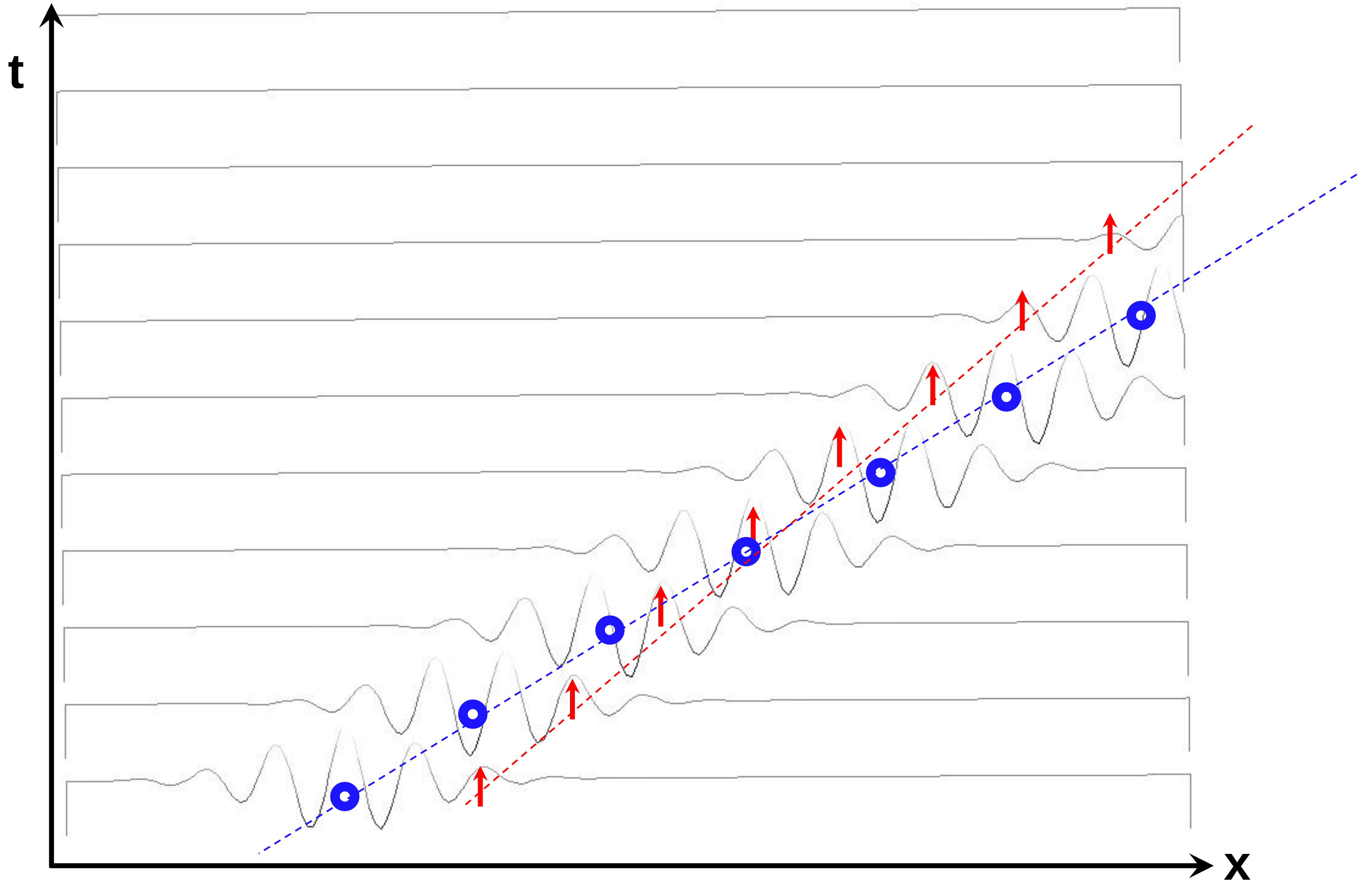


Diagramme spatio-temporel



$v_g = 3/2 v_\varphi$

Spectral analysis

Fourier transform:

$$u(x,t) = \frac{1}{2}\int_0^{+\infty} \hat{u}(k)\,\mathrm{e}^{\mathrm{i}(kx-\omega(k)t)}\mathrm{d}k + c.c.$$

$\hat{u}(k)$ is given by Fourier transform at time t=0

Carrier/enveloppe :

$$u(x,t) = \frac{1}{2}A(x,t)\,\mathrm{e}^{\mathrm{i}(k_0x-\omega_0t)} + c.c.$$

Enveloppe :

$$A(x,t) = \int_0^{\infty} \hat{u}(k)\,\mathrm{e}^{\mathrm{i}(k-k_0)x-\mathrm{i}(\omega-\omega_0)t}\mathrm{d}k.$$

Spectral analysis

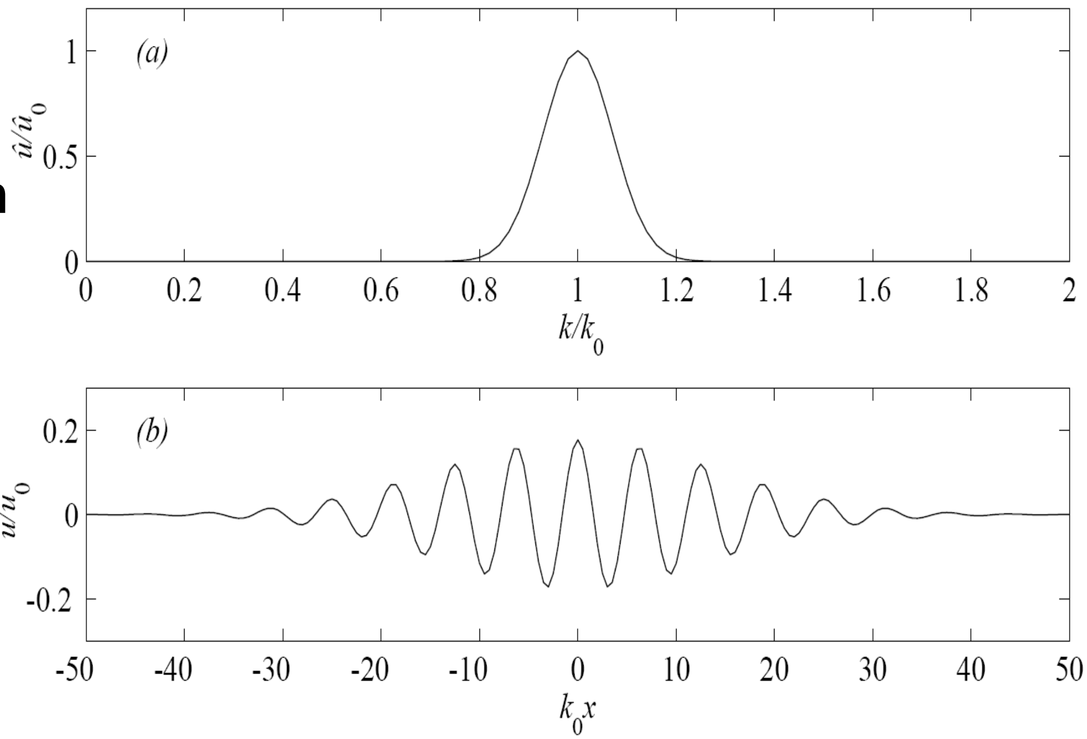
Gaussian spectrum: $$\hat{u}(k) = u_0\, \mathrm{e}^{-\sigma^2 (k-k_0)^2}$$

Initial enveloppe : $$A(x,0) = \frac{u_0\sqrt{\pi}}{2\sigma}\, \mathrm{e}^{-\frac{x^2}{4\sigma^2}}$$

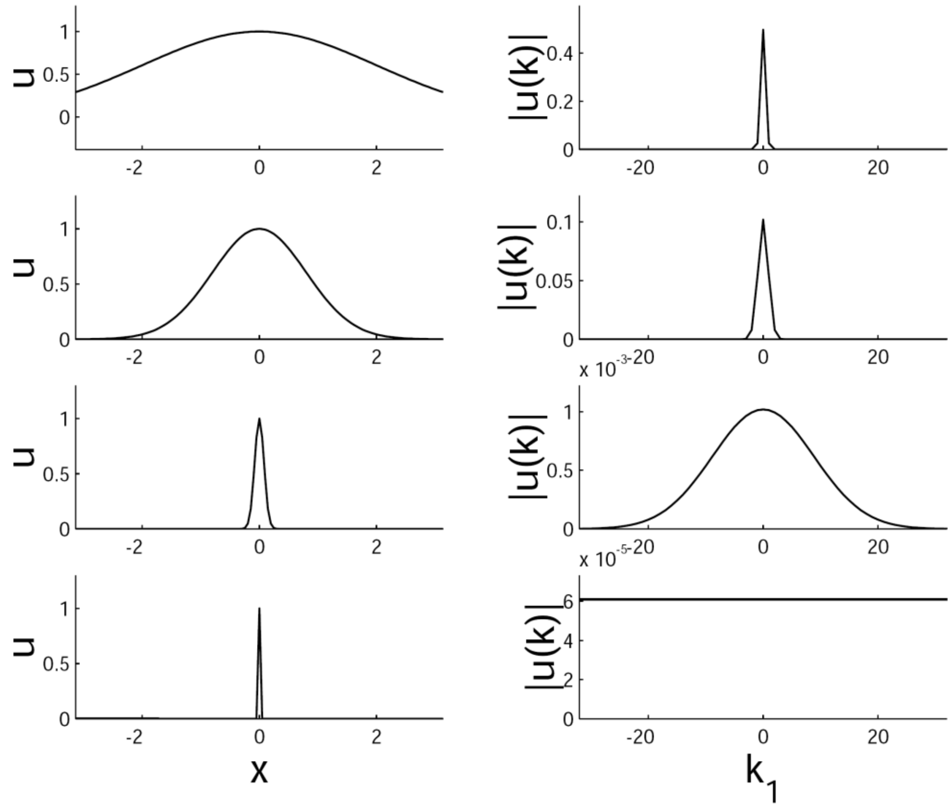
Gaussian spectrum

spectrum

wave

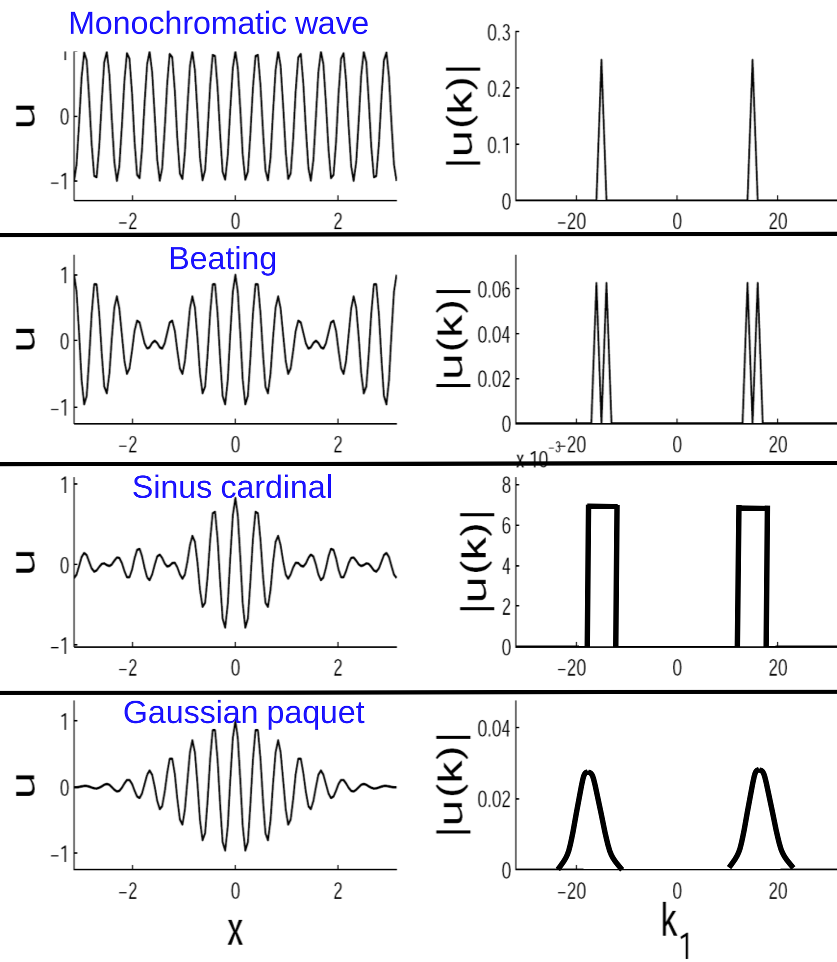


Paquets d'ondes gaussiens



$u_0(x)$	$\widehat{u}_0(k_1)$
$\exp\left(-\frac{x^2}{2\,\sigma^2}\right)$	$\frac{\sigma}{\sqrt{2\pi}}\;\exp\left(-\frac{\sigma^2\,k^2}{2}\right)$
$\delta(x)$	$\frac{1}{2\pi}$

Waves and spectrum



$2\ \cos(k_0\, x)$	$\delta(k_1+k_0)+\delta(k_1-k_0)$
$\cos[(k_0-\kappa)x]+\cos[(k_0+\kappa)x]$ $=2\ \cos(k_0\, x)\ \cos(\kappa\, x)$	$\delta(k_1+k_0+\kappa)+\delta(k_1-k_0-\kappa)$ $+\delta(k_1+k_0-\kappa)+\delta(k_1-k_0+\kappa)$
$\int_{k_0-\kappa}^{k_0+\kappa}\cos(k_1\, x)\ dk_1$ $=4\,\kappa\ \cos(k_0\, x)\ \frac{\sin(\kappa\, x)}{\kappa\, x}$	$=1$ pour $\lvert k_1 \pm k_0\rvert \leq \kappa$ $=0$ sinon
$2\,\kappa\sqrt{2\pi}\ \cos(k_0\, x)\exp\left(-\frac{\kappa^2\, x^2}{2}\right)$	$\exp\left[-\frac{(k_1+k_0)^2}{2\,\kappa^2}\right]+\exp\left[-\frac{(k_1-k_0)^2}{2\,\kappa^2}\right]$

Spectral analysis

$$A(x,t) = \int_0^\infty \hat{u}(k)\, \mathrm{e}^{\mathrm{i}(k-k_0)x - \mathrm{i}(\omega-\omega_0)t} \mathrm{d}k.$$

Spectral analysis

$$A(x,t) = \int_0^\infty \hat{u}(k)\, \mathrm{e}^{\mathrm{i}(k-k_0)x - \mathrm{i}(\omega-\omega_0)t} \mathrm{d}k.$$

Definition of group velocity $\omega - \omega_0 = c_g(k - k_0), \qquad c_g = \frac{\partial \omega}{\partial k}(k_0)$

Spectral analysis

$$A(x,t) = \int_0^\infty \hat{u}(k)\, \mathrm{e}^{\mathrm{i}(k-k_0)x-\mathrm{i}(\omega-\omega_0)t} \mathrm{d}k.$$

$$\hat{u}(k) = u_0\, \mathrm{e}^{-\sigma^2 (k-k_0)^2}$$

Spectral analysis

Gaussian spectrum: $$\hat{u}(k) = u_0 \, \mathrm{e}^{-\sigma^2 (k-k_0)^2}$$

Initial enveloppe : $$A(x,0) = \frac{u_0 \sqrt{\pi}}{2\sigma} \, \mathrm{e}^{-\frac{x^2}{4\sigma^2}}$$

Spectral analysis

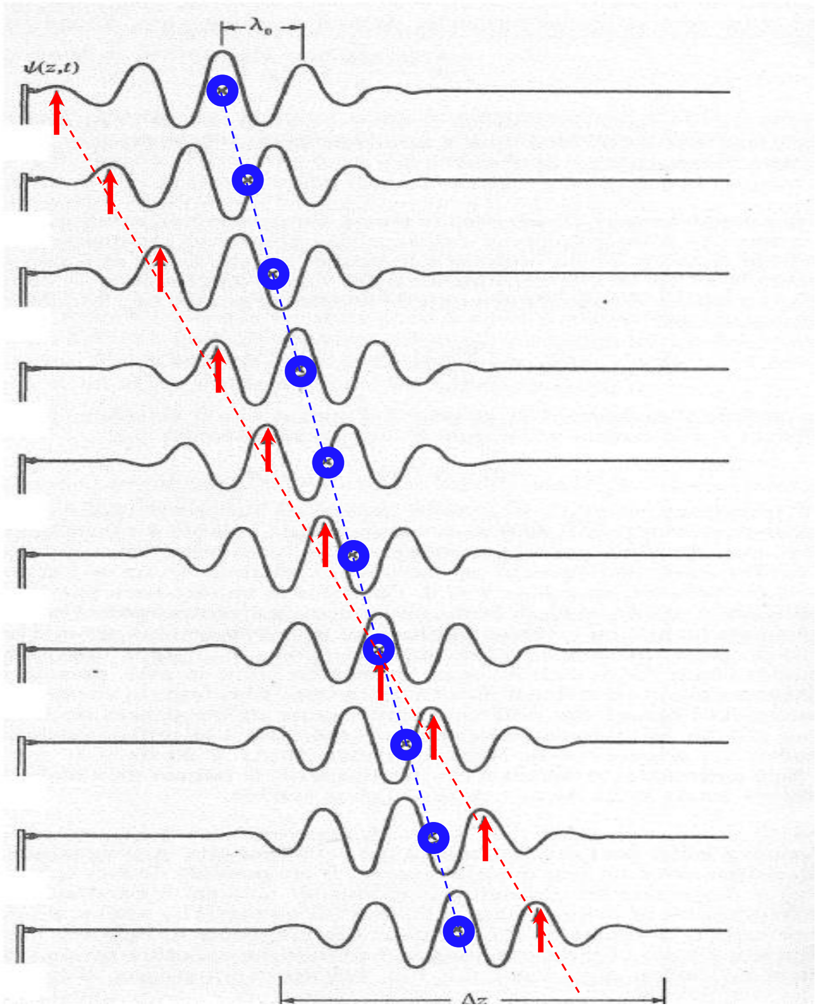
$$A(x,t) = \int_0^\infty \hat{u}(k)\, \mathrm{e}^{\mathrm{i}(k-k_0)x - \mathrm{i}(\omega-\omega_0)t} \mathrm{d}k.$$

Definition of group velocity $\omega - \omega_0 = c_g(k - k_0), \qquad c_g = \frac{\partial \omega}{\partial k}(k_0)$

$$\hat{u}(k) = u_0\, \mathrm{e}^{-\sigma^2 (k-k_0)^2}$$

$$A(x,t) = \frac{u_0 \sqrt{\pi}}{2\sigma}\, \mathrm{e}^{-\frac{(x-c_g t)^2}{4\sigma^2}}$$

Group velocity



Wave packet

Spectral analysis

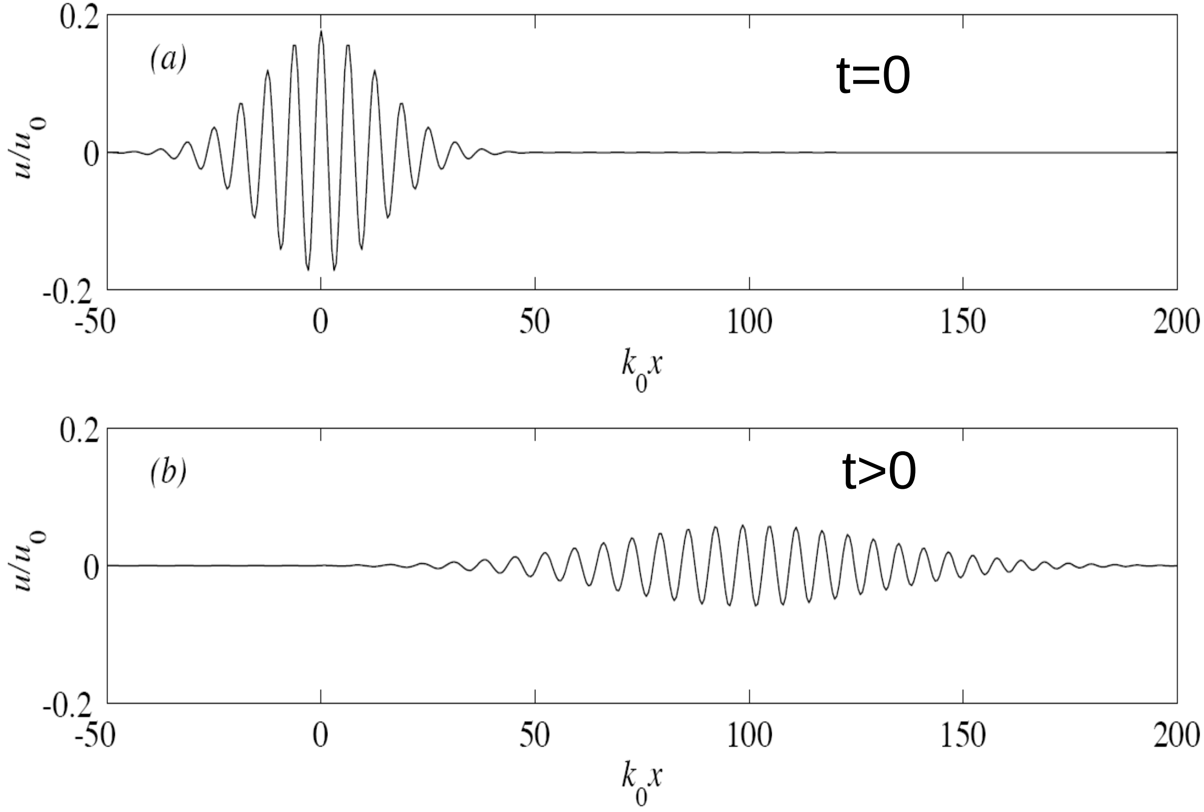
Higher order development

$$\omega - \omega_0 = c_g(k - k_0) + \frac{\omega_0''}{2}(k - k_0)^2$$

$$c_g = \frac{\partial \omega}{\partial k}(k_0), \qquad \omega_0'' = \frac{\partial^2 \omega}{\partial k^2}(k_0)$$

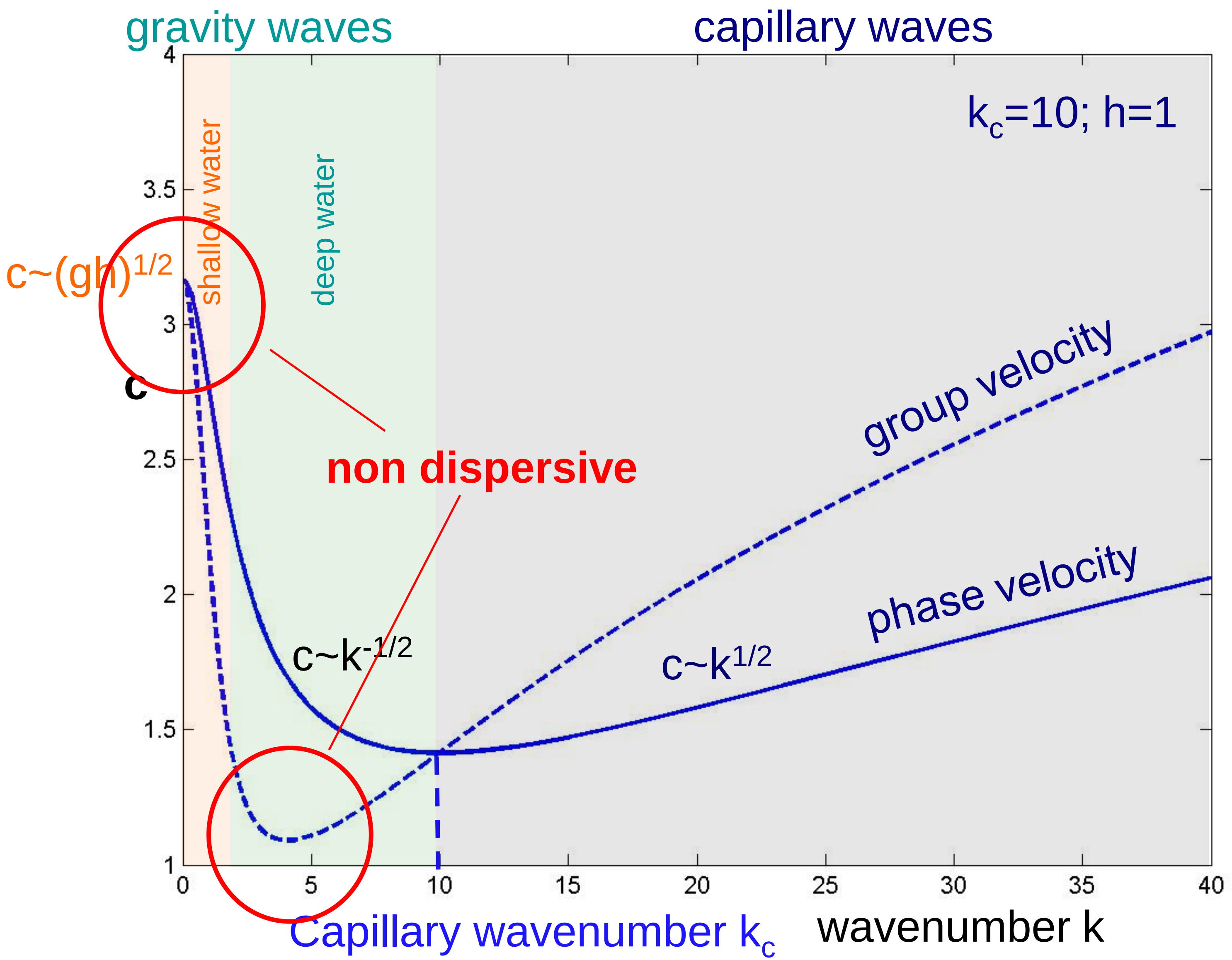
$$A(x,t) = \frac{u_0}{2}\sqrt{\frac{\pi}{\sigma^2 + \frac{1}{2}\mathrm{i}\omega_0''t}}\,\exp\left(-\frac{(x - c_g t)^2}{4(\sigma^2 + \frac{1}{2}\mathrm{i}\omega_0''t)}\right)$$

Wave packet dispersion

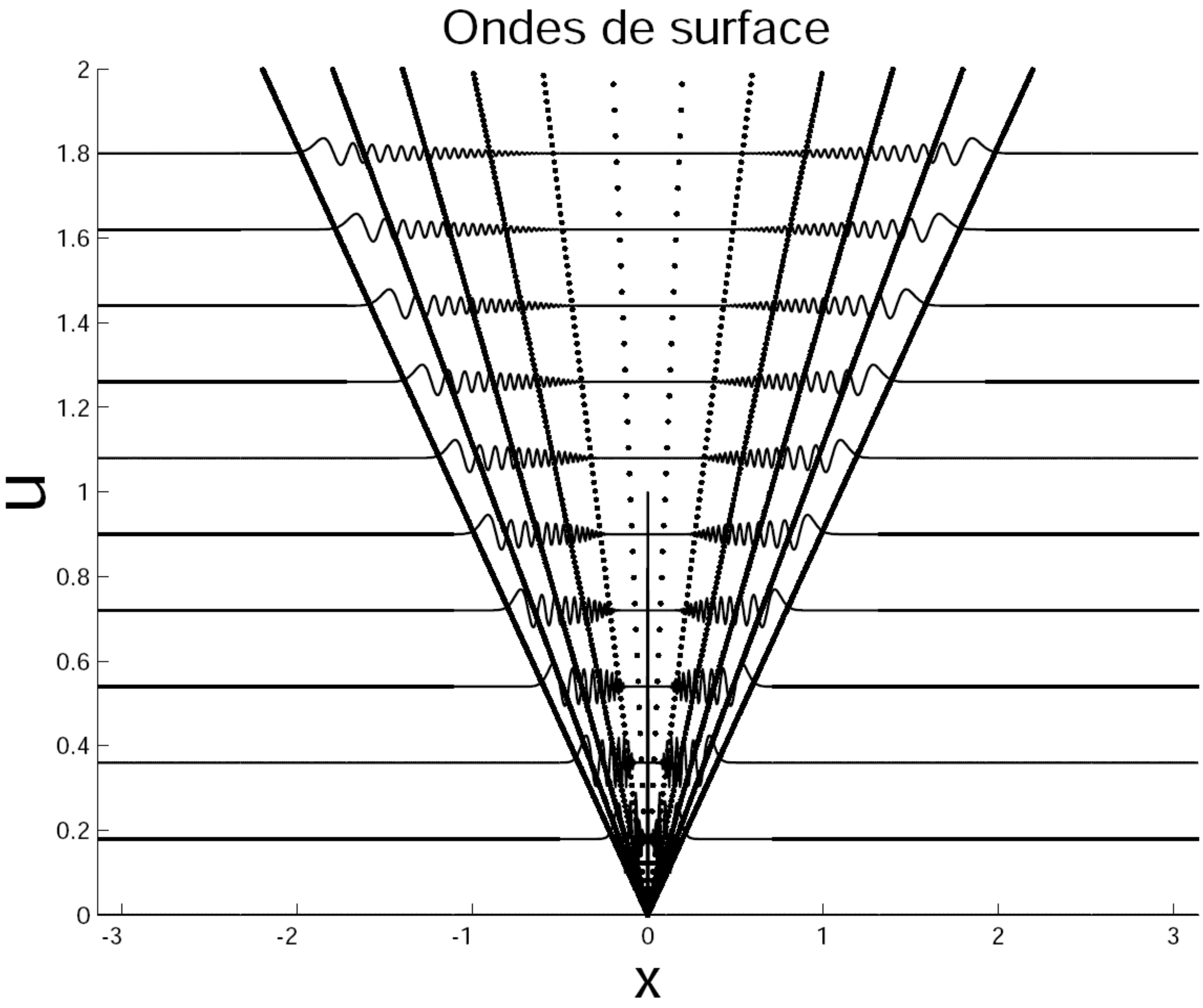


Onde correspondant à l'enveloppe pour $\sigma^{-1}k_0 = 0{,}1$ *et* $\omega_0'' = 4c_g/k_0$ *: (a), instant initial* $t = 0$ *; (b),* $c_g t = 100/k_0$.

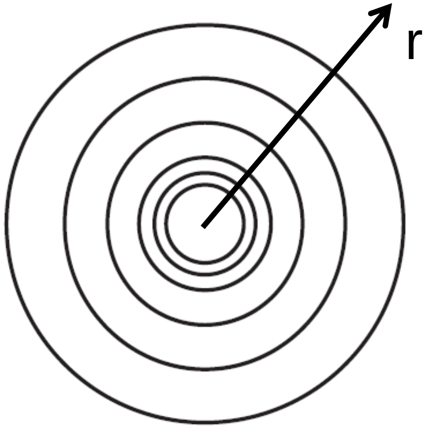
Dispersion relation



Dispersion



Dispersion



Waves with k reach r at time t=r/v(k)

For deep gravity waves: $v_{deep/gravity} \sim \pm\frac{1}{2}\sqrt{\frac{g}{k}}$

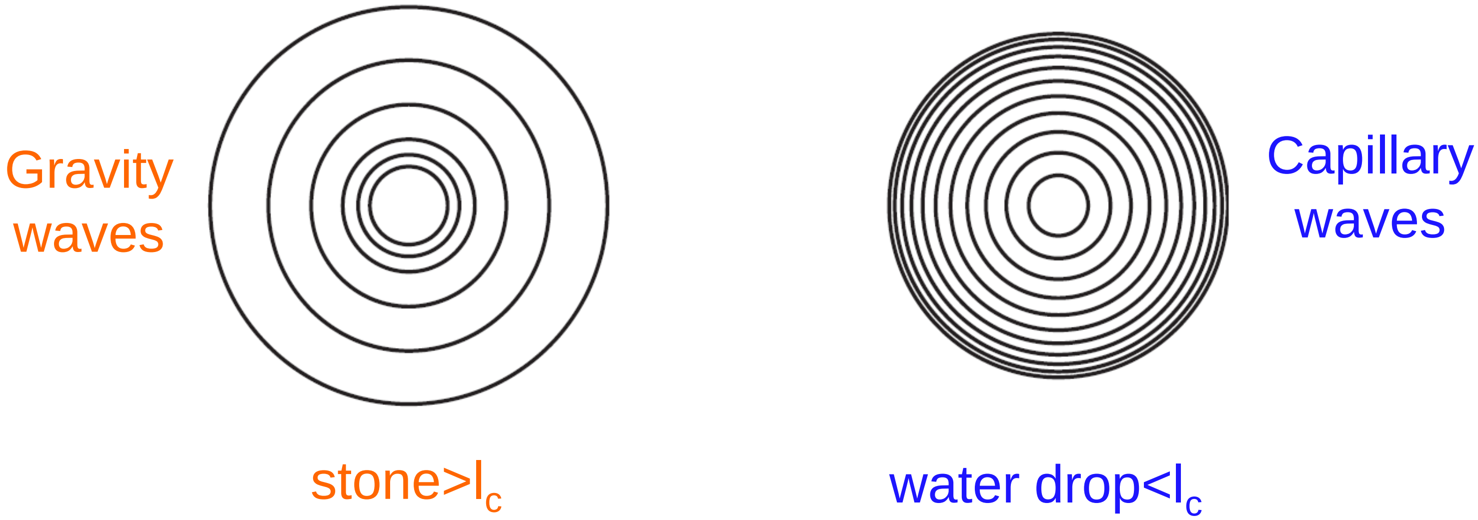
$k=gt^2/4r^2$

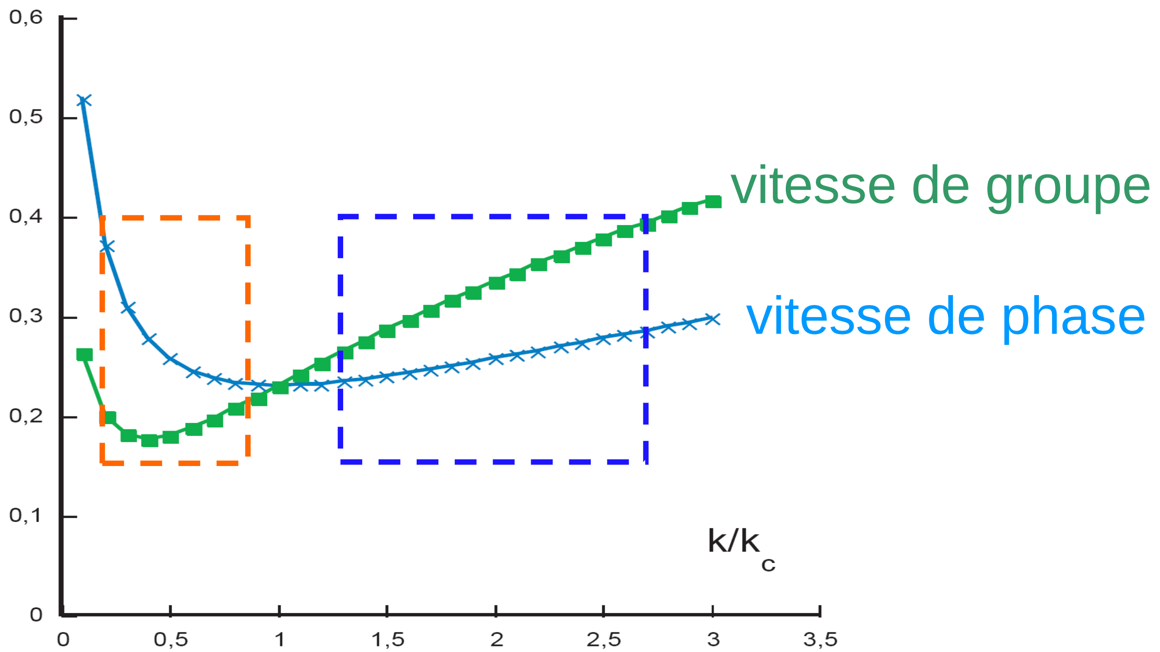
Since $\omega_{deep/gravity} \sim \pm\sqrt{gk}$

$\omega=gt/2r$

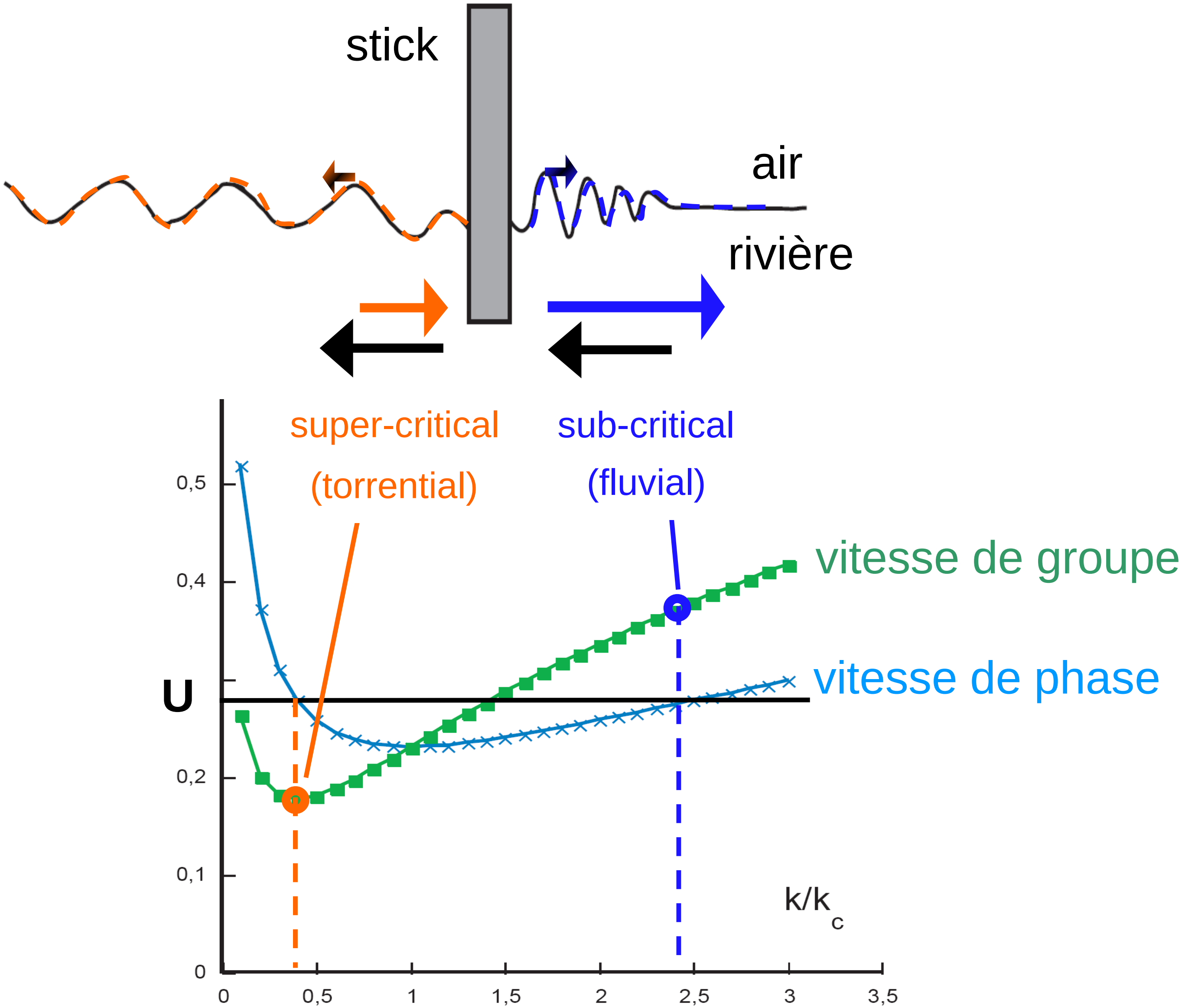
⇨The frequency increases with time

"Ronds dans l'eau"

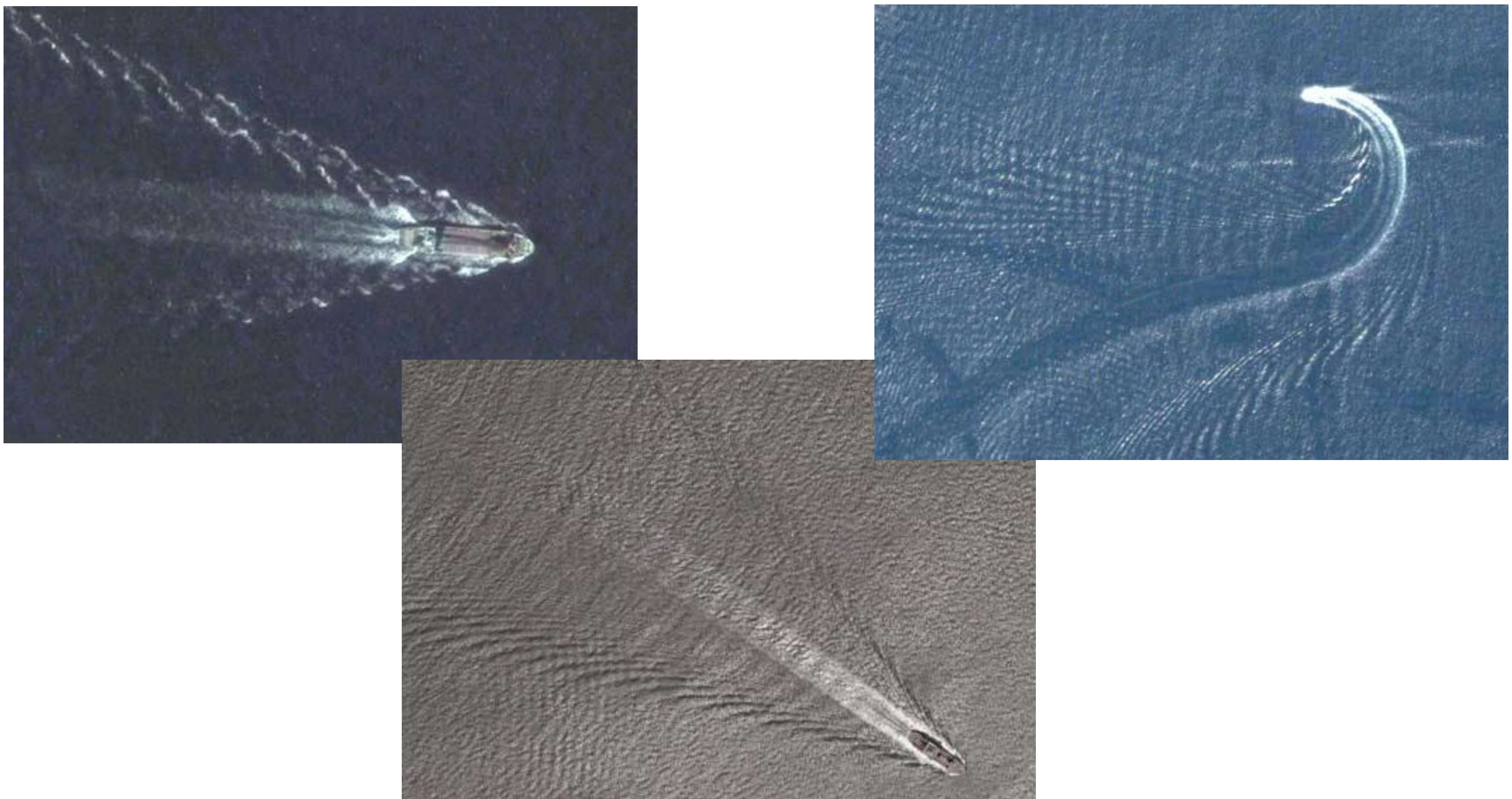




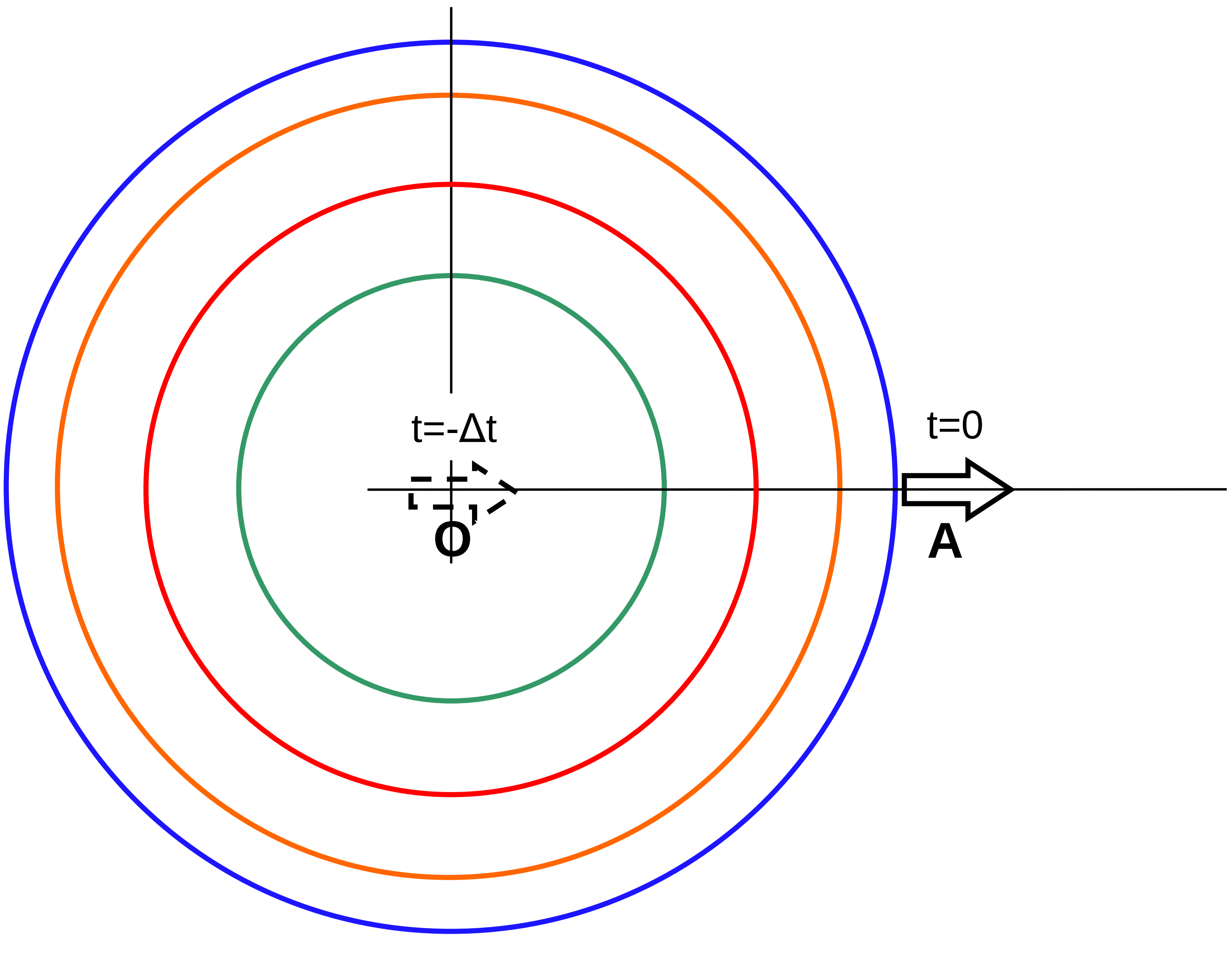
Waves created by an obstacle in a river



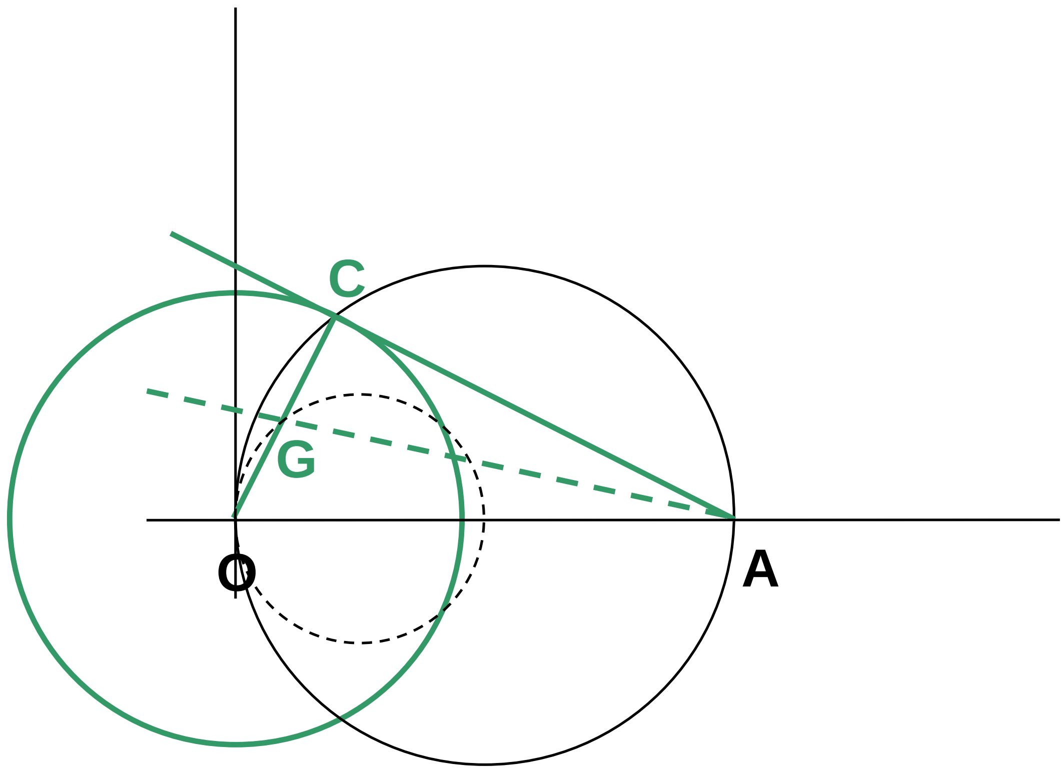
Kelvin wakes



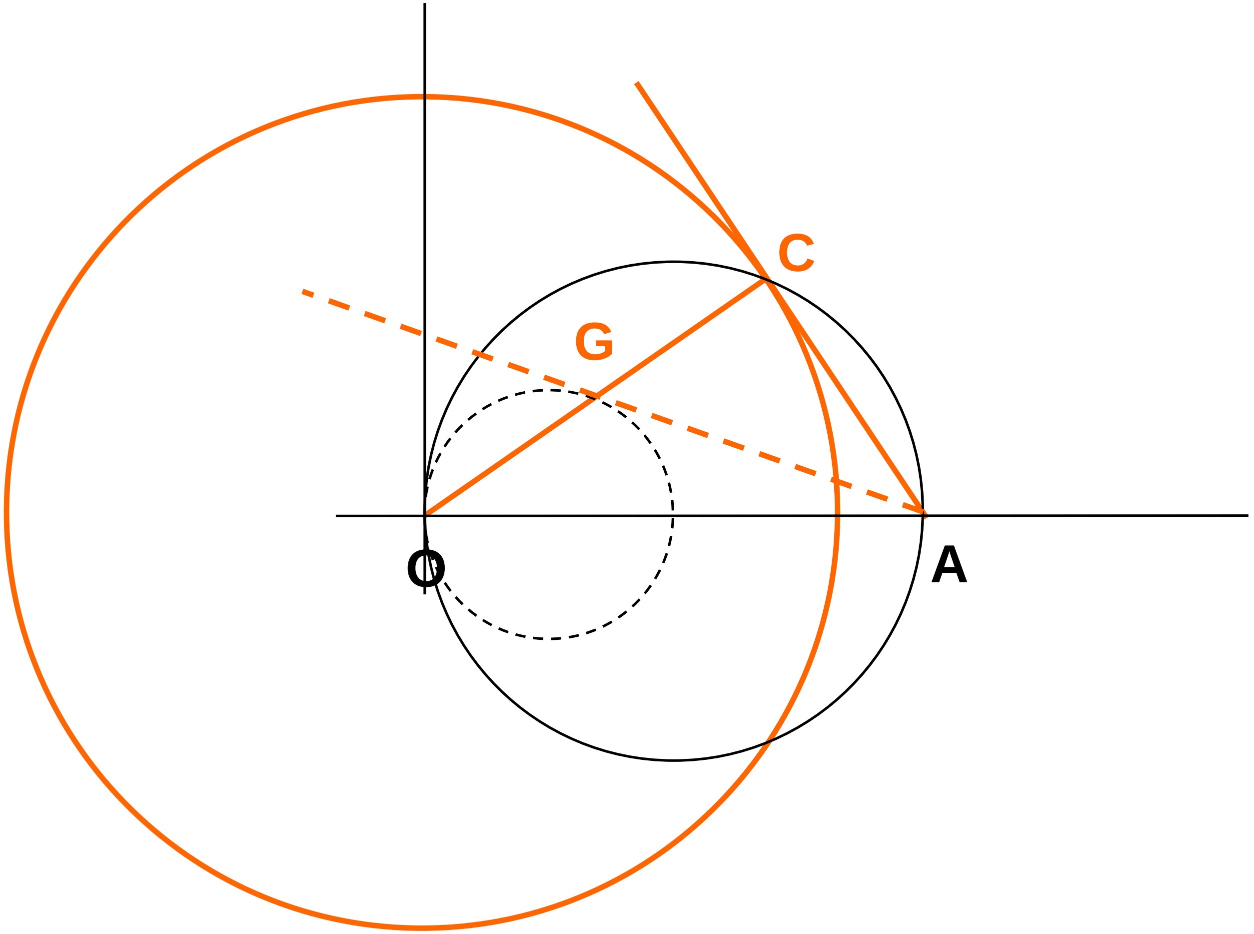
Kelvin wake



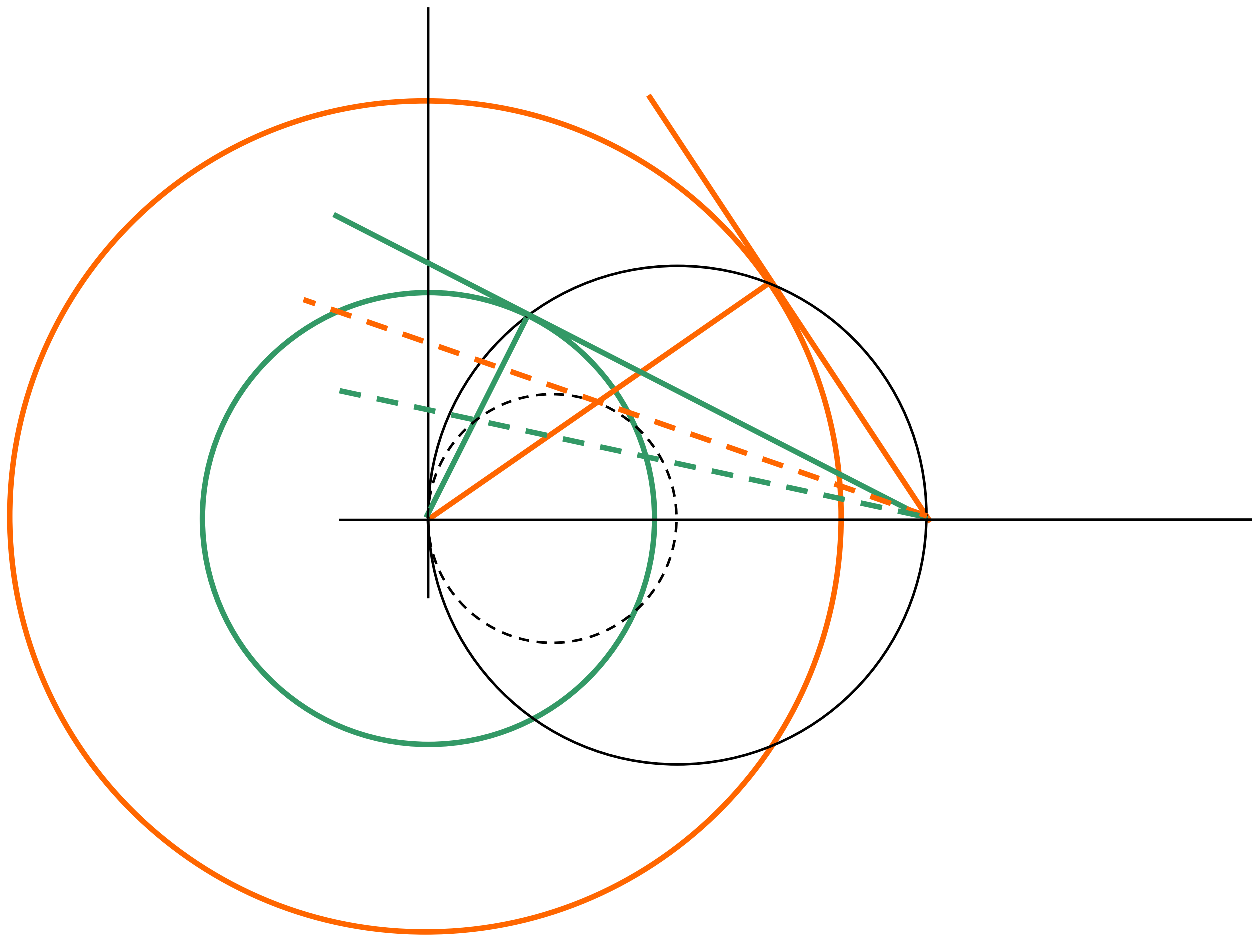
Waves created by a ship



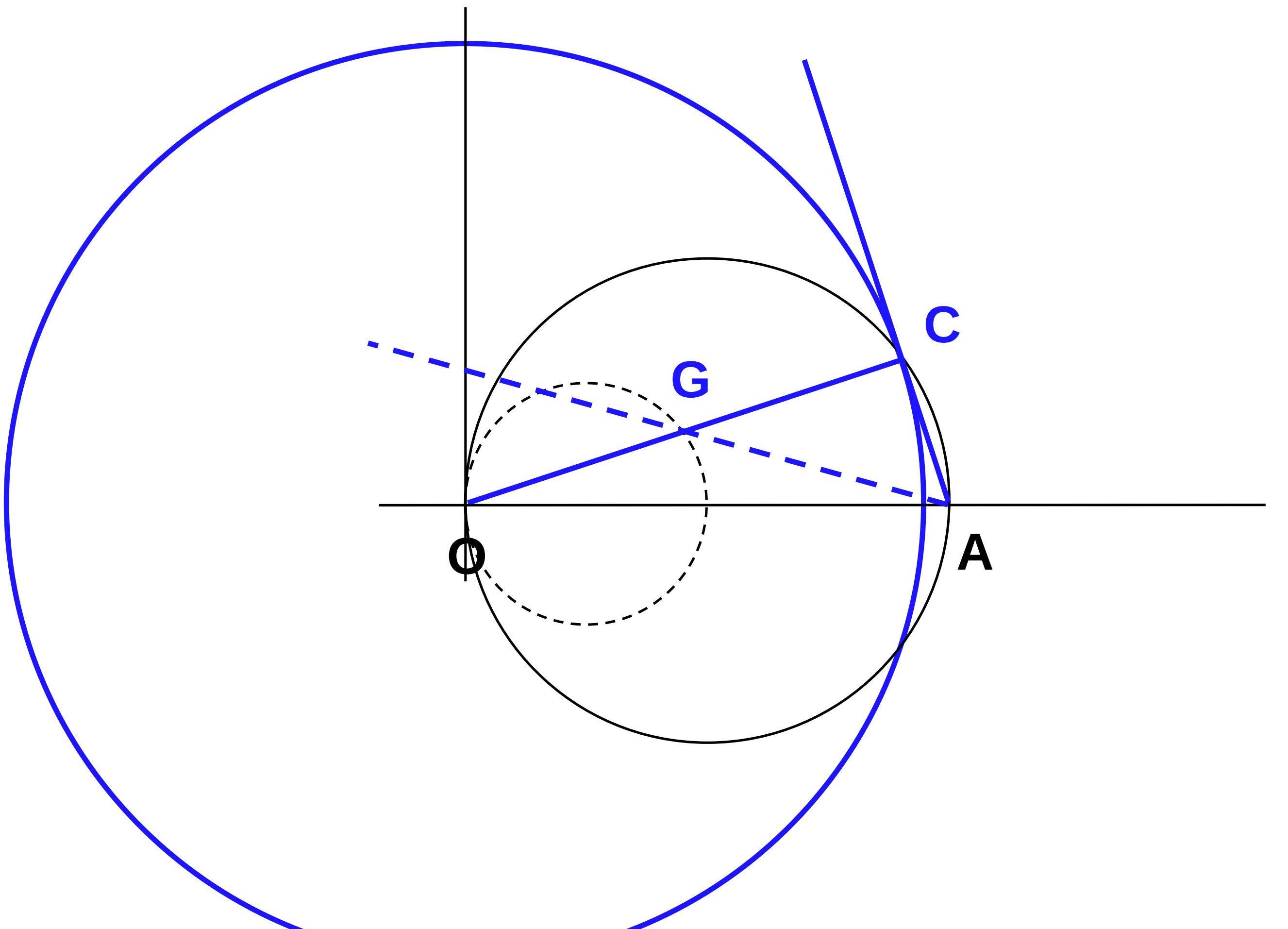
Waves created by a ship



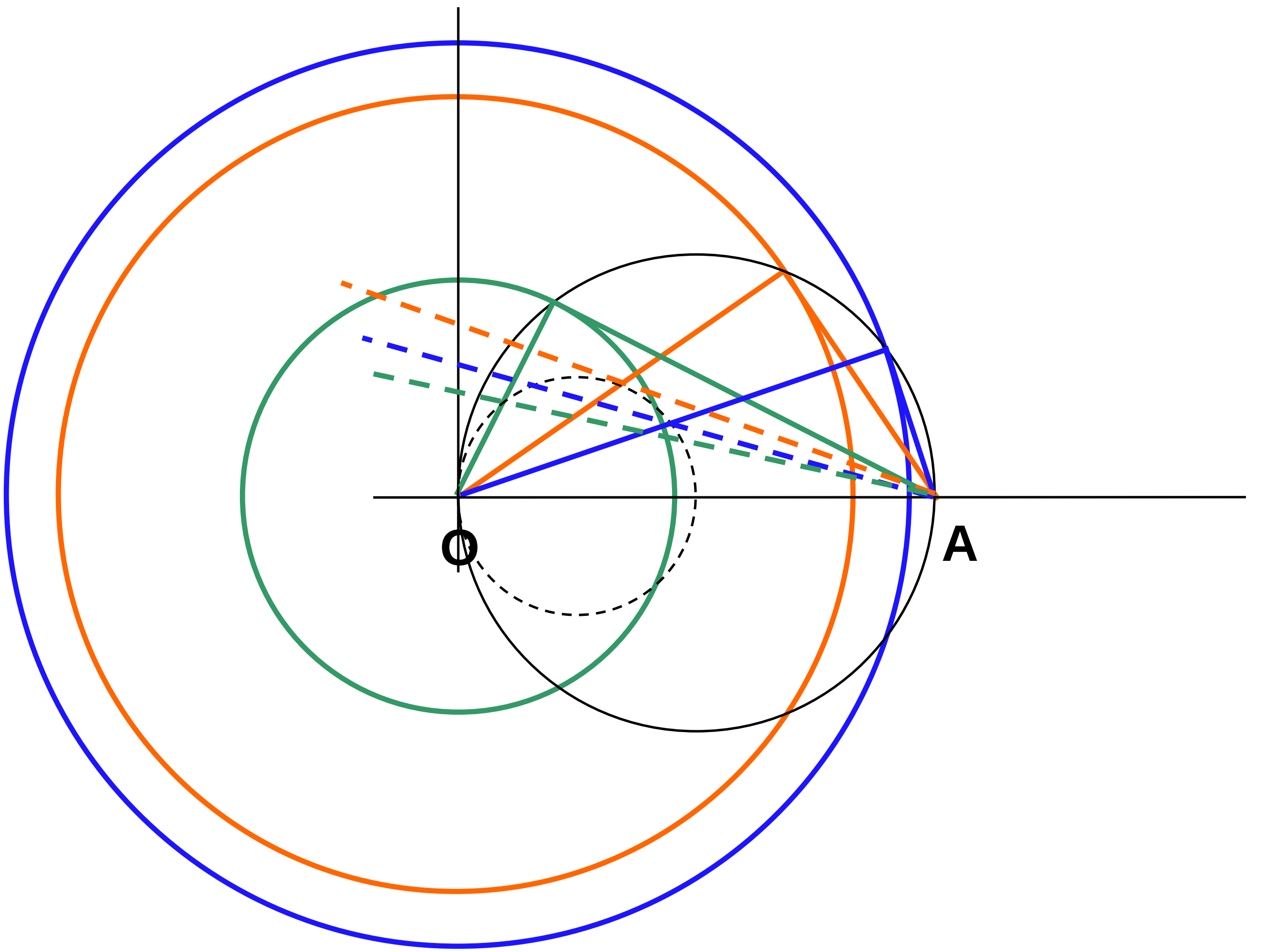
Waves created by a ship



Waves created by a ship



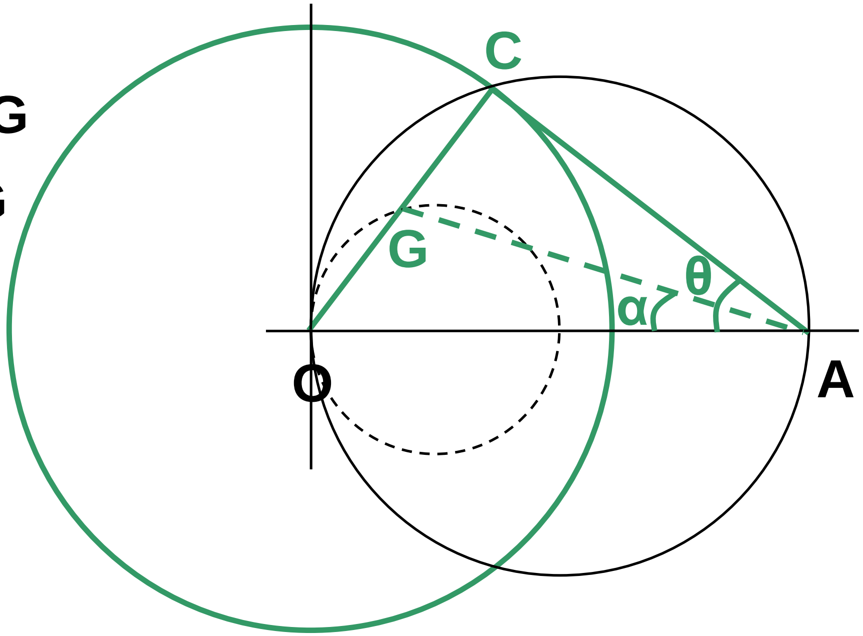
Waves created by a ship



Waves created by a ship

$\sin(\alpha)/OG=\cos(\theta)/AG$

$\sin(\theta-\alpha)AG=GC=OG$

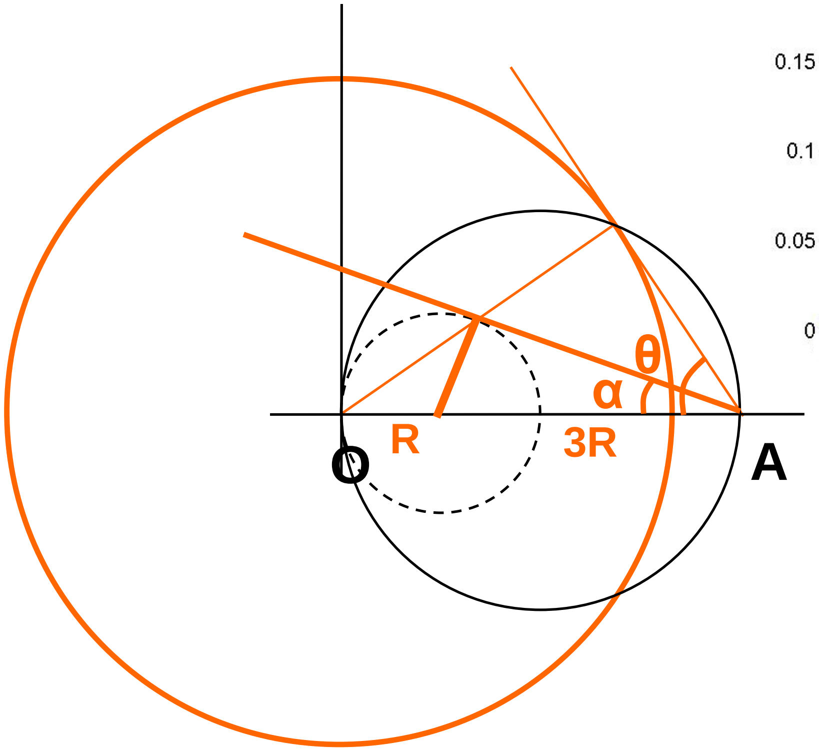


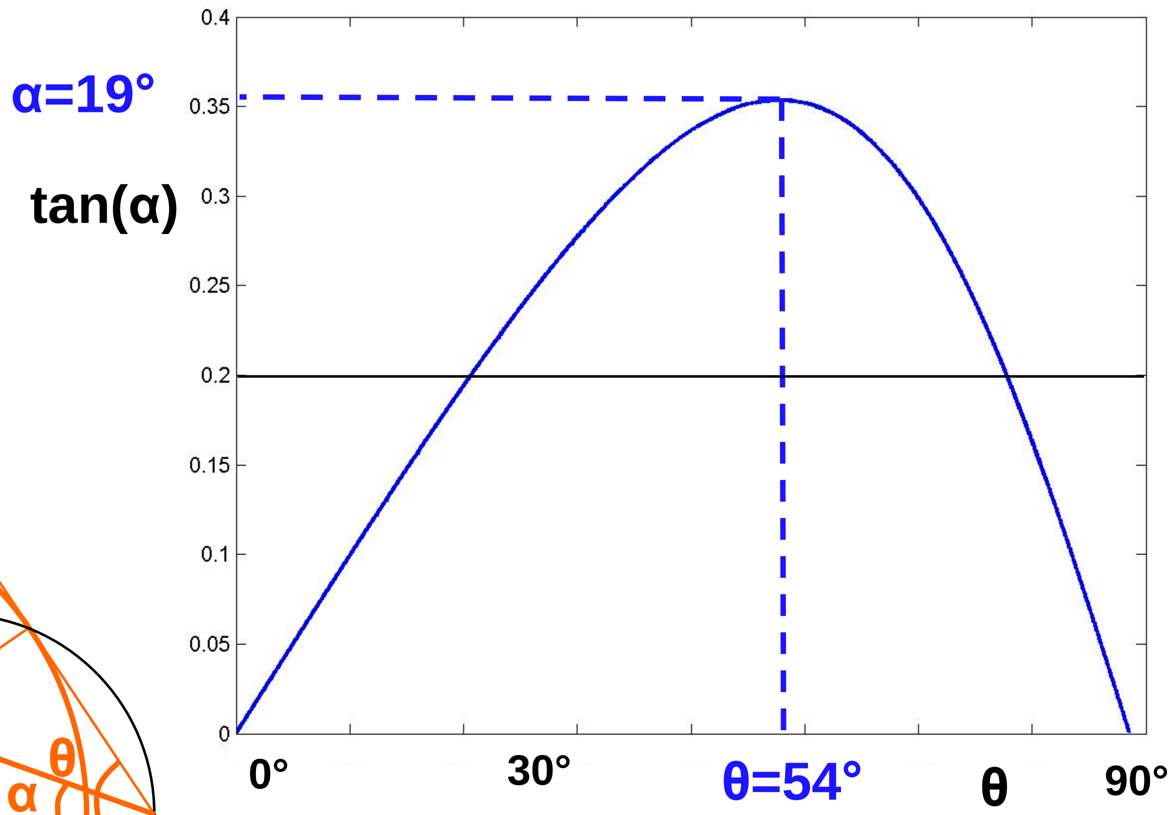
⇨ **$\sin(\alpha)=\cos(\theta)\sin(\theta-\alpha)$**

⇨ **$\sin(\alpha)=\cos(\theta)(\sin(\theta)\cos(\alpha)+\cos(\theta)\sin(\alpha))$**

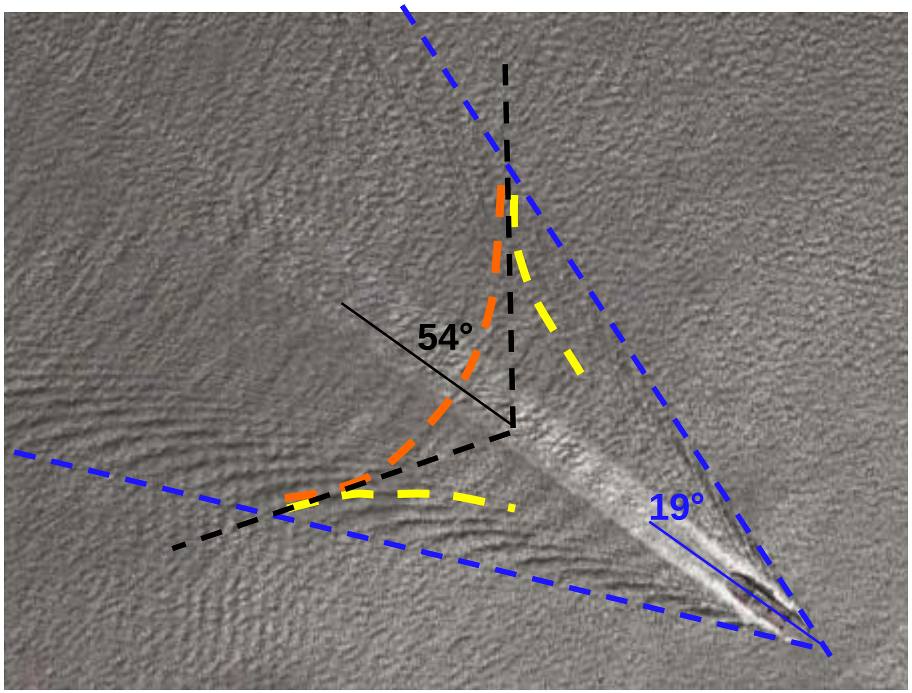
⇨ **$\tan(\alpha)=\cos(\theta)\sin(\theta)/(1+\cos^2(\theta))$**

Waves created by a ship

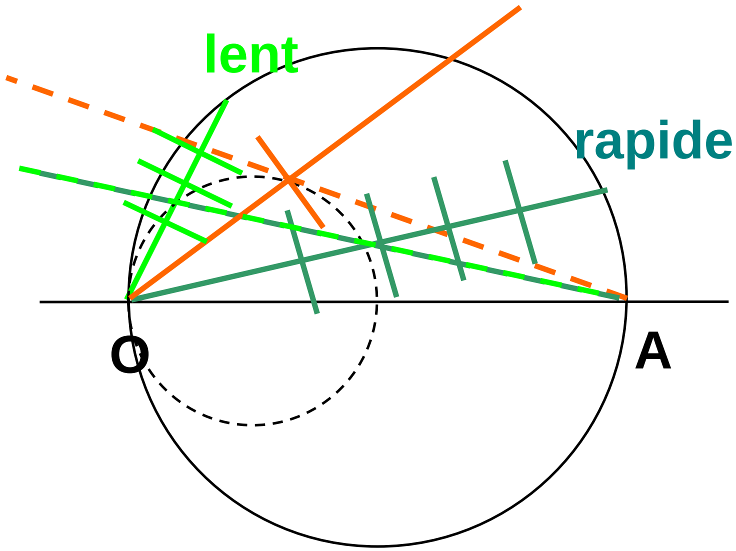


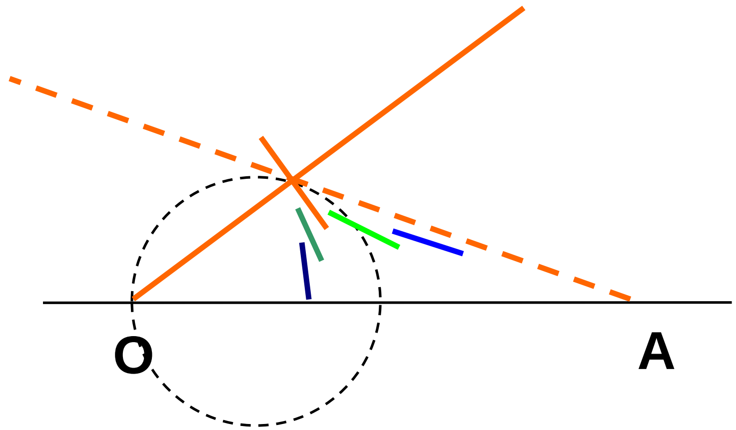


Waves created by a ship

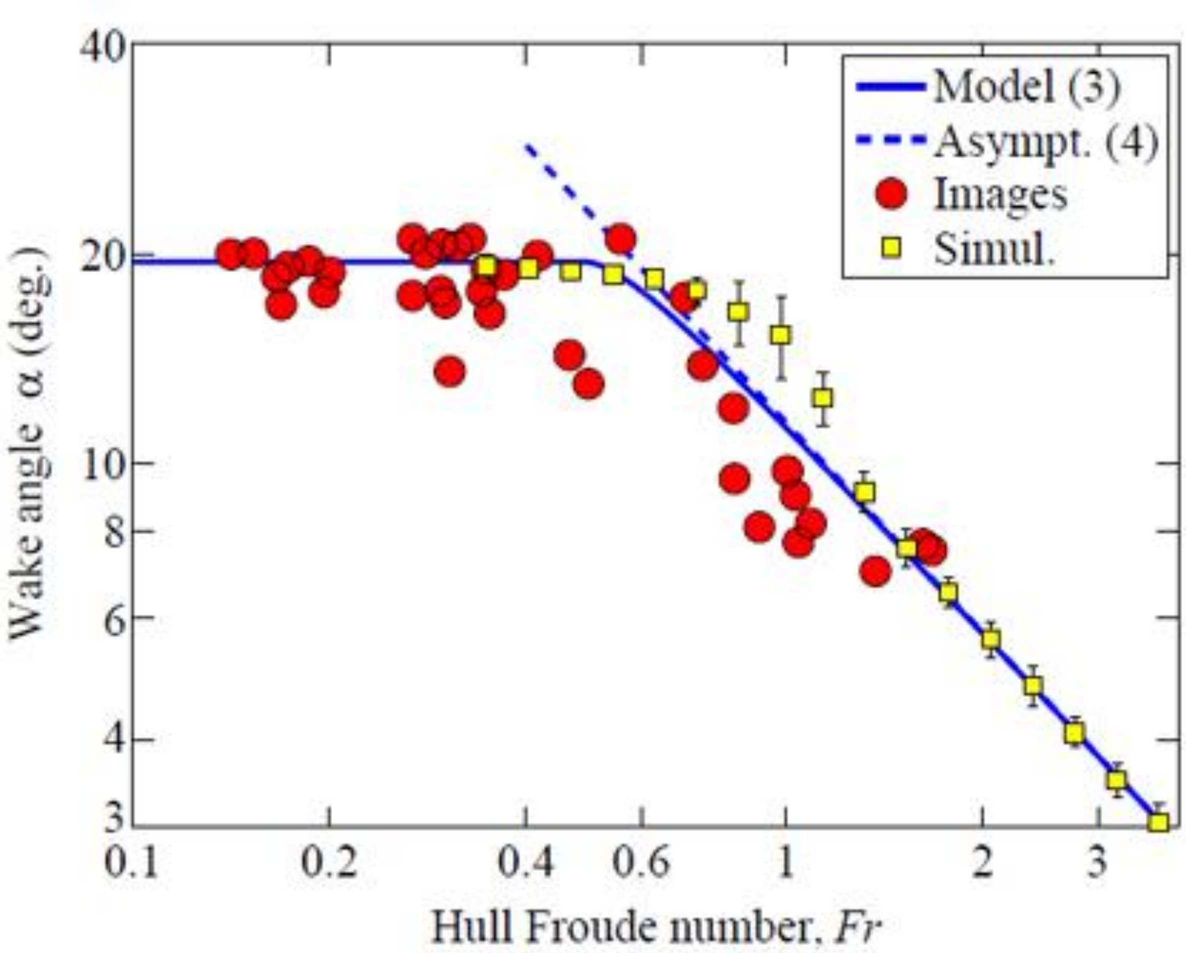


Waves created by a ship





Careful google-earth data analysis by Moisy and Rabaud (2013)



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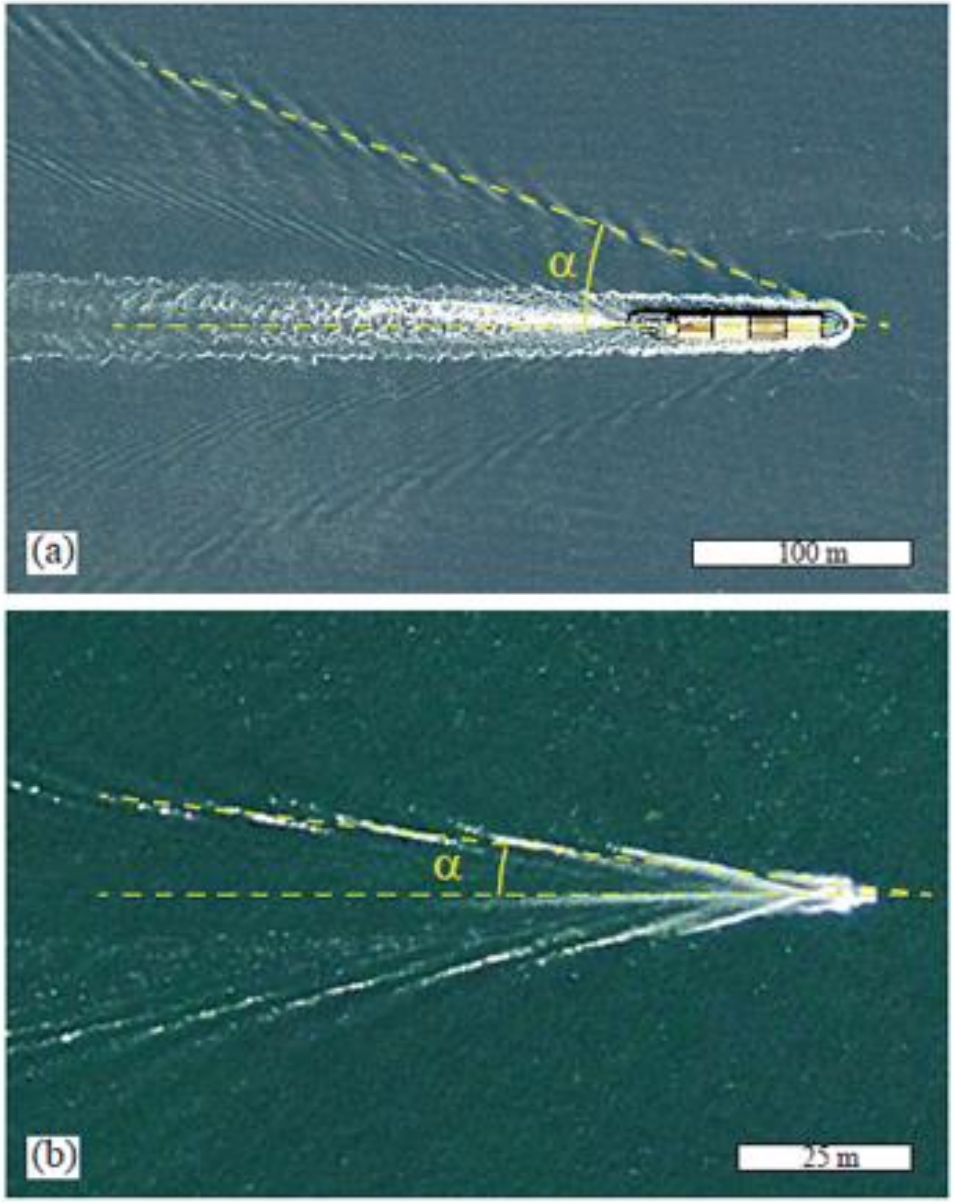


Froude number

$$Fr = \frac{U}{c}$$

c: celerity of wave

$$Fr = \frac{U}{\sqrt{gd}}$$

d: size of ship

$$Fr = \frac{U}{\sqrt{gl}}$$

shallow water

Froude decomposition

$$C_T(F, Re) = C_D(Re) + C_W(F, Re)$$

Total drag | Drag in absence of free surface | Wave drag

Froude hypothesis

$$C_T(F, Re) = C_D(Re) + C_W(F, \cancel{Re})$$

Total drag

Drag in absence of free surface

Wave drag

Alternative expression

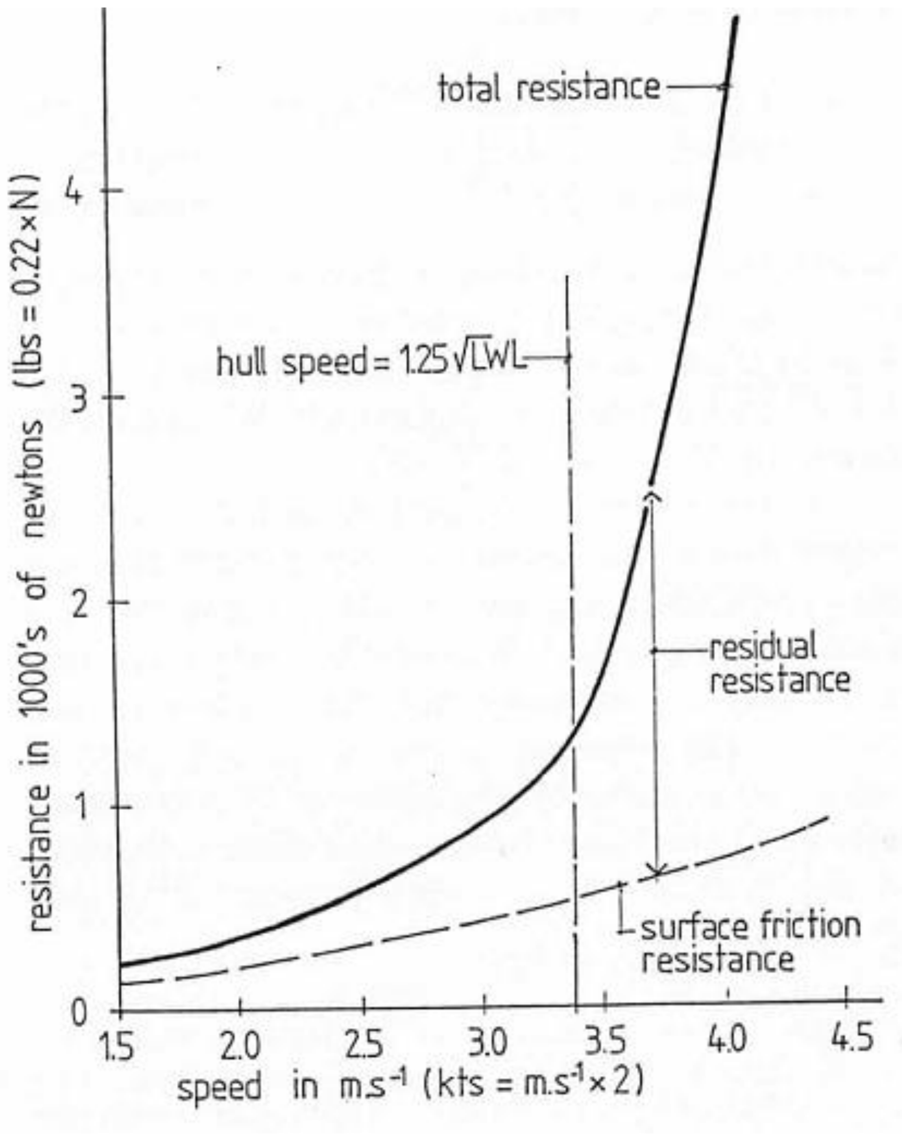
$$C_T(F, Re) = C_F(Re) + C_P(F, Re)$$

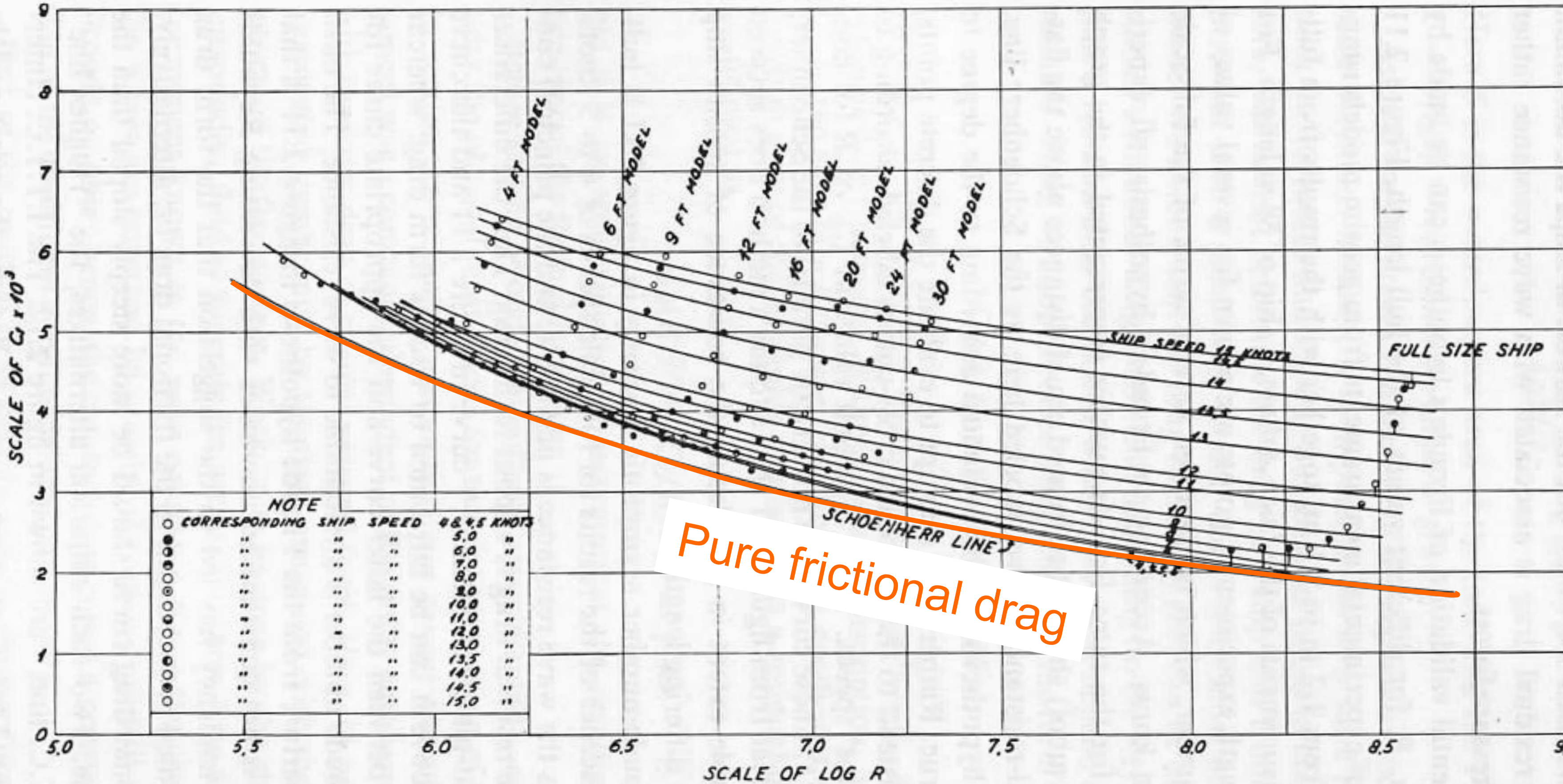
Total drag Friction drag Pressure drag

$$C_P(F, Re) \approx C_W(F)$$

Pressure drag Wave drag (Residual drag)

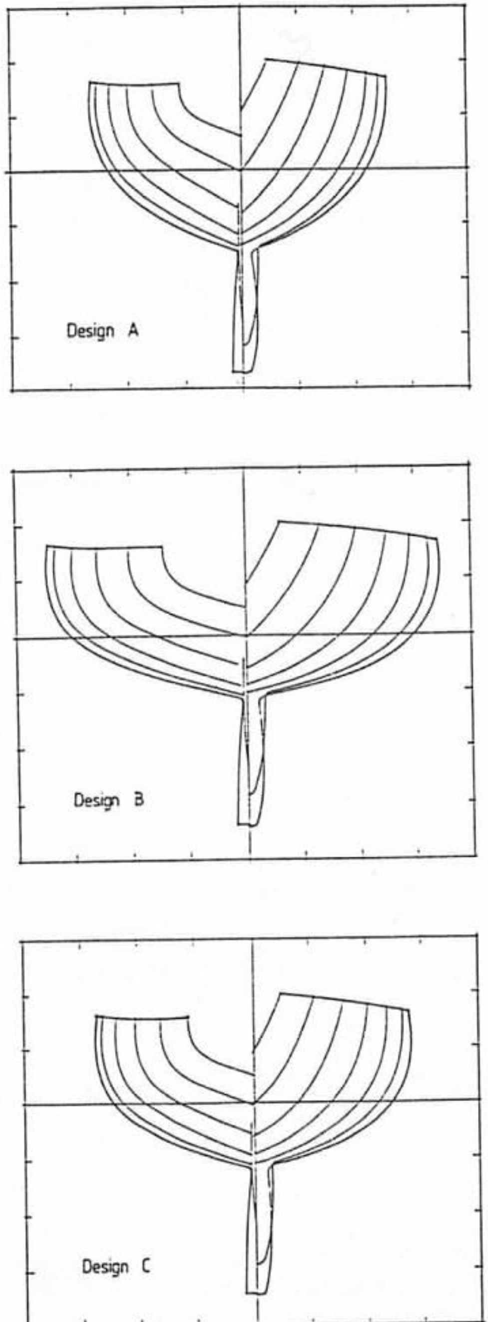
Friction resistance and residual resistance

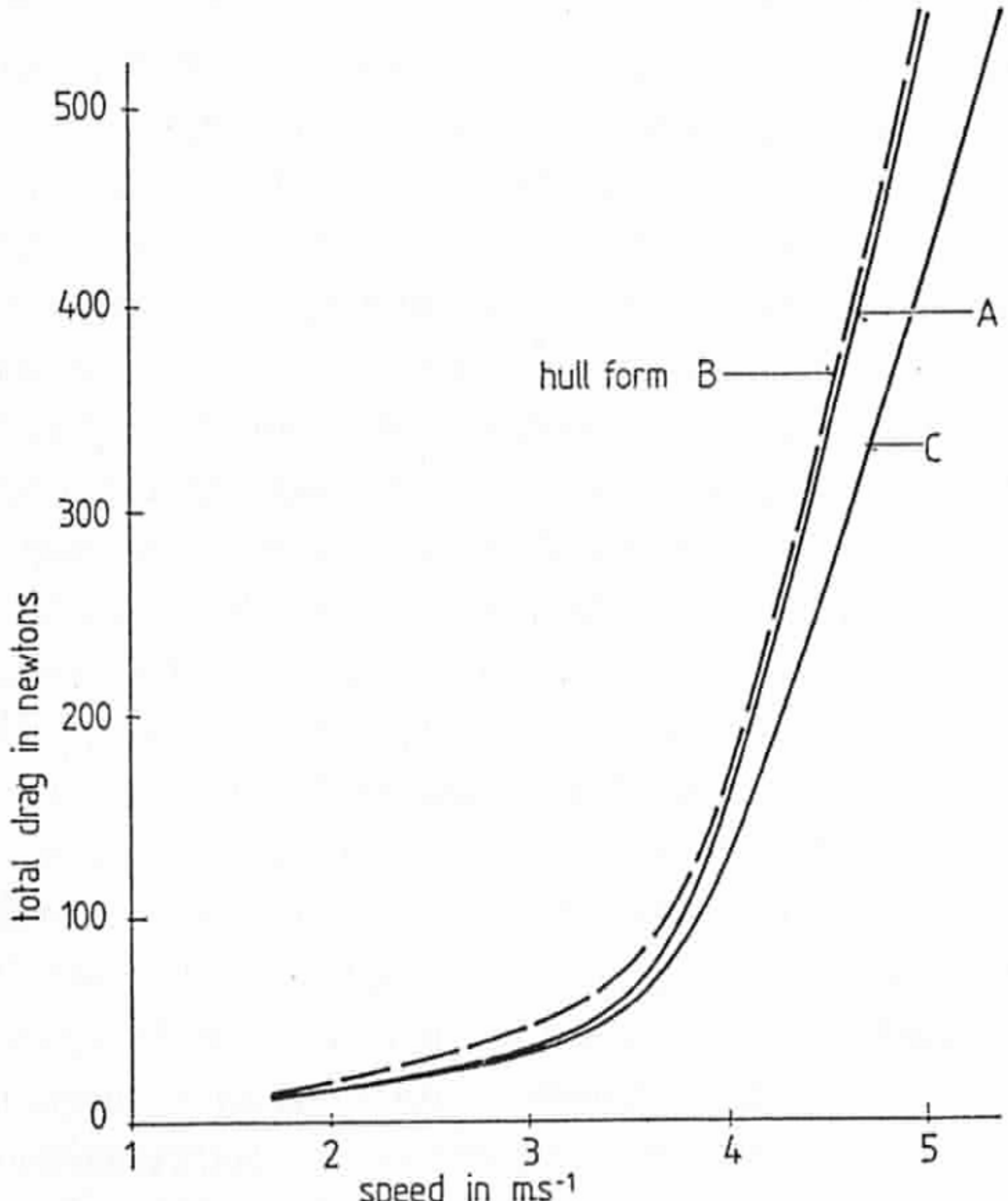




2.11

Total drag coefficients of the *Lucy Ashton* and several geosim models of the same vessel (from Troost and Zakay 1951). The faired curves represent constant values of the Froude number and, if Froude's hypothesis were strictly valid, these would be parallel with spacing independent of the Reynolds number. Note that, even for this small full-scale vessel (58 m long), there is a large gap between the largest model results and the full-scale results.

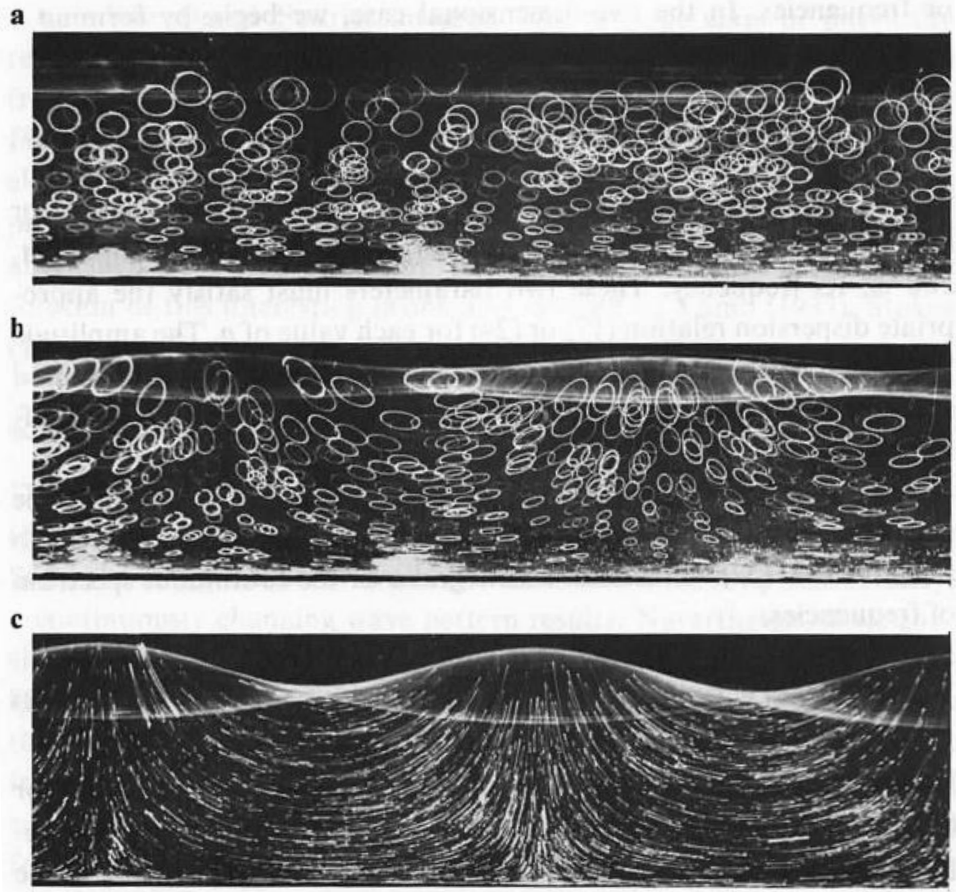

Design A
Design B
Design C


total drag in newtons
500
400
300
200
100
0
1
2
3
4
5
speed in ms-1
hull form B
A
C

Standing waves

$$\eta = A\cos(kx - \omega t) + A\cos(kx + \omega t)$$
$$= 2A\cos kx\cos\omega t,$$

Reflected waves



6.7
Particle trajectories in a plane progressive wave (a), a partial reflected wave (b), and a standing wave (c). These correspond respectively to a reflection coefficient of 0, 0.38, and 1.0 in equation (56). Note that the reflection coefficient can be measured from the maximum and minimum of the envelope, using (56). These photographs are based on time exposures, and are reproduced from a more extensive series of observations made by Ruellan and Wallet (1950).

Reflected waves

$$\eta = A\mathrm{Re}[e^{-ikx+i\omega t} + Re^{ikx+i\omega t}]$$

R: complex reflection coefficient

$$\eta = A\mathrm{Re}[e^{-ikx+i\omega t}(1 + \cdot Re^{2ikx})]$$

Wave energy

$$KE + PE = \rho \iiint_{\mathcal{V}} \left(\frac{1}{2} V^2 + gy \right) d\mathcal{V}$$

$$E = \rho \int_{-h}^{\eta} \left(\frac{1}{2} V^2 + gy \right) dy = \frac{1}{2} \rho \int_{-h}^{\eta} V^2 \, dy + \frac{1}{2} \rho g \left(\eta^2 - \cancel{h^2} \right).$$

Neglect potential energy of fluid at rest

Wave energy

$$KE + PE = \rho \iiint_{\mathcal{V}} \left(\frac{1}{2} V^2 + gy\right) d\mathcal{V}$$

$$E = \rho \int_{-h}^{\eta} \left(\frac{1}{2} V^2 + gy\right) dy = \frac{1}{2}\rho \int_{-h}^{\eta} V^2 \, dy + \frac{1}{2}\rho g \left(\eta^2 - h^2\right).$$

Neglect potential energy of fluid at rest

$$E = \frac{\rho \omega^2 A^2}{4k} e^{2k\eta} + \frac{1}{2}\rho g \eta^2.$$

Wave energy

$$KE + PE = \rho \iiint_{\mathcal{V}} \left(\frac{1}{2}V^2 + gy\right) d\mathcal{V}$$

$$E = \rho \int_{-h}^{\eta} \left(\frac{1}{2}V^2 + gy\right) dy = \frac{1}{2}\rho \int_{-h}^{\eta} V^2\, dy + \frac{1}{2}\rho g\left(\eta^2 - h^2\right).$$

Neglect potential energy of fluid at rest

$$E = \frac{\rho\omega^2 A^2}{4k} e^{2k\eta} + \frac{1}{2}\rho g \eta^2.$$

Consider A<<1

$$E = \tfrac{1}{4}\rho g A^2 + \tfrac{1}{2}\rho g A^2 \cos^2(kx - \omega t)$$

Wave energy

$$KE + PE = \rho \iiint_{\mathcal{V}} \left(\frac{1}{2} V^2 + gy\right) d\mathcal{V}$$

$$E = \rho \int_{-h}^{\eta} \left(\frac{1}{2} V^2 + gy\right) dy = \frac{1}{2}\rho \int_{-h}^{\eta} V^2 \, dy + \frac{1}{2}\rho g \left(\eta^2 - \cancel{h}^2\right).$$

Neglect potential energy of fluid at rest

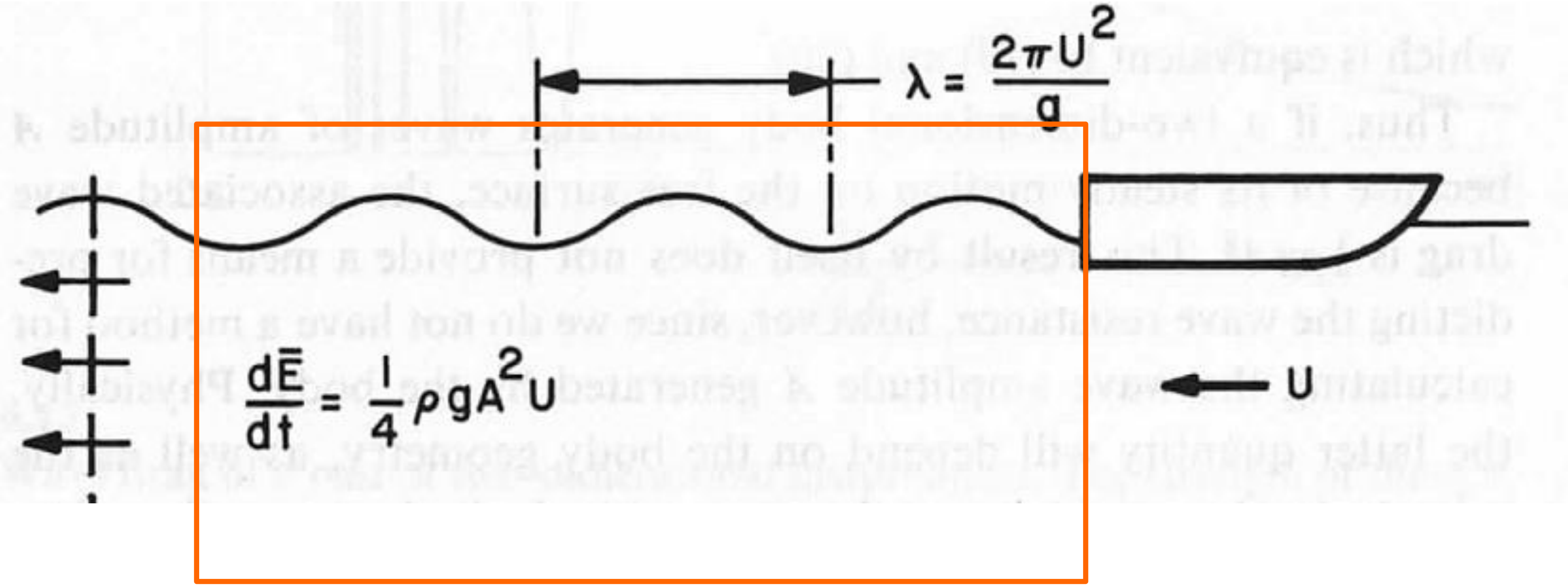
$$E = \tfrac{1}{4}\rho g A^2 + \tfrac{1}{2}\rho g A^2 \cos^2(kx - \omega t)$$

On average: $\bar{E} = \tfrac{1}{2}\rho g A^2$

Wave energy flux

One can show that the energy travels at the group velocity…

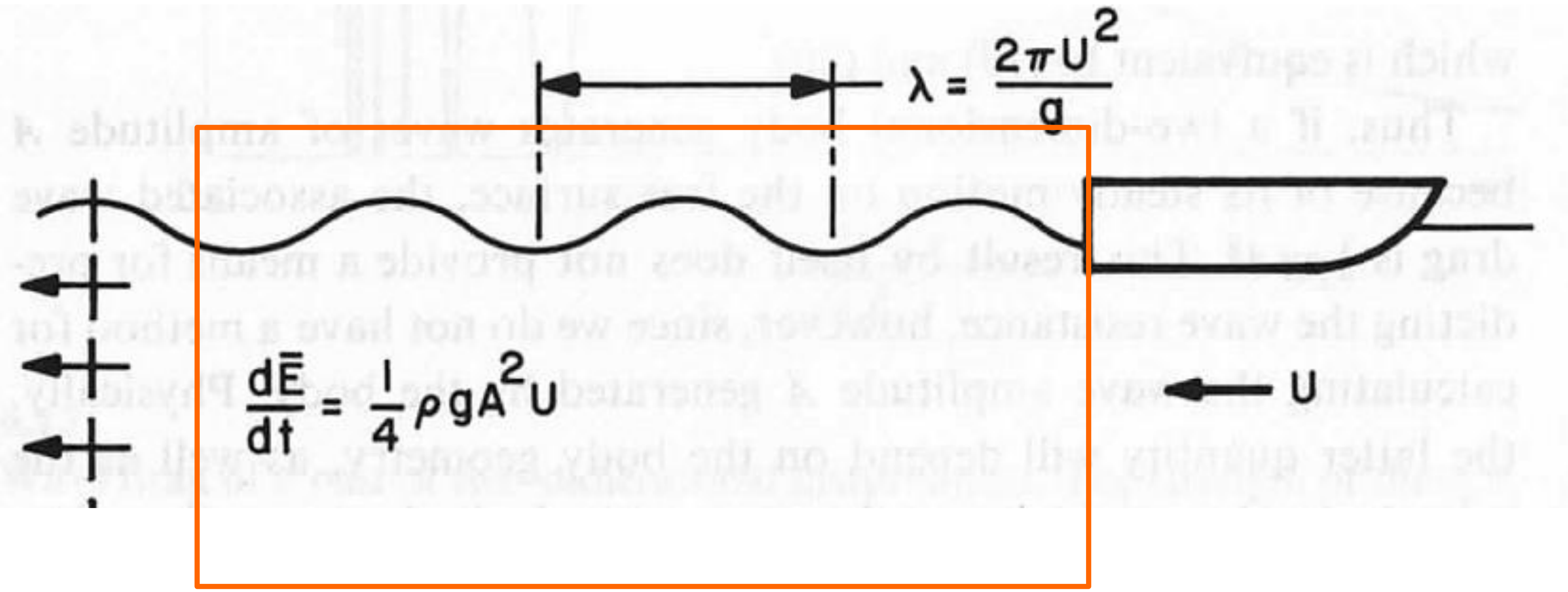
Energy balance (in ship's frame)



Energy input : -work of drag= -D.U

Energy output: (Vg-U)E

Energy balance (in ship's frame)



Energy input : -work of drag= -D.U

Energy output: (Vg-U)E

Deep water: Vg=U/2

D=E/2

$$D = \tfrac{1}{4}\rho g A^2$$

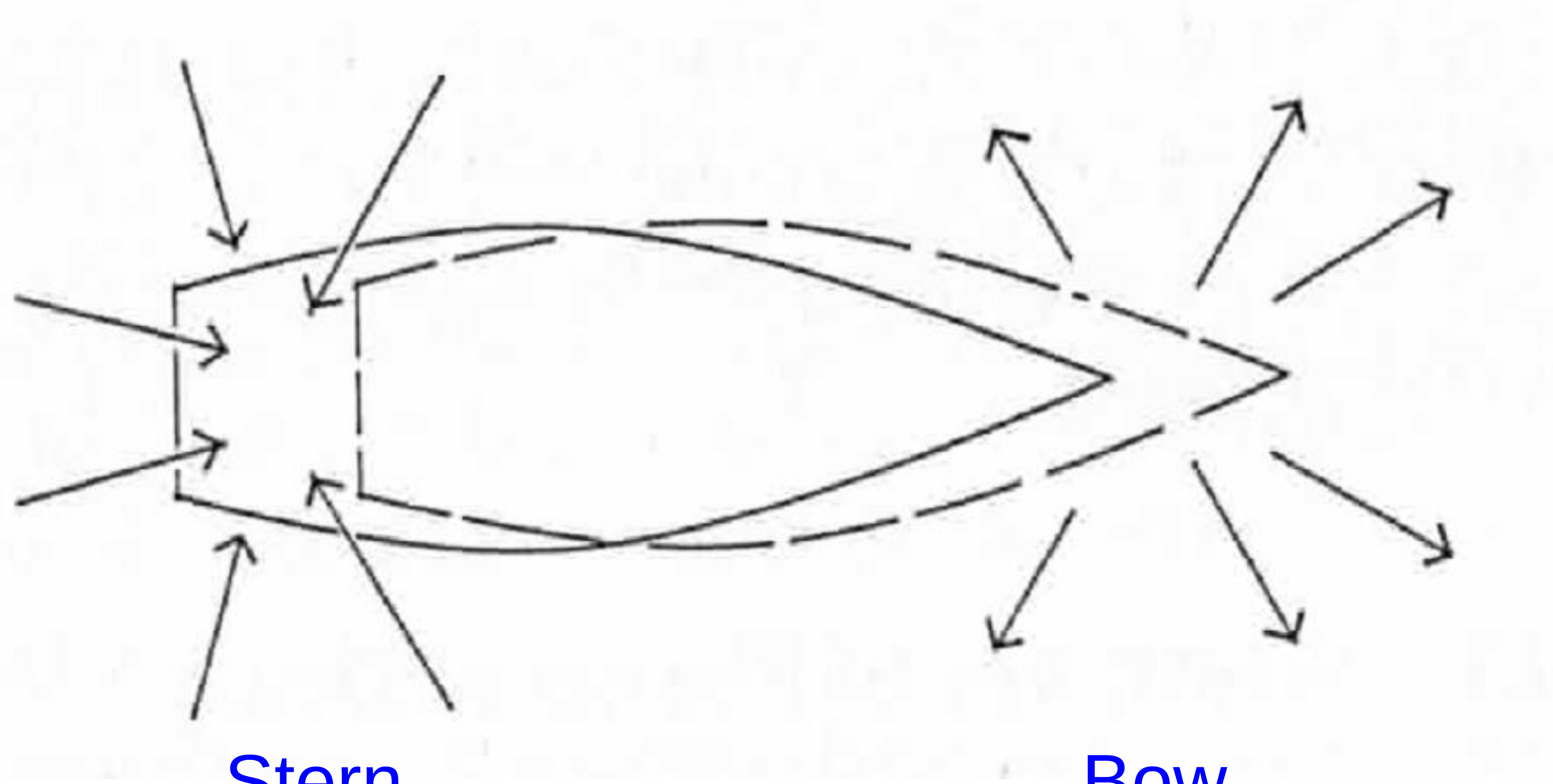
Wave production from bow

displacement $\eta = a\cos(kx + \varepsilon)$

dispersion relation $k = g/U^2$

wave amplitude a

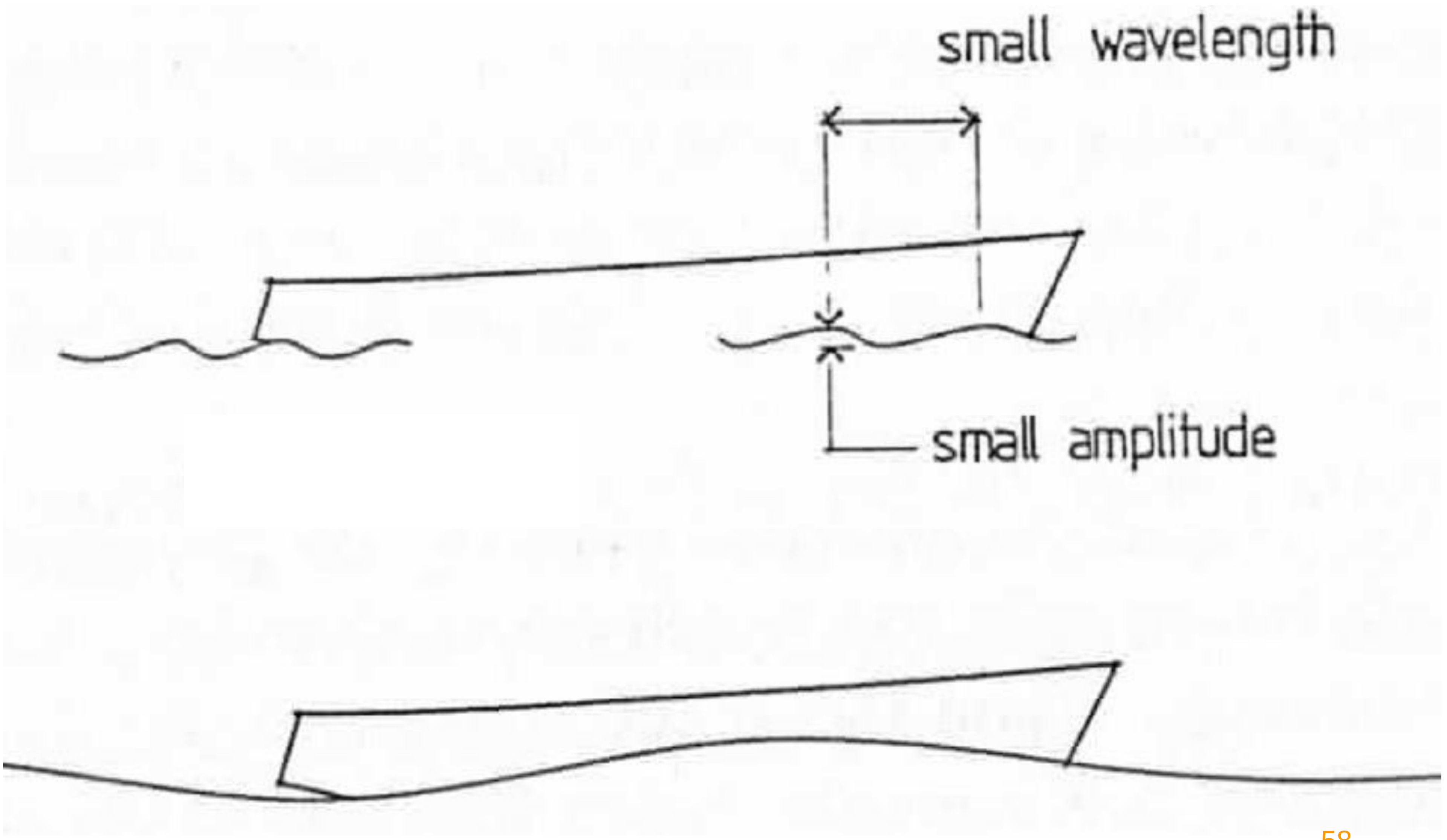
wave energy $\frac{1}{4}\rho g a^2$



Stern
"Hole"

Bow
"Bump"

Wavelength selection as a function of velocity
Which of these two ships has the highest velocity?



Superimpose wave of opposite amplitude from stern located at a distance l

displacement

$$\eta = a\cos(kx + \varepsilon) - a\cos(kx + \varepsilon + kl)$$
$$= \mathrm{Re}\{ae^{i(kx+\varepsilon)}(1 - e^{ikl})\}.$$

dispersion relation

$$k = g/U^2$$

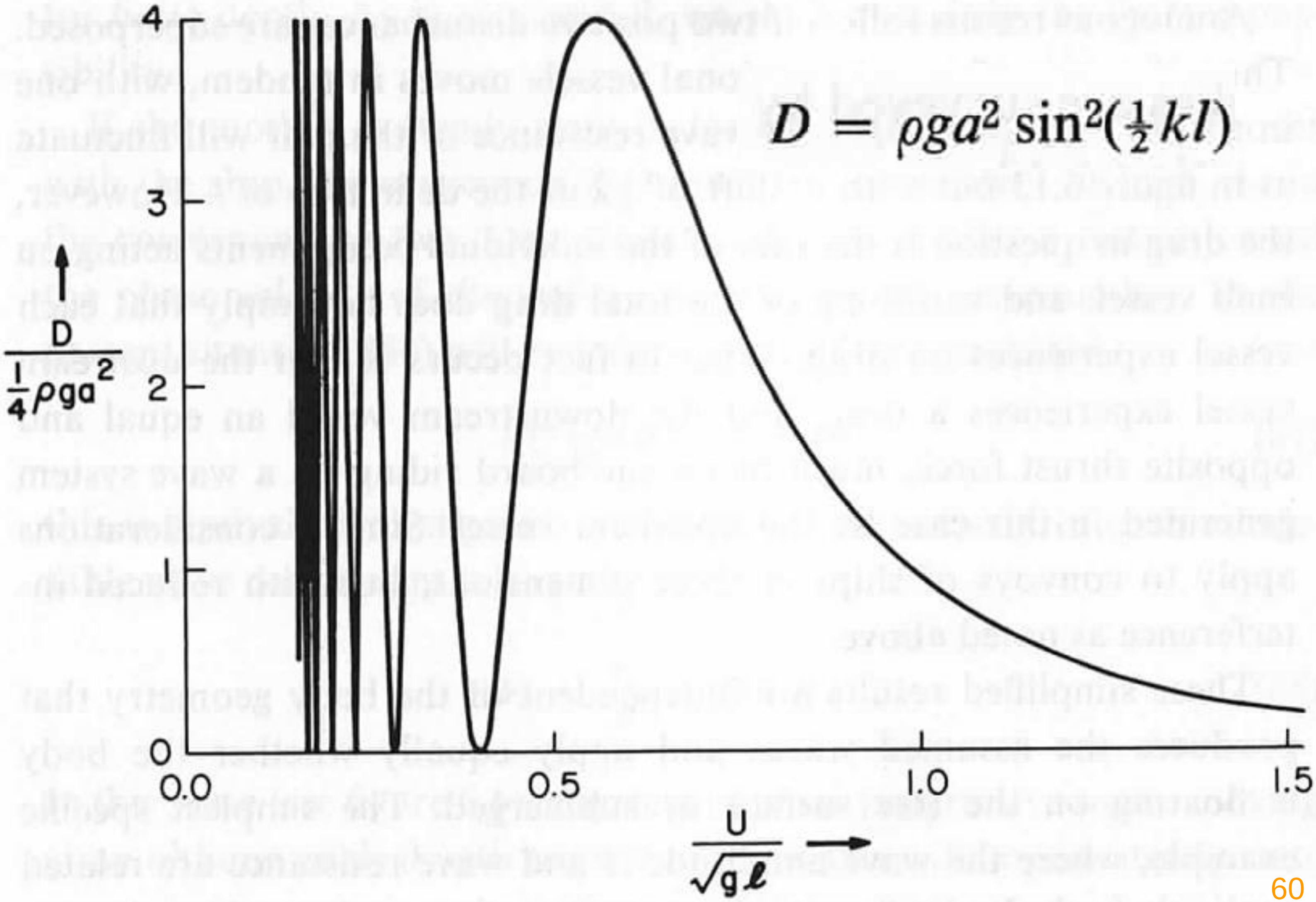
wave amplitude

$$A = a\left|1 - e^{ikl}\right| = 2a\left|\sin(\tfrac{1}{2}kl)\right|$$

wave energy

$$D = \rho g a^2 \sin^2(\tfrac{1}{2}kl)$$

Superimpose wave of opposite amplitude from stern located at a distance l



The missing ingredient is now to determine the wave amplitude!

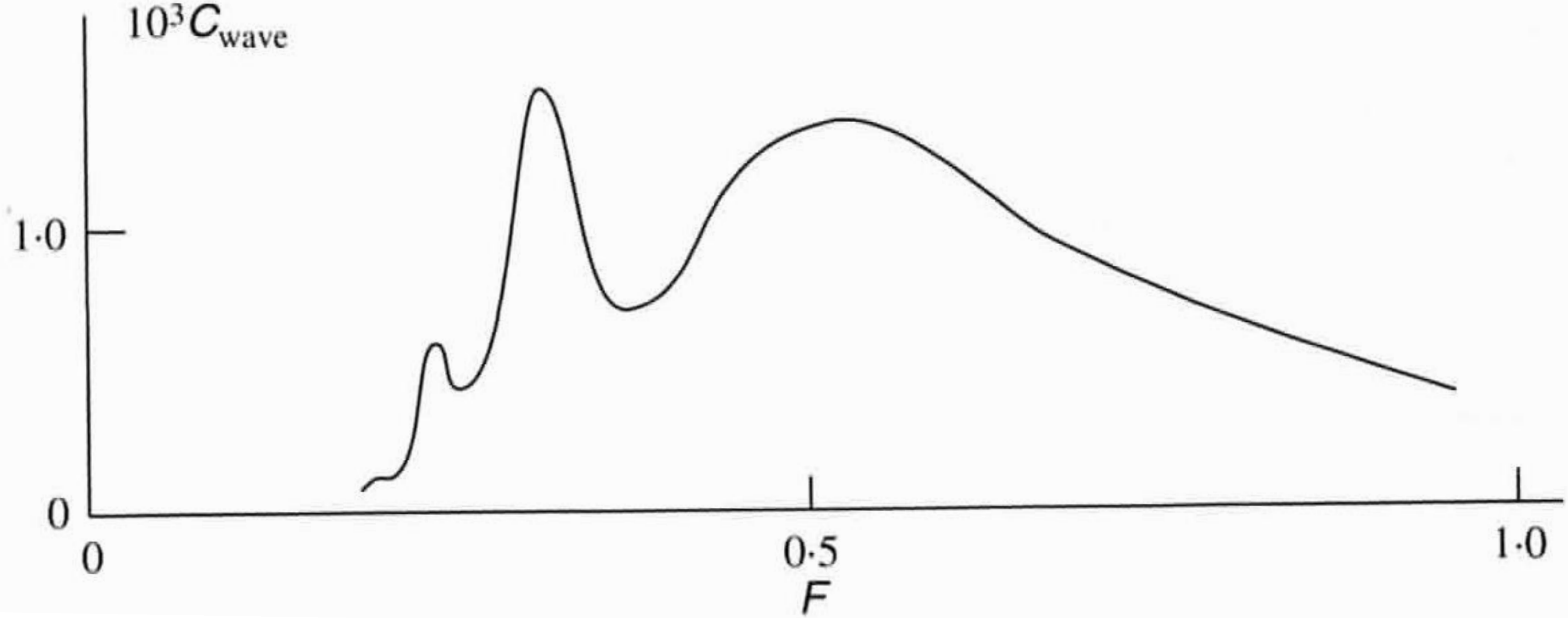
Combination of theoretical and empirical results.

Generalisation to 3D

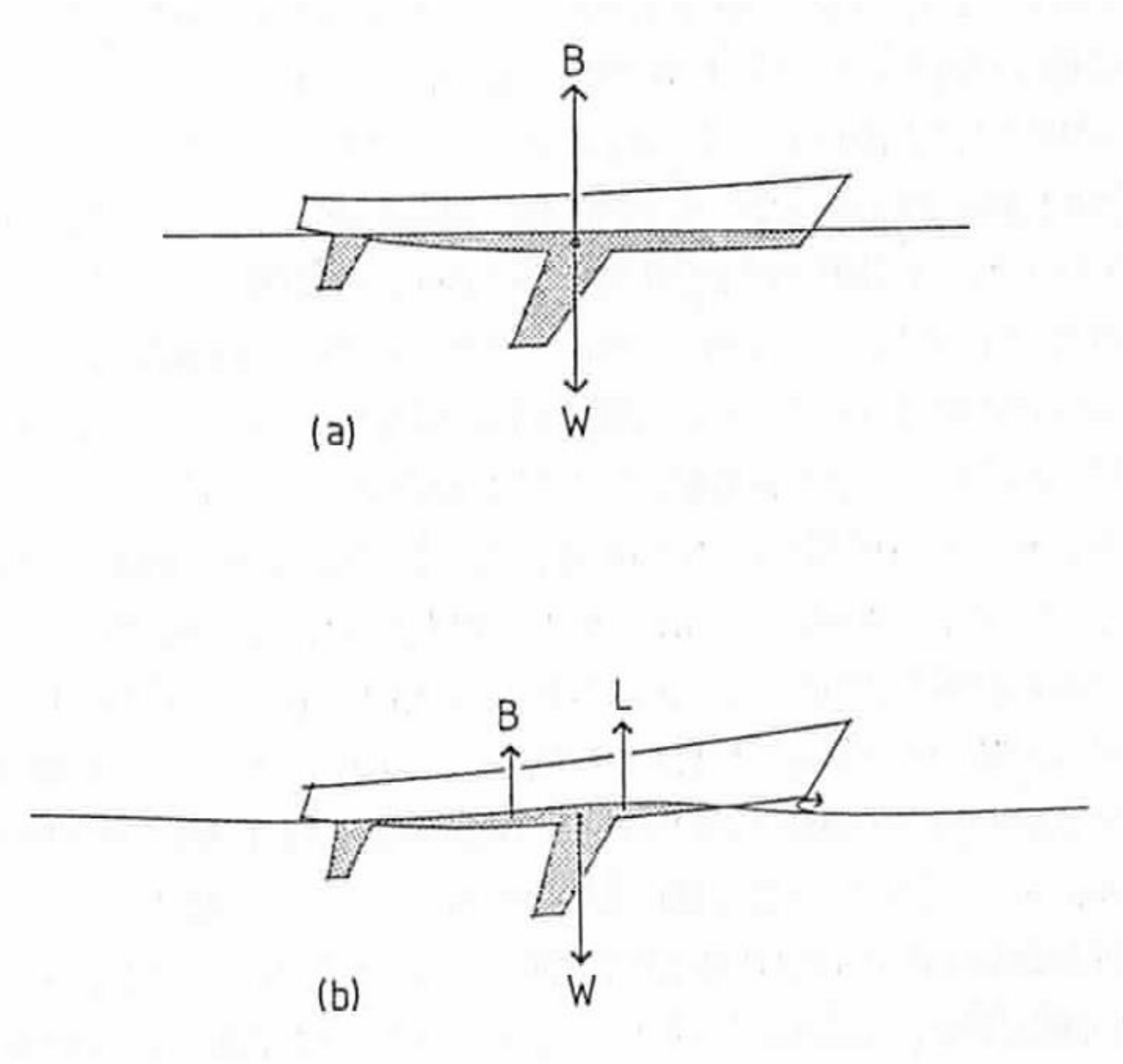
$$\frac{dE}{dt} = \frac{1}{2}\rho g \int_{-\infty}^{\infty} A^2(V_g \cos\theta - U)\, dz.$$

$$D = \frac{1}{2}\pi\rho U^2 \int_{-\pi/2}^{\pi/2} |A(\theta)|^2 \cos^3\theta \; d\theta.$$

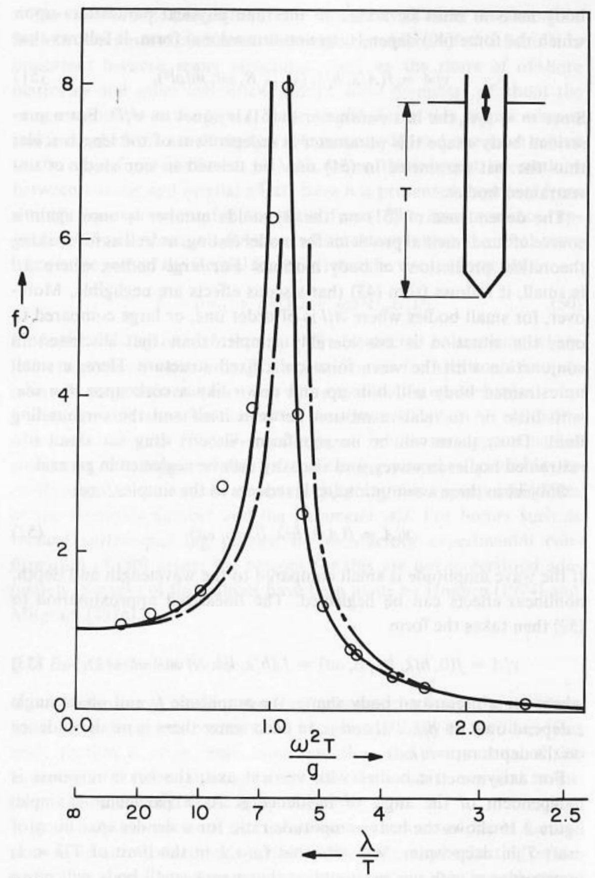
Less interference


$10^3 C_{wave}$
1·0
0
0
0·5
1·0
F

This picture is in reality complicated by the floating position of the ship



Which opens the field of dynamical reaction of ships to waves….



2.16
Heave response of a slender spar buoy in regular waves, from Adee and Bai (1970). The ordinate is the ratio of heave amplitude to wave amplitude, and the buoy is a circular cylinder, with a conical bottom, as shown to scale in the sketch. The dashed line is a theoretical prediction that neglects the hydrodynamic forces due to the motions of the body. The solid line includes a correction for the added mass. The circles denote experimental measurements.