

Hydrodynamics 14

Waves



Dispersion relation

$$\omega^2 = \tanh(kH) \left(\frac{\gamma k^3}{\rho} + gk \right)$$

Capillary wavenumber: $k_c = \sqrt{\rho g / \gamma}$

Length scale: $\tilde{k} = k / k_c$

Time scale: $\tilde{\omega} = \omega / \sqrt{g k_c}$

One single non-dimensional parameter $\tilde{H} = H k_c$

$$\tilde{\omega}^2 = \tanh(\tilde{k} \tilde{H}) \left(\tilde{k}^3 + \tilde{k} \right)$$

Dispersion relation

	gravity $\tilde{k} \ll 1$	capillary $\tilde{k} \gg 1$
shallow water $\tilde{k} \ll 1/\tilde{H}$	$\omega_{\text{shallow/gravity}} \sim \pm k \sqrt{gH}$ $c_{\text{shallow/gravity}} \sim \pm \sqrt{gH}$ $v_{\text{shallow/gravity}} \sim \pm \sqrt{gH}$	$\omega_{\text{shallow/capillary}} \sim \pm k^2 \sqrt{\gamma H / \rho}$ $c_{\text{shallow/capillary}} \sim \pm k \sqrt{\gamma H / \rho}$ $v_{\text{shallow/capillary}} \sim \pm 2k \sqrt{\gamma H / \rho}$
Deep water $\tilde{k} \gg 1/\tilde{H}$	$\omega_{\text{deep/gravity}} \sim \pm \sqrt{gk}$ $c_{\text{deep/gravity}} \sim \pm \sqrt{\frac{g}{k}}$ $v_{\text{deep/gravity}} \sim \pm \frac{1}{2} \sqrt{\frac{g}{k}}$	$\omega_{\text{deep/capillary}} \sim \pm k^{3/2} \sqrt{\gamma / \rho}$ $c_{\text{deep/capillary}} \sim \pm k^{1/2} \sqrt{\gamma / \rho}$ $v_{\text{deep/capillary}} \sim \pm 3/2 k^{1/2} \sqrt{\gamma / \rho}$

Dispersion relation

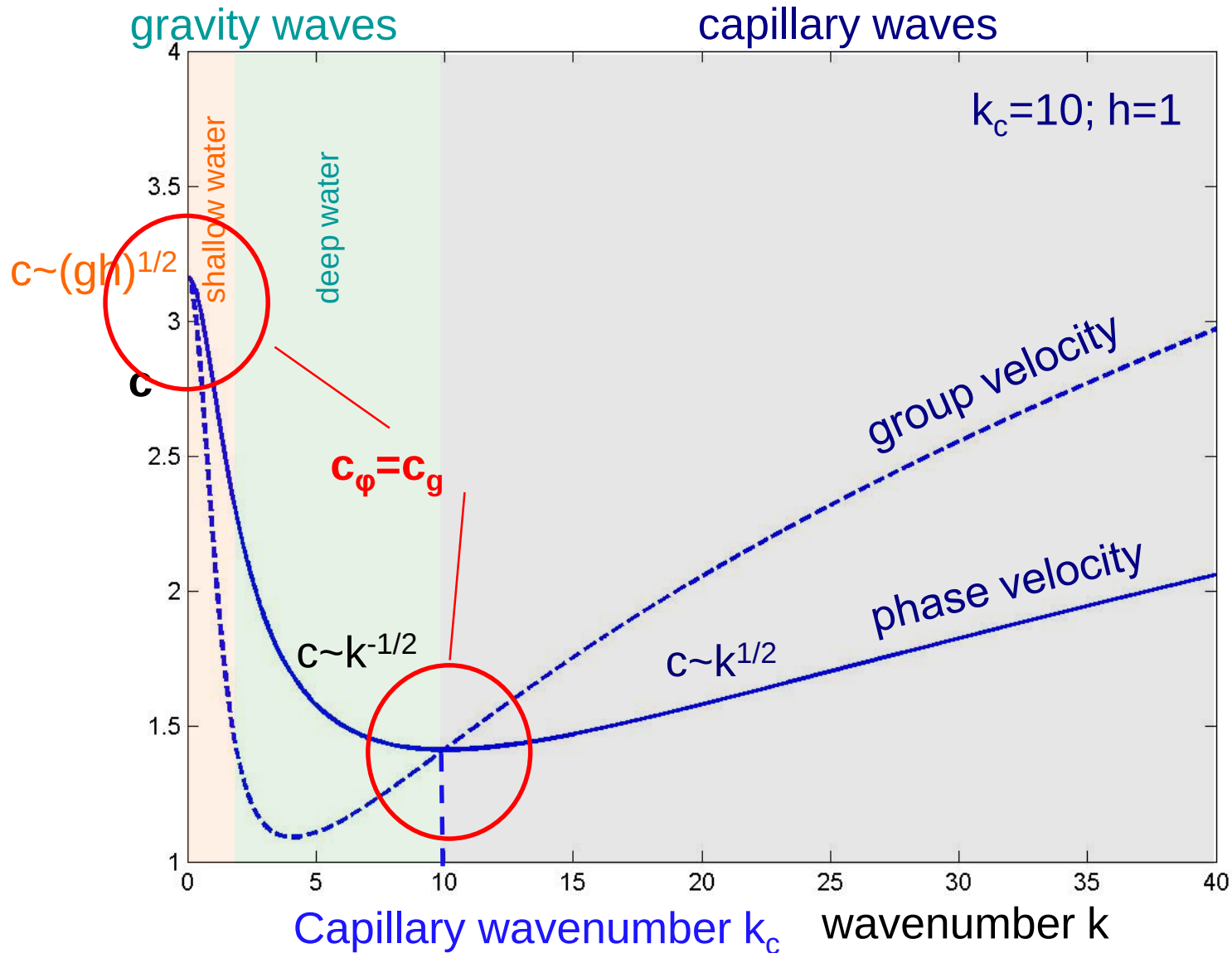


Diagramme spatio-temporel

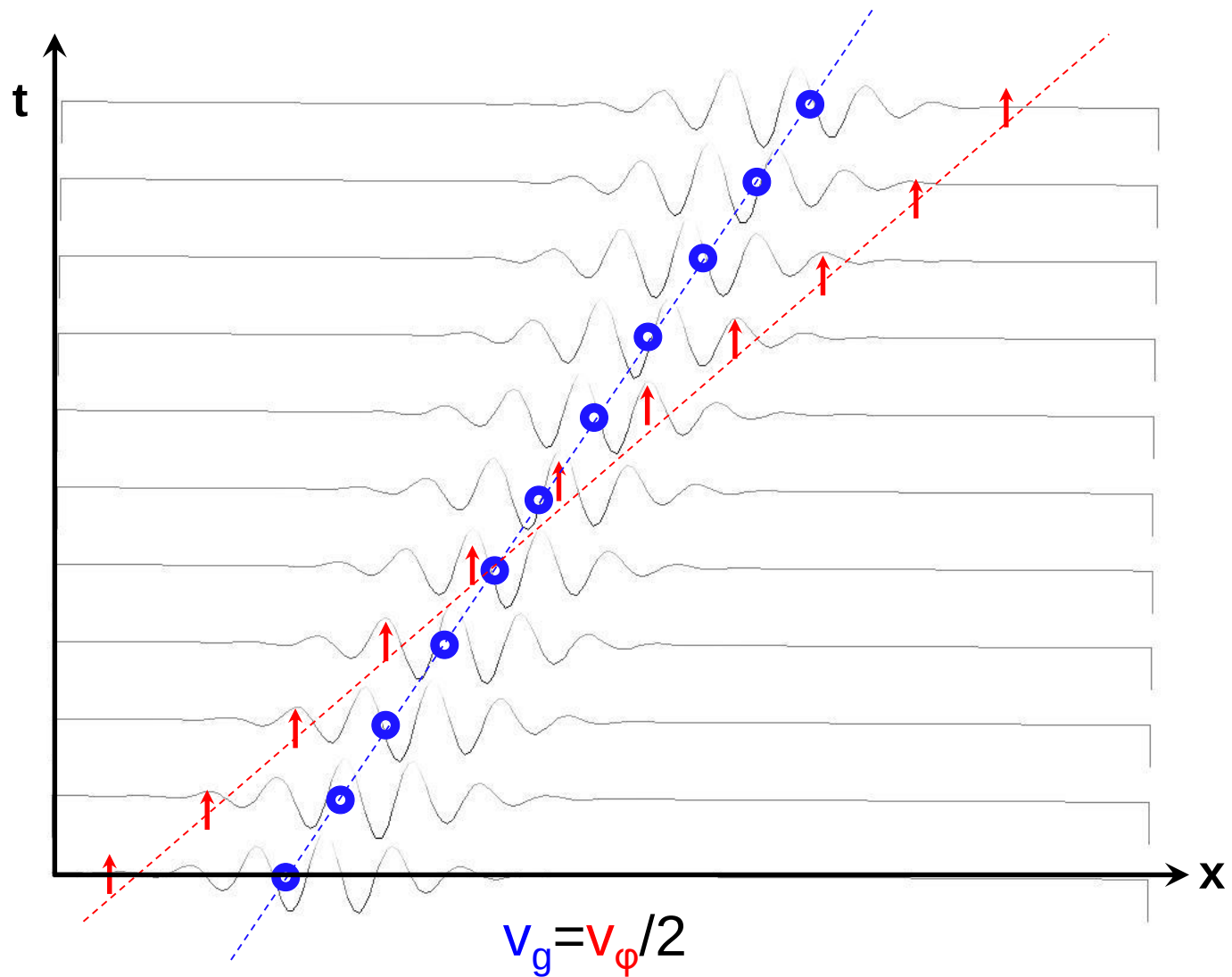


Diagramme spatio-temporel

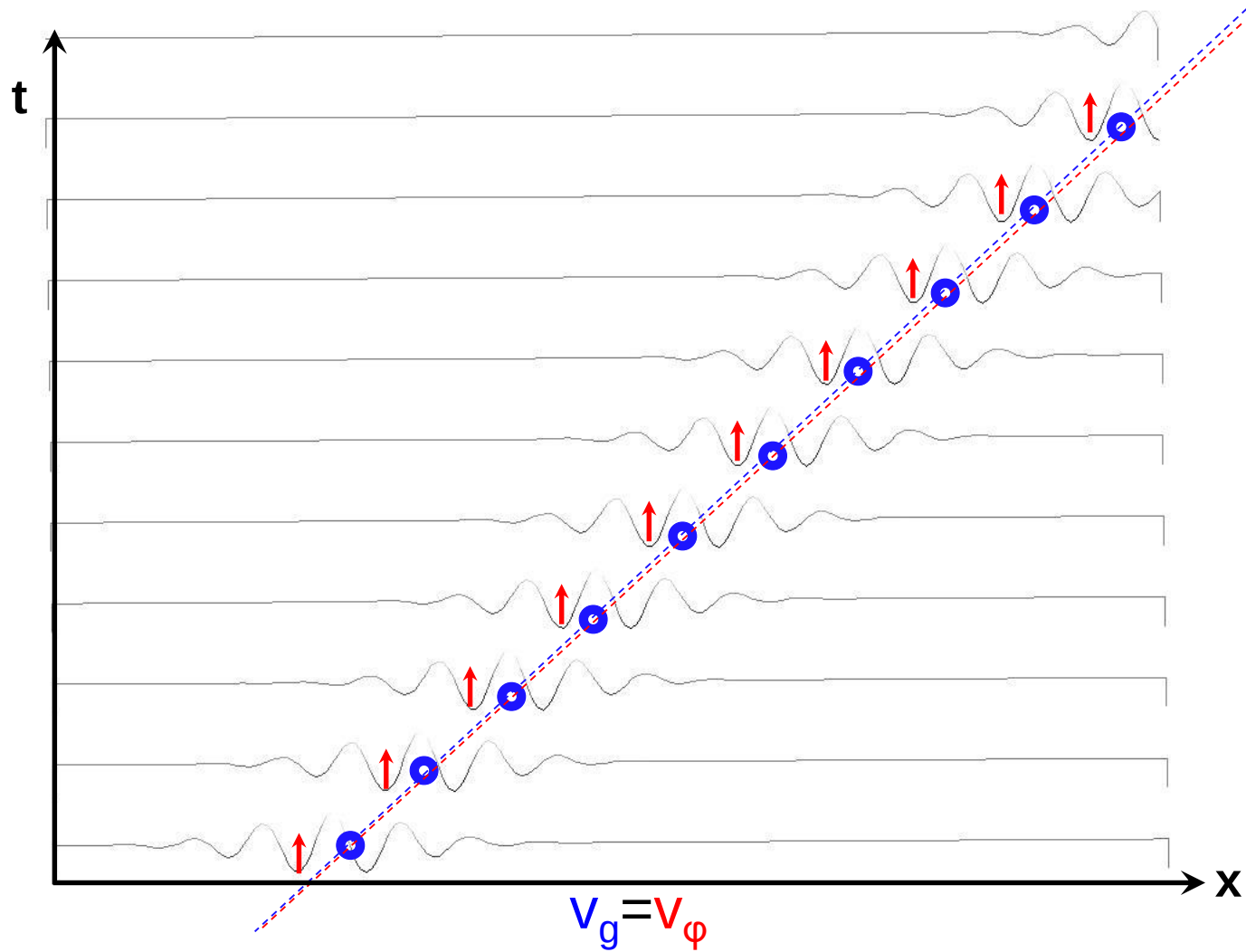
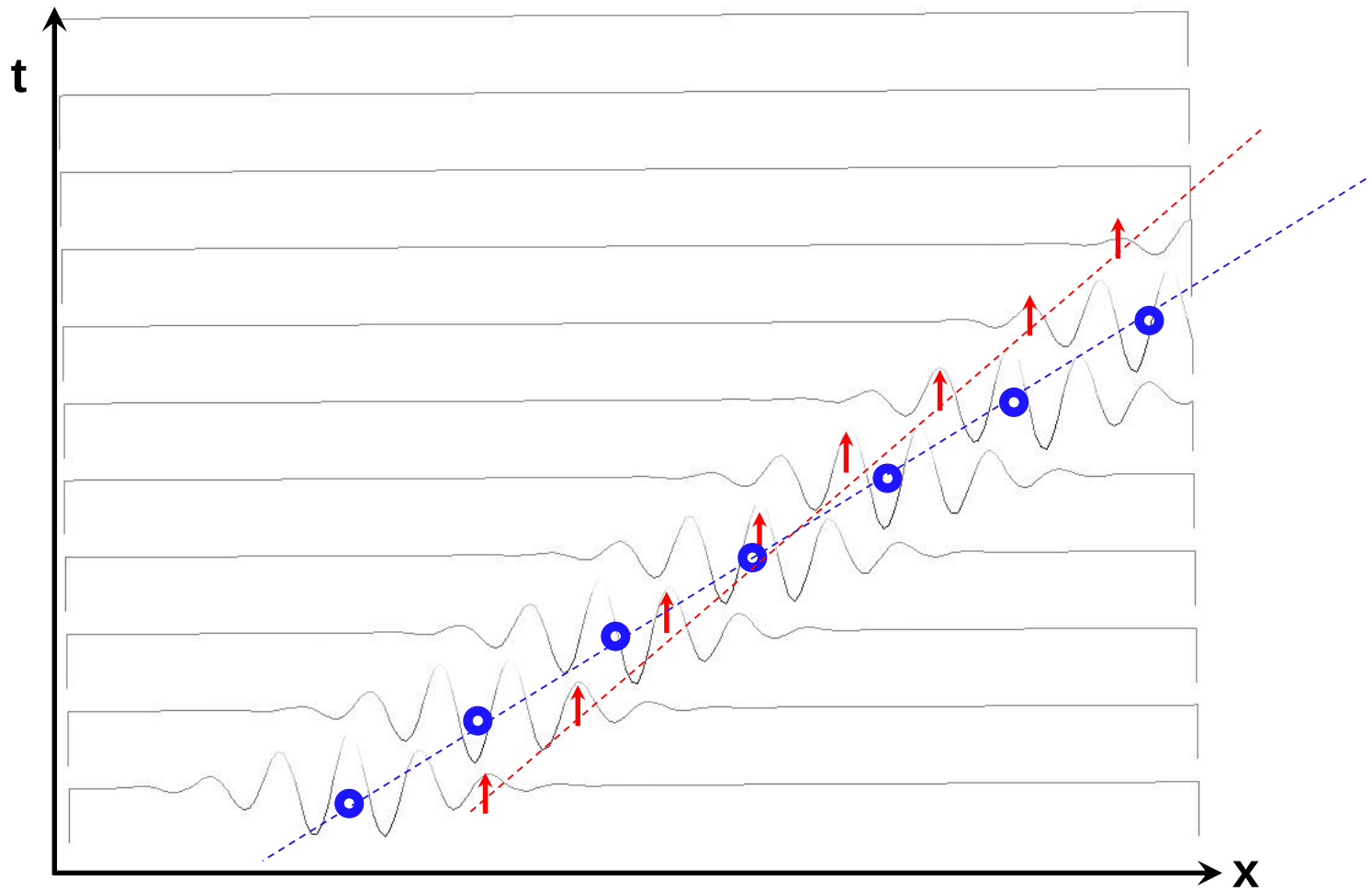


Diagramme spatio-temporel



$$v_g = \frac{3}{2} v_\phi$$

Spectral analysis

Fourier transform:

$$u(x, t) = \frac{1}{2} \int_0^{+\infty} \hat{u}(k) e^{i(kx - \omega(k)t)} dk + c.c.$$

$\hat{u}(k)$ is given by Fourier transform at time $t=0$

Carrier/enveloppe :

$$u(x, t) = \frac{1}{2} A(x, t) e^{i(k_0 x - \omega_0 t)} + c.c.$$

Enveloppe :

$$A(x, t) = \int_0^{\infty} \hat{u}(k) e^{i(k-k_0)x - i(\omega - \omega_0)t} dk.$$

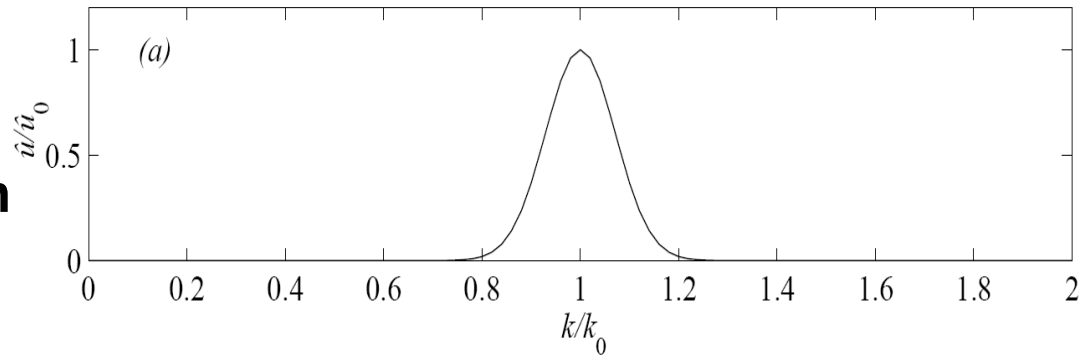
Spectral analysis

Gaussian spectrum: $\hat{u}(k) = u_0 e^{-\sigma^2(k-k_0)^2}$

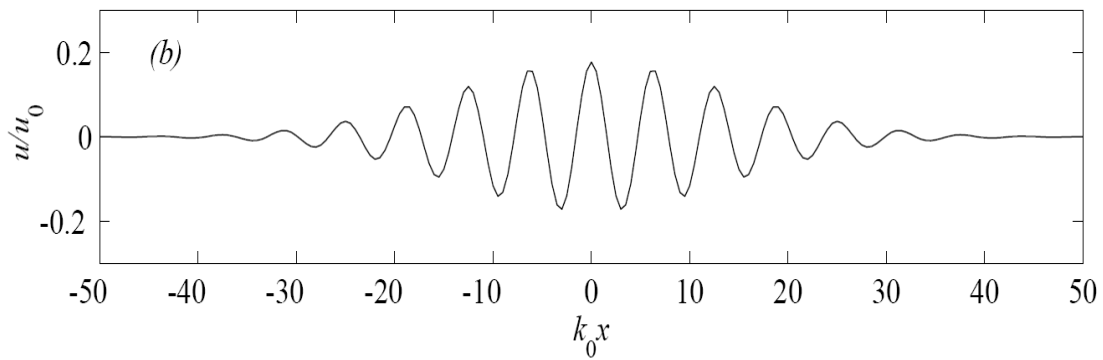
Initial envelope : $A(x, 0) = \frac{u_0 \sqrt{\pi}}{2\sigma} e^{-\frac{x^2}{4\sigma^2}}$

Gaussian spectrum

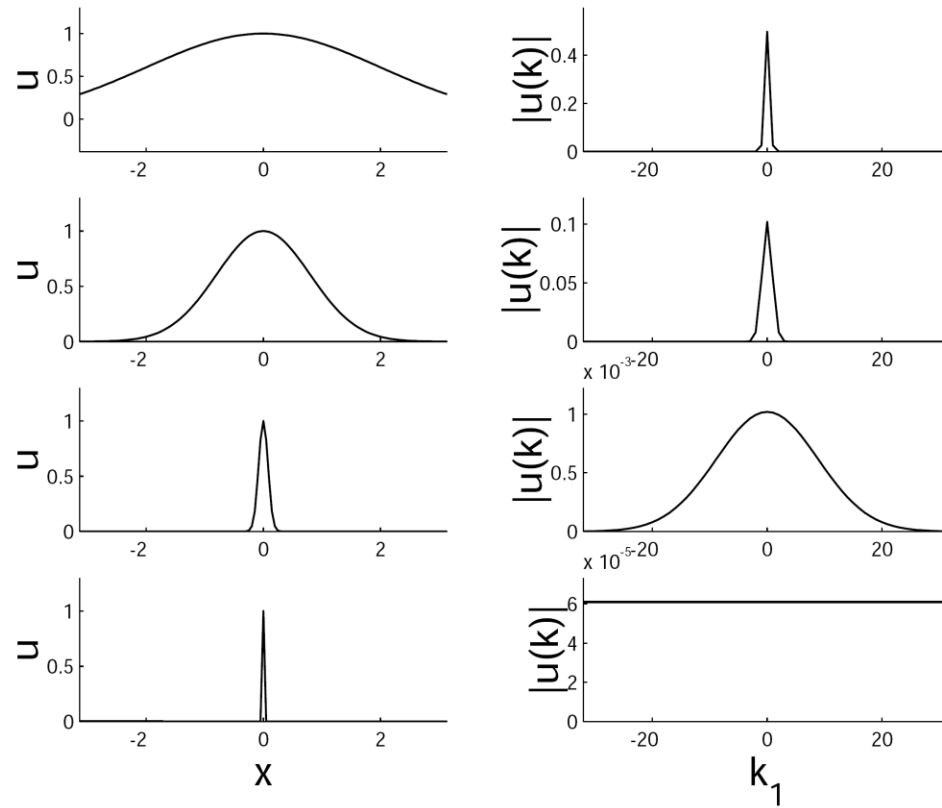
spectrum



wave

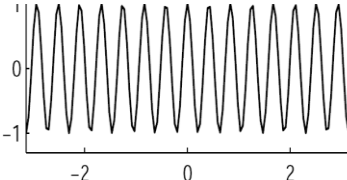
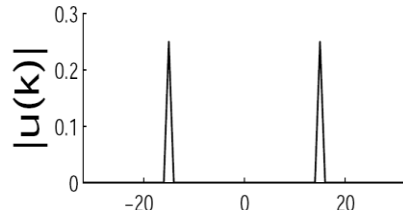
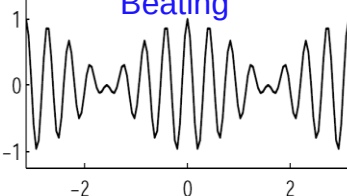
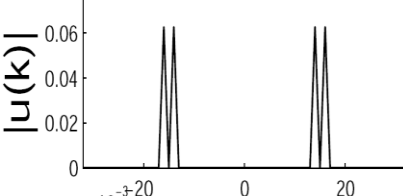
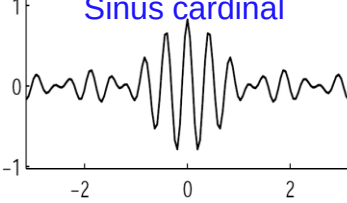
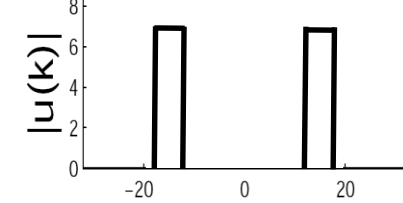
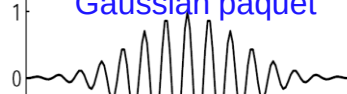


Paquets d'ondes gaussiens



$u_0(x)$	$\hat{u}_0(k_1)$
$\exp\left(-\frac{x^2}{2\sigma^2}\right)$	$\frac{\sigma}{\sqrt{2\pi}} \exp\left(-\frac{\sigma^2 k^2}{2}\right)$
$\delta(x)$	$\frac{1}{2\pi}$

Waves and spectrum

$2 \cos(k_0 x)$	Monochromatic wave  	$\delta(k_1 + k_0) + \delta(k_1 - k_0)$
$\cos[(k_0 - \kappa)x] + \cos[(k_0 + \kappa)x]$ $= 2 \cos(k_0 x) \cos(\kappa x)$	Beating  	$\delta(k_1 + k_0 + \kappa) + \delta(k_1 - k_0 - \kappa)$ $+ \delta(k_1 + k_0 - \kappa) + \delta(k_1 - k_0 + \kappa)$
$\int_{k_0 - \kappa}^{k_0 + \kappa} \cos(k_1 x) dk_1$ $= 4 \kappa \cos(k_0 x) \frac{\sin(\kappa x)}{\kappa x}$	Sinus cardinal  	$= 1 \text{ pour } k_1 \pm k_0 \leq \kappa$ $= 0 \text{ sinon}$
$2 \kappa \sqrt{2\pi} \cos(k_0 x) \exp\left(-\frac{\kappa^2 x^2}{2}\right)$	Gaussian paquet  	$\exp\left[-\frac{(k_1 + k_0)^2}{2 \kappa^2}\right] + \exp\left[-\frac{(k_1 - k_0)^2}{2 \kappa^2}\right]$

Spectral analysis

$$A(x, t) = \int_0^\infty \hat{u}(k) e^{i(k-k_0)x - i(\omega - \omega_0)t} dk.$$

Spectral analysis

$$A(x, t) = \int_0^\infty \hat{u}(k) e^{i(k-k_0)x - i(\omega - \omega_0)t} dk.$$

Definition of group velocity $\omega - \omega_0 = c_g(k - k_0), \quad c_g = \frac{\partial \omega}{\partial k}(k_0)$

Spectral analysis

$$A(x, t) = \int_0^{\infty} \hat{u}(k) e^{i(k-k_0)x - i(\omega - \omega_0)t} dk.$$

$$\hat{u}(k) = u_0 e^{-\sigma^2 (k-k_0)^2}$$

Spectral analysis

Gaussian spectrum: $\hat{u}(k) = u_0 e^{-\sigma^2(k-k_0)^2}$

Initial envelope : $A(x, 0) = \frac{u_0 \sqrt{\pi}}{2\sigma} e^{-\frac{x^2}{4\sigma^2}}$

Spectral analysis

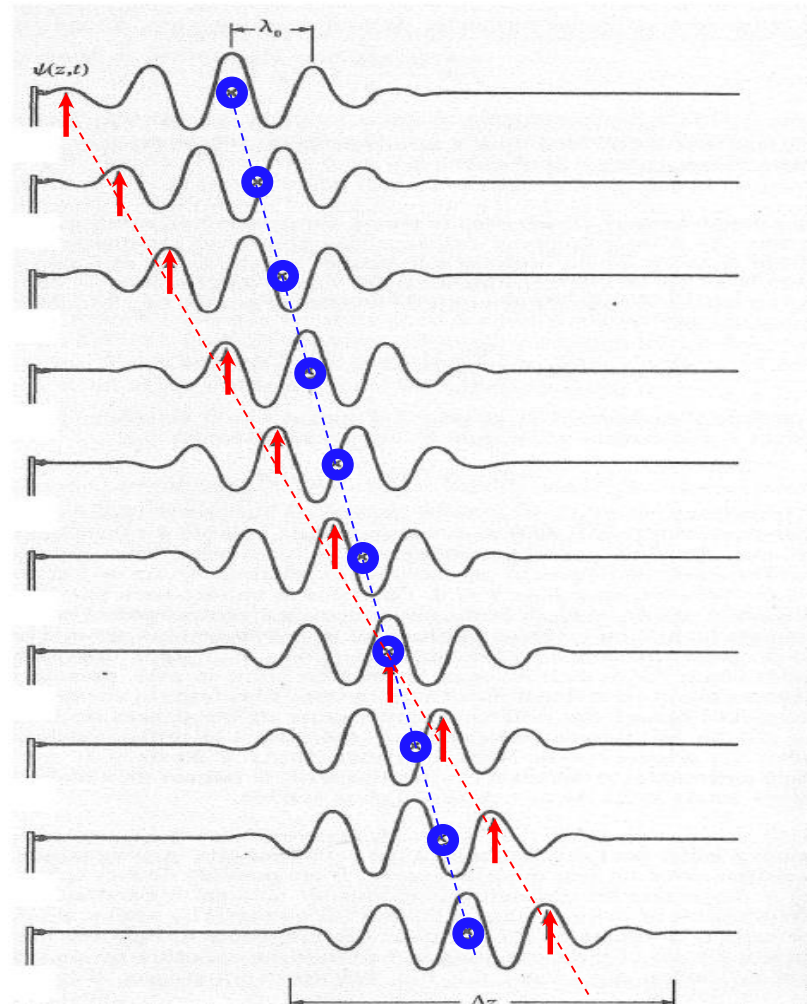
$$A(x, t) = \int_0^\infty \hat{u}(k) e^{i(k-k_0)x - i(\omega - \omega_0)t} dk.$$

Definition of group velocity $\omega - \omega_0 = c_g(k - k_0), \quad c_g = \frac{\partial \omega}{\partial k}(k_0)$

$$\hat{u}(k) = u_0 e^{-\sigma^2(k-k_0)^2}$$

$$A(x, t) = \frac{u_0 \sqrt{\pi}}{2\sigma} e^{-\frac{(x - c_g t)^2}{4\sigma^2}}$$

Group velocity



Wave packet

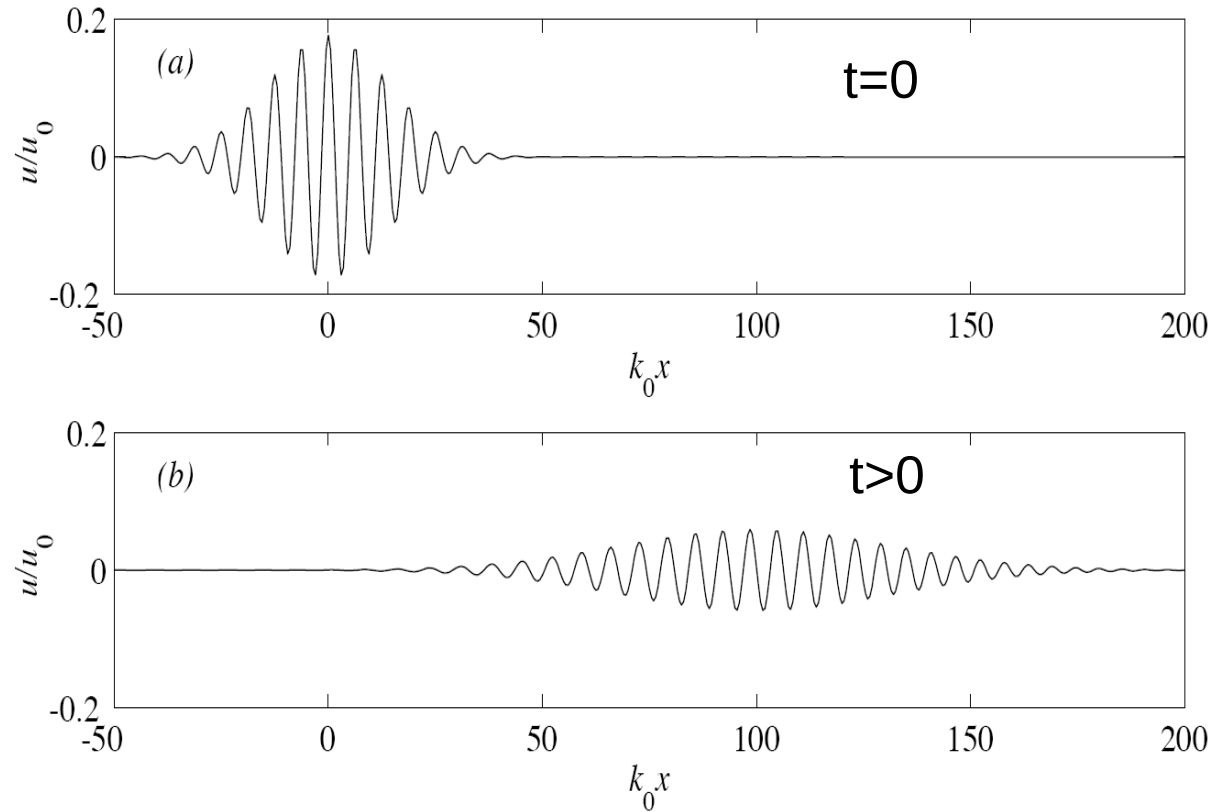
Spectral analysis

Higher order
development

$$\omega - \omega_0 = c_g(k - k_0) + \frac{\omega_0''}{2}(k - k_0)^2$$
$$c_g = \frac{\partial \omega}{\partial k}(k_0), \quad \omega_0'' = \frac{\partial^2 \omega}{\partial k^2}(k_0)$$

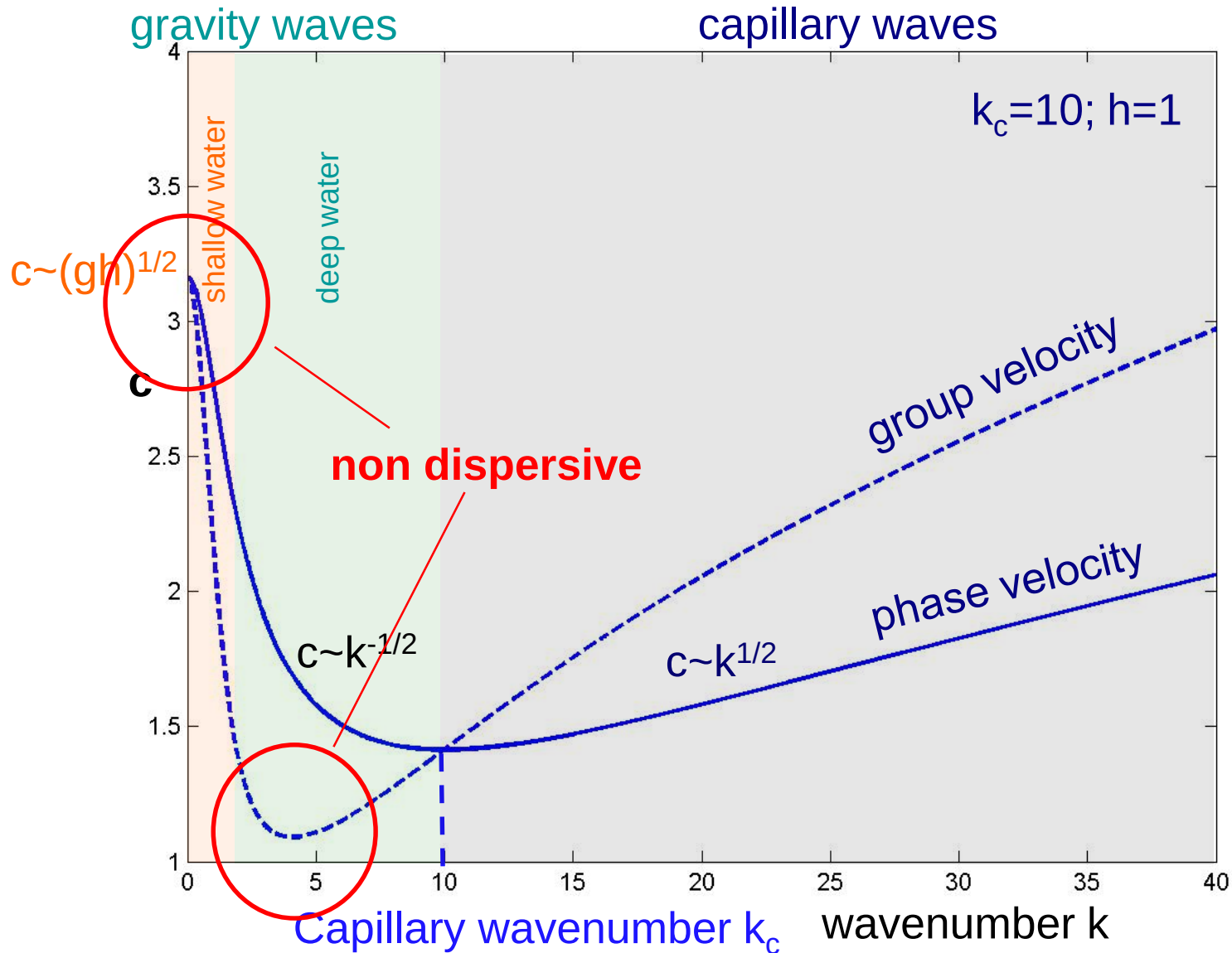
$$A(x, t) = \frac{u_0}{2} \sqrt{\frac{\pi}{\sigma^2 + \frac{1}{2}i\omega_0''t}} \exp\left(-\frac{(x - c_g t)^2}{4(\sigma^2 + \frac{1}{2}i\omega_0''t)}\right)$$

Wave packet dispersion

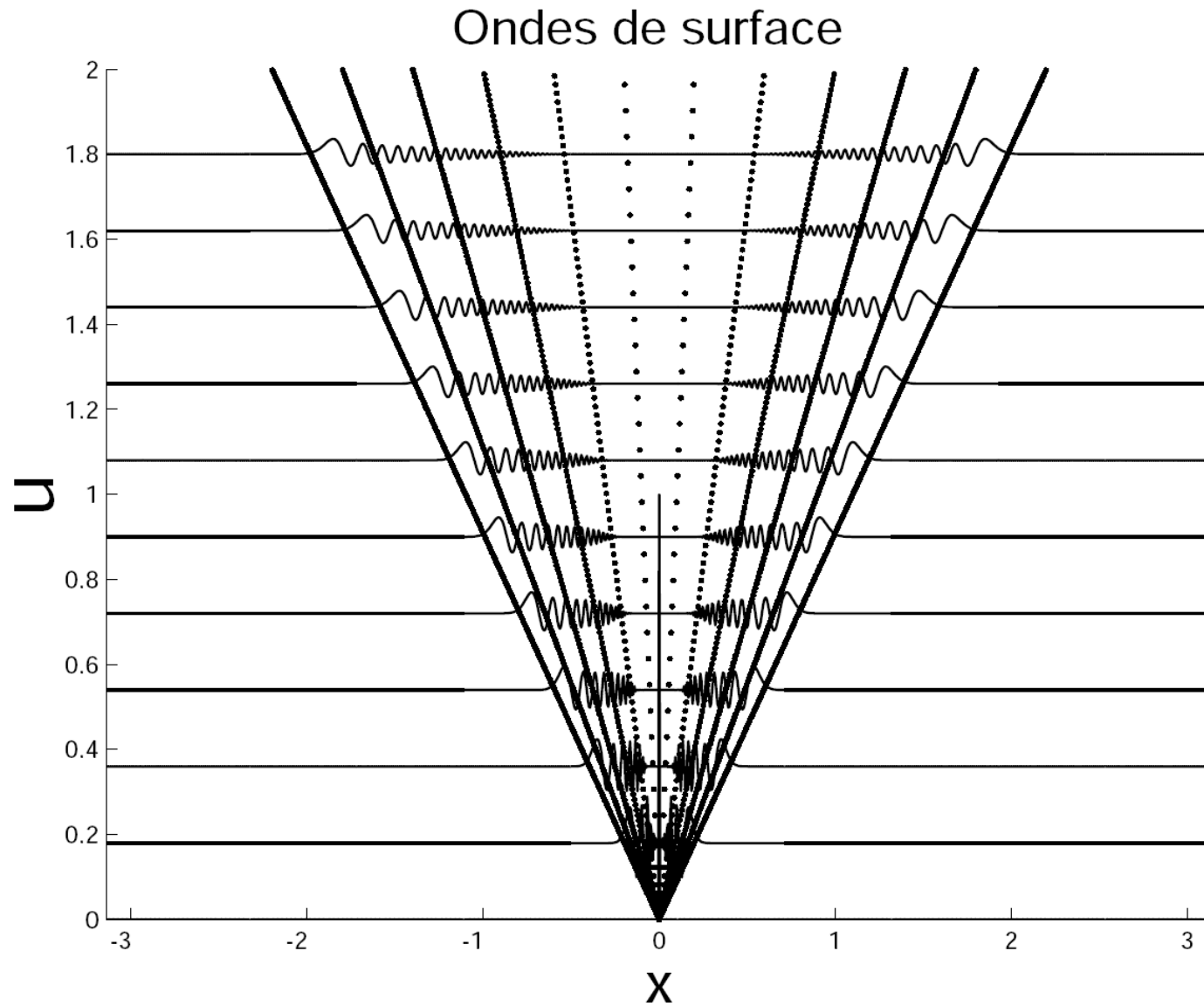


Onde correspondant à l'enveloppe $\hat{u} = \hat{u}_0 \exp(i k_0 x - i \omega_0 t)$ pour $\sigma^{-1} k_0 = 0,1$ et $\omega_0'' = 4c_g/k_0$: (a), instant initial $t = 0$; (b), $c_g t = 100/k_0$.

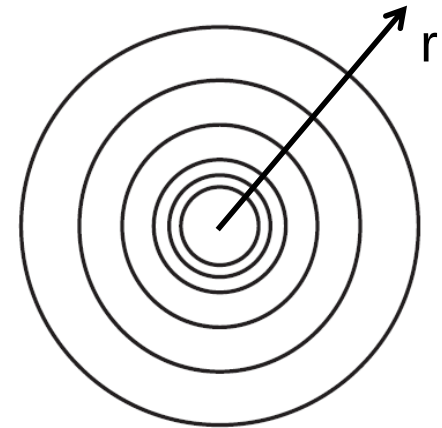
Dispersion relation



Dispersion



Dispersion



Waves with k reach r at time $t=r/v(k)$

For deep gravity waves: $v_{\text{deep/gravity}} \sim \pm \frac{1}{2} \sqrt{\frac{g}{k}}$

$$k = gt^2/4r^2$$

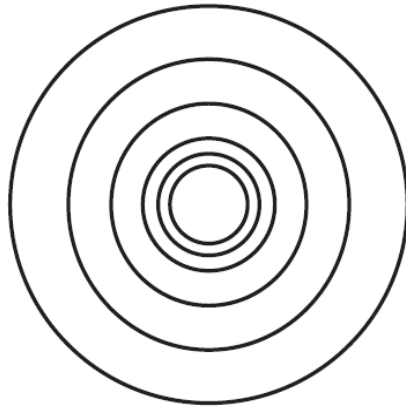
Since $\omega_{\text{deep/gravity}} \sim \pm \sqrt{gk}$

$$\omega = gt/2r$$

⇒ The frequency increases with time

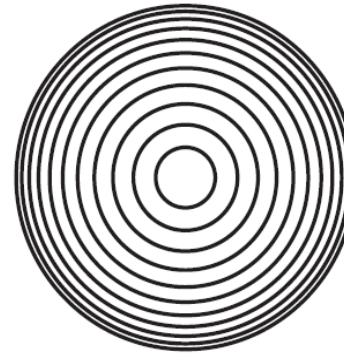
“Ronds dans l’eau”

Gravity
waves

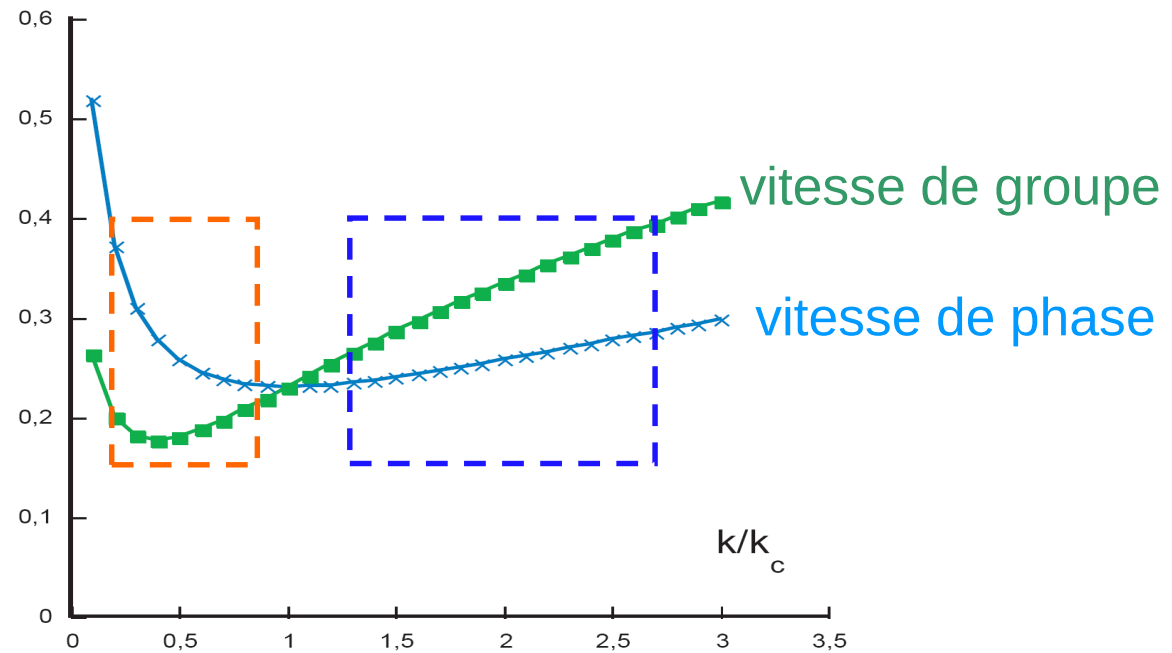


stone $> l_c$

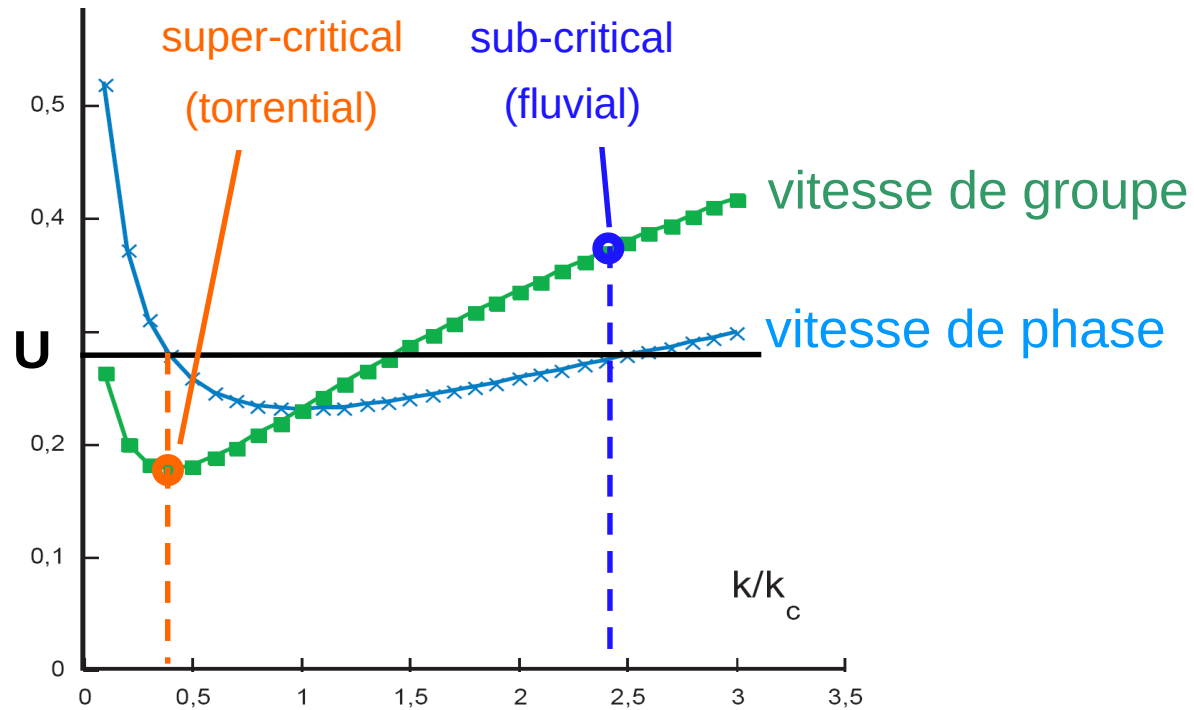
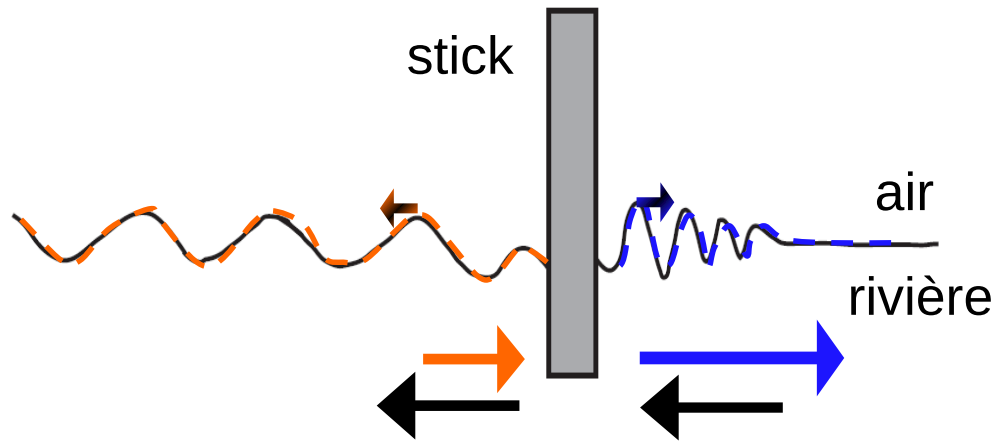
Capillary
waves



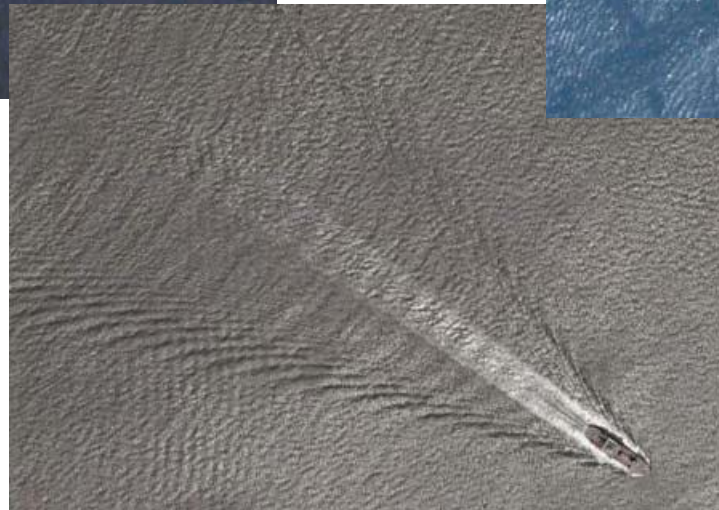
water drop $< l_c$



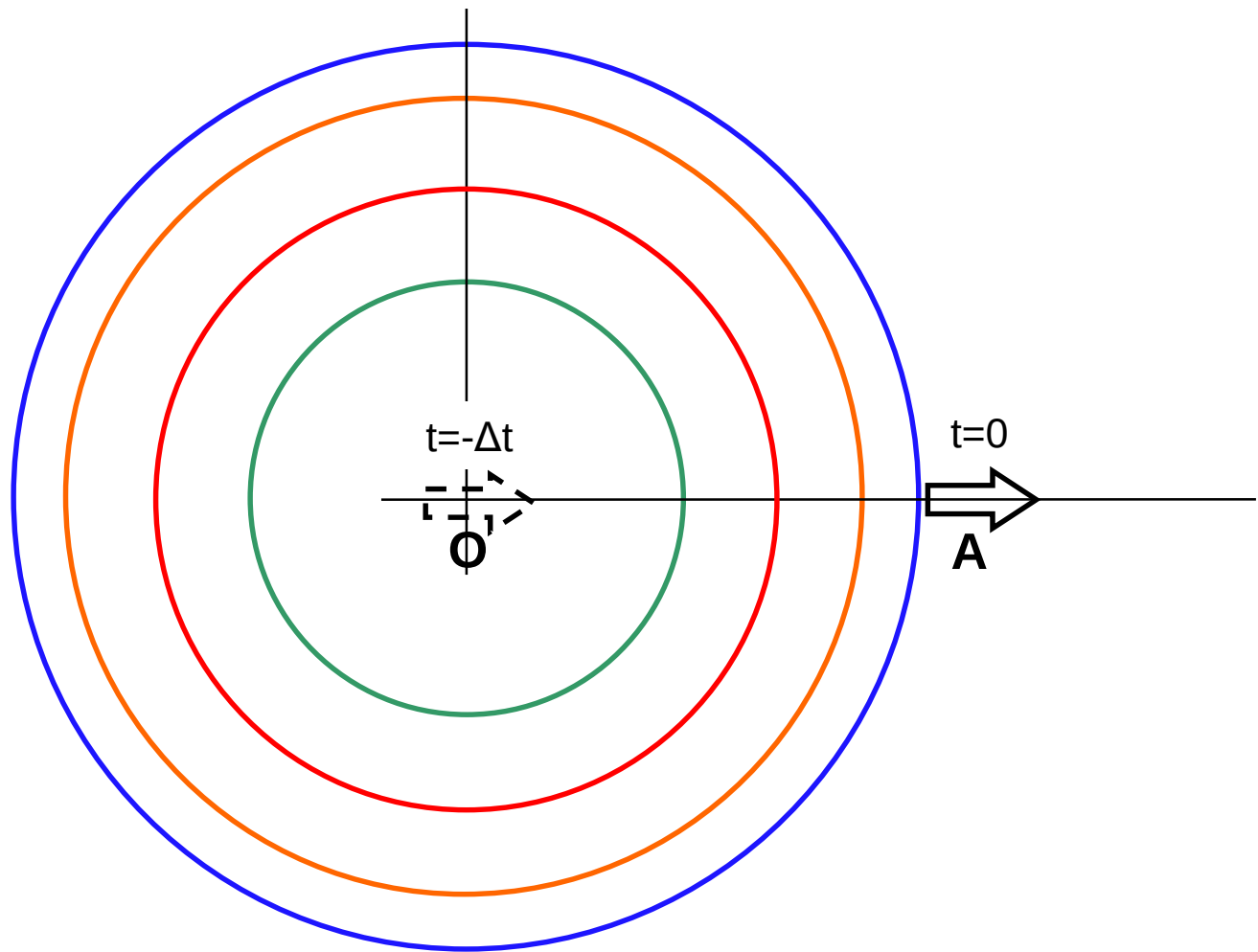
Waves created by an obstacle in a river



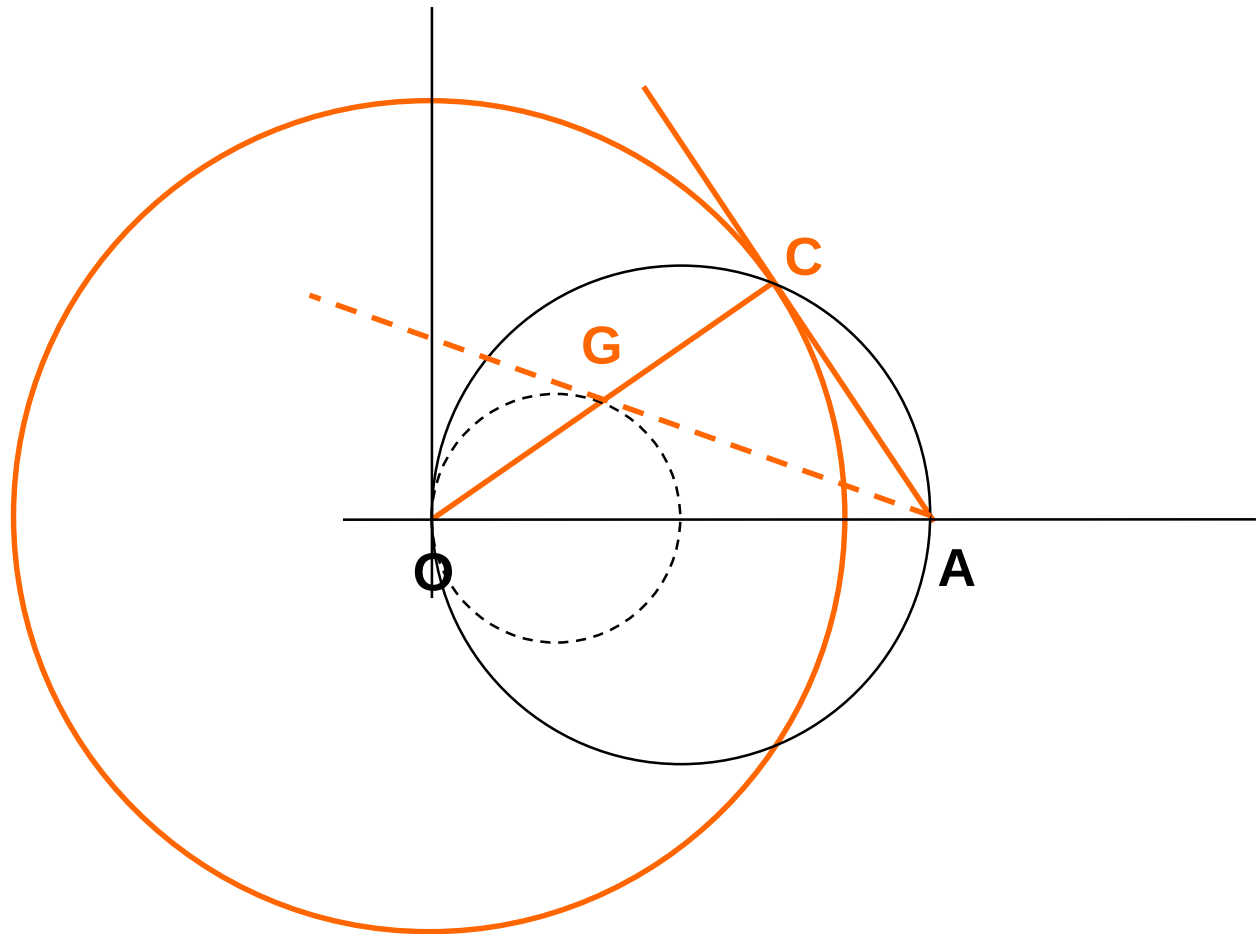
Kelvin wakes



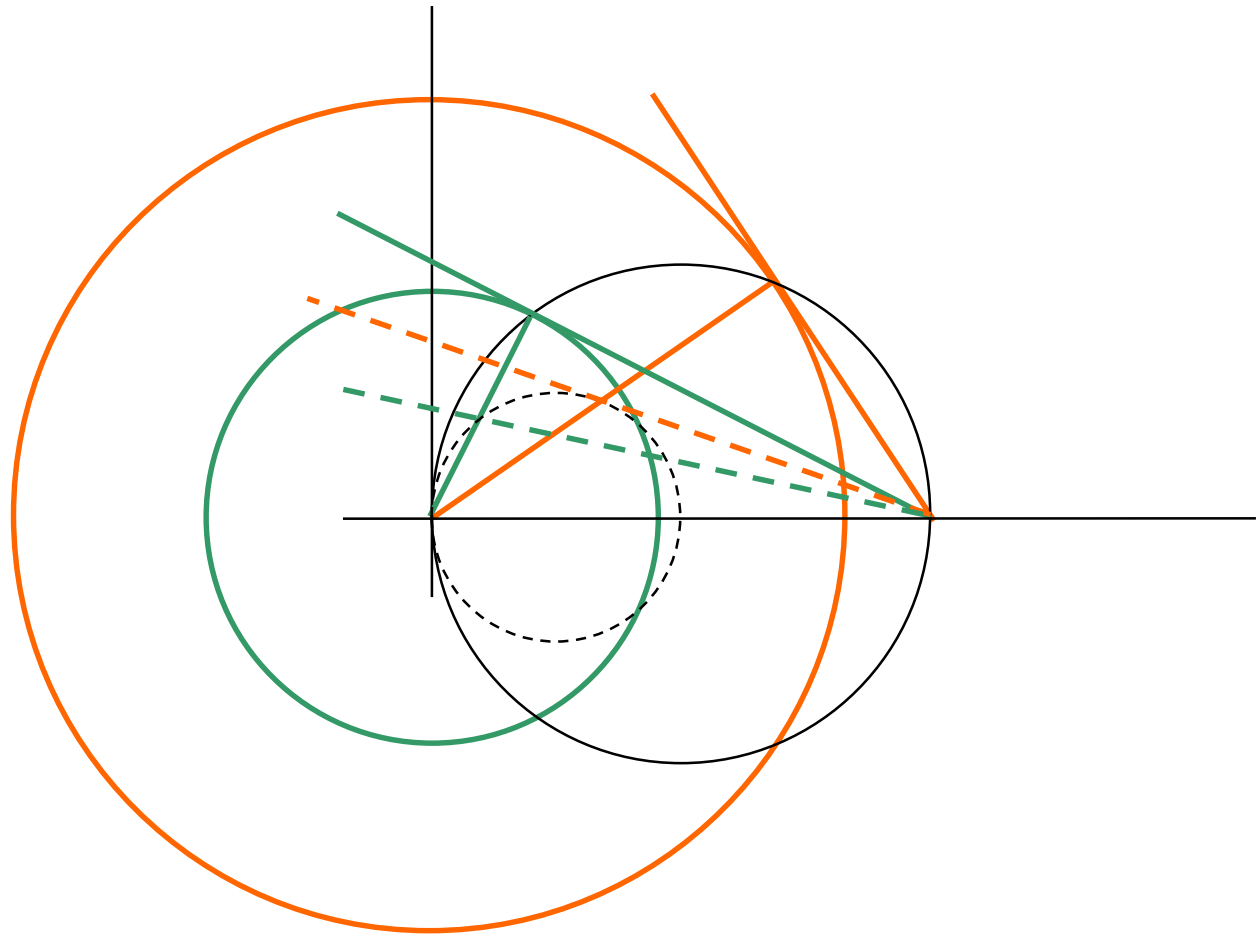
Kelvin wake



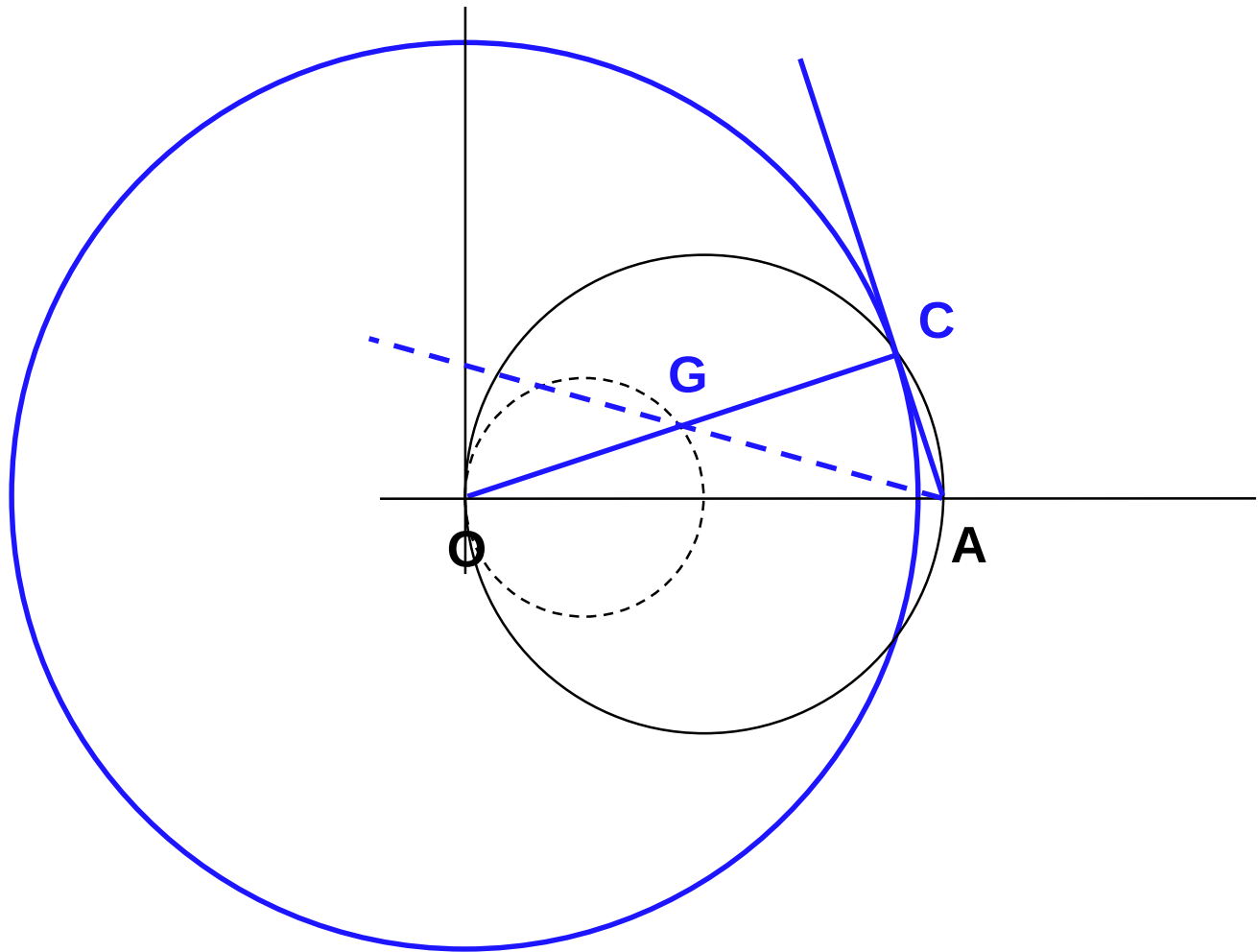
Waves created by a ship



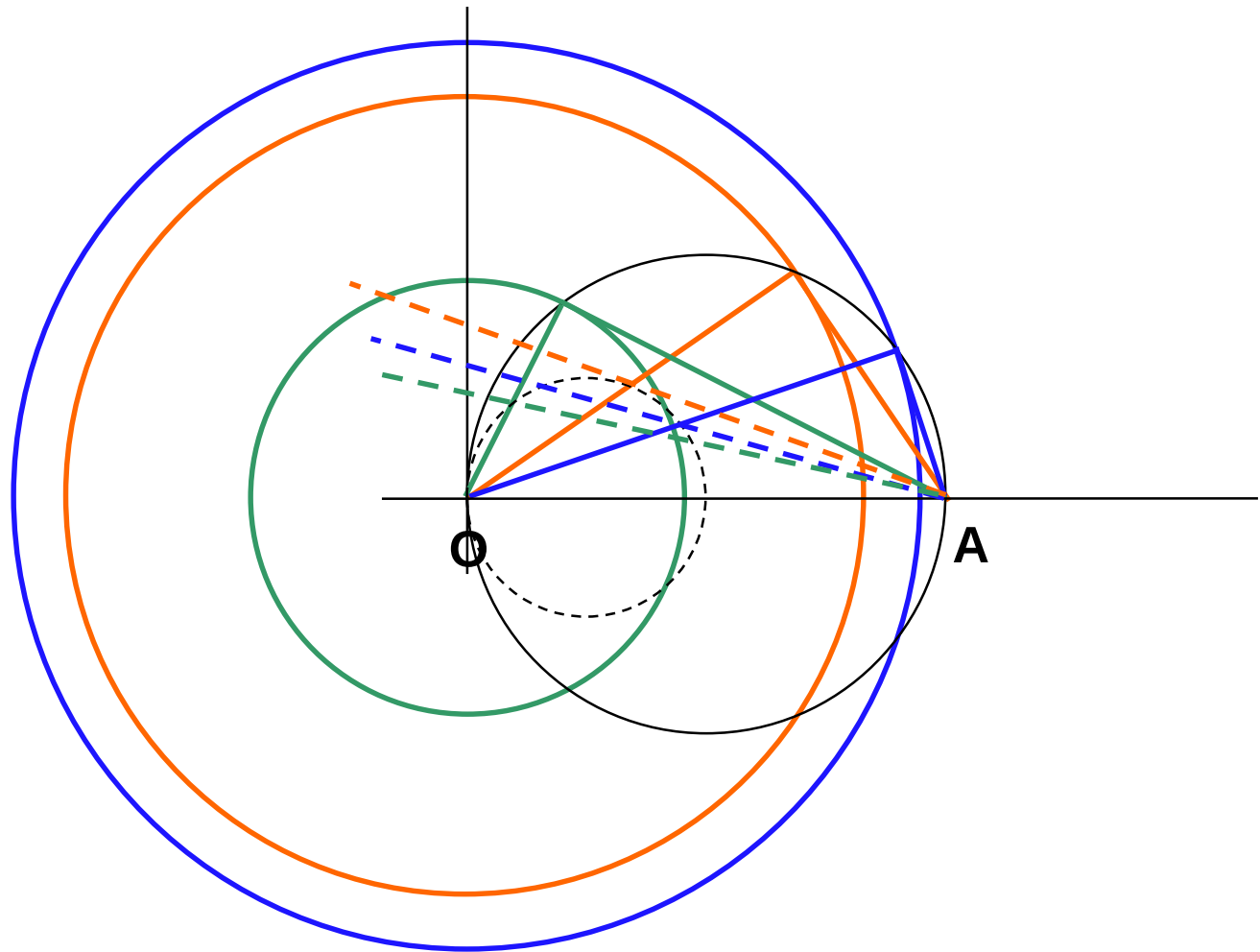
Waves created by a ship



Waves created by a ship



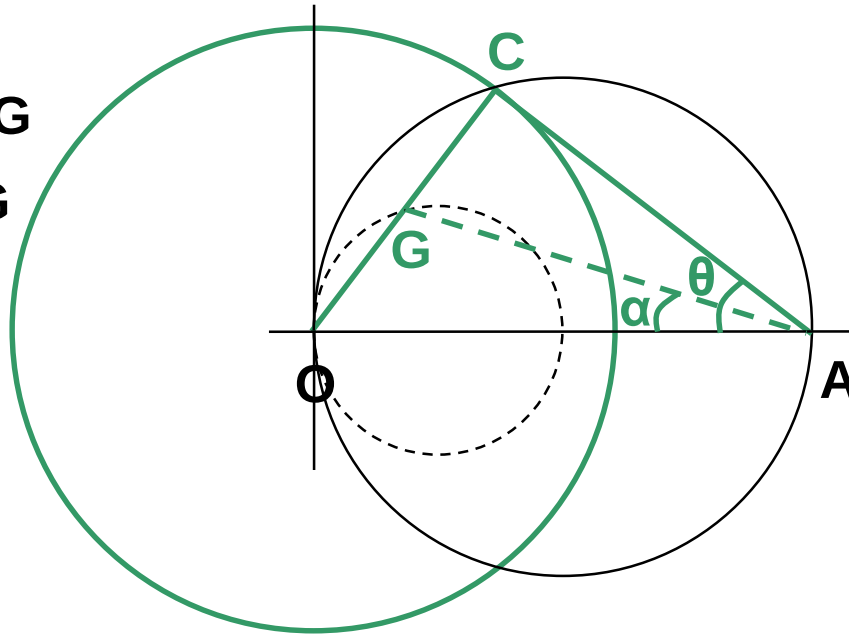
Waves created by a ship



Waves created by a ship

$$\sin(\alpha)/OG = \cos(\theta)/AG$$

$$\sin(\theta - \alpha)AG = GC = OG$$

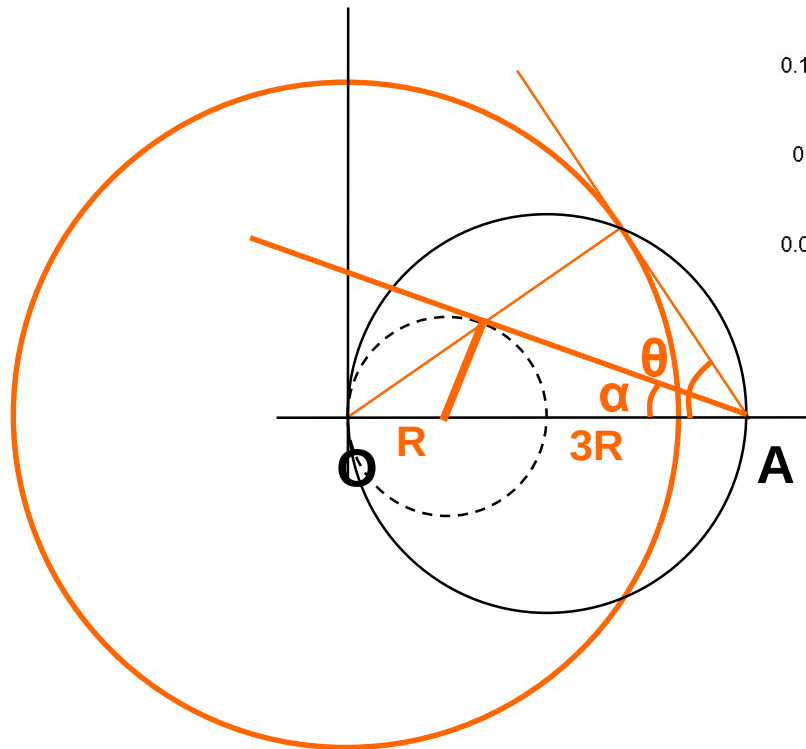


$$\Rightarrow \sin(\alpha) = \cos(\theta) \sin(\theta - \alpha)$$

$$\Rightarrow \sin(\alpha) = \cos(\theta) (\sin(\theta) \cos(\alpha) + \cos(\theta) \sin(\alpha))$$

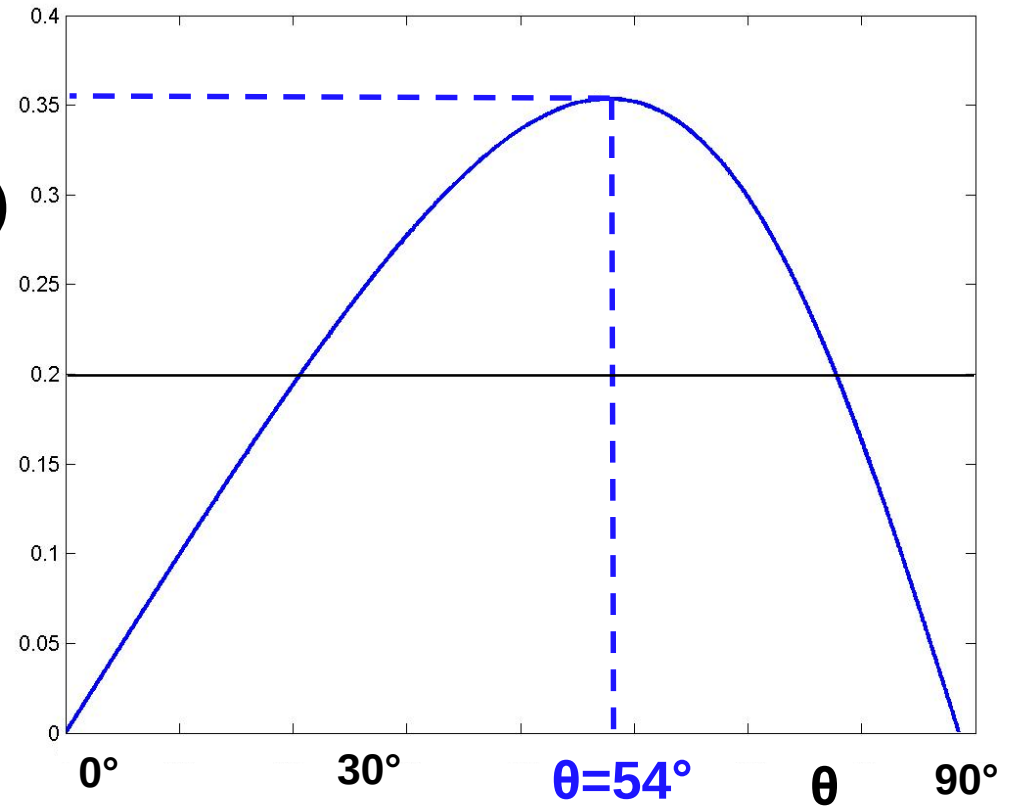
$$\Rightarrow \tan(\alpha) = \cos(\theta) \sin(\theta) / (1 + \cos^2(\theta))$$

Waves created by a ship

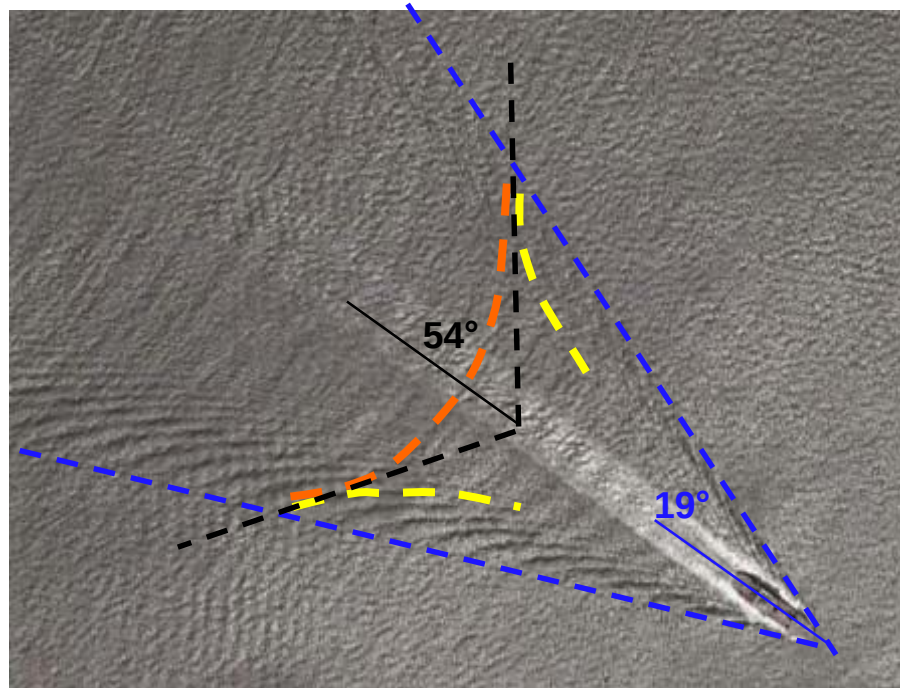


$$\alpha = 19^\circ$$

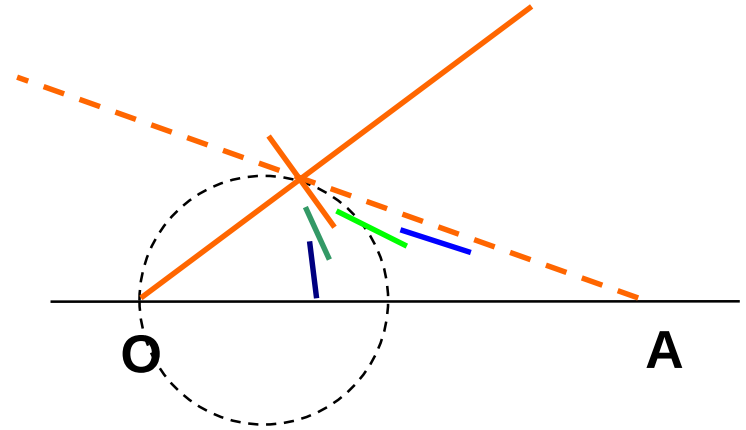
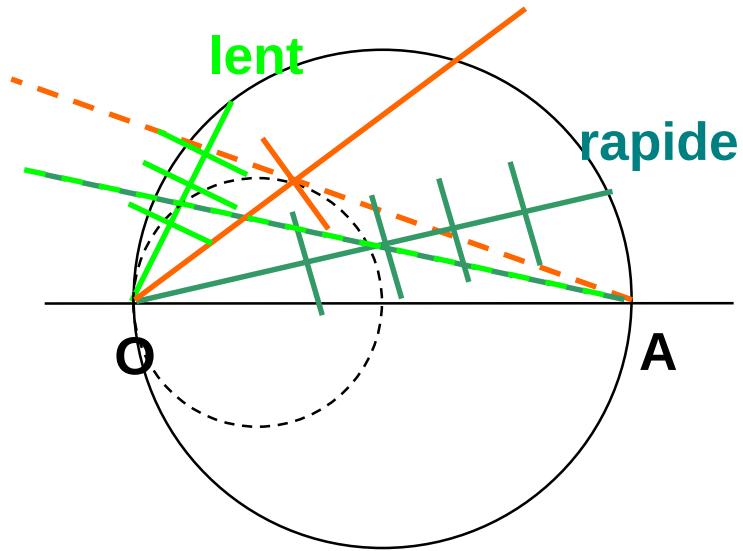
$$\tan(\alpha)$$



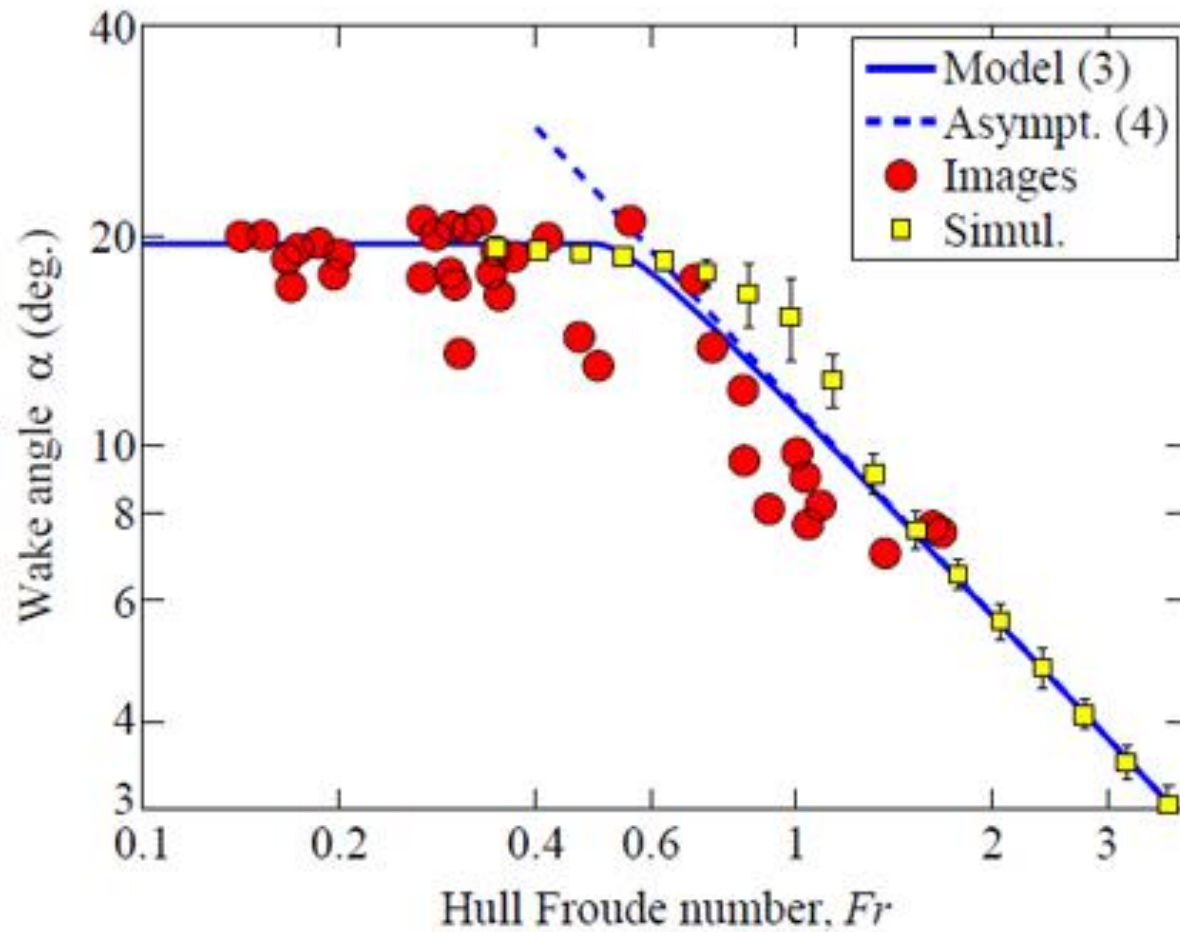
Waves created by a ship



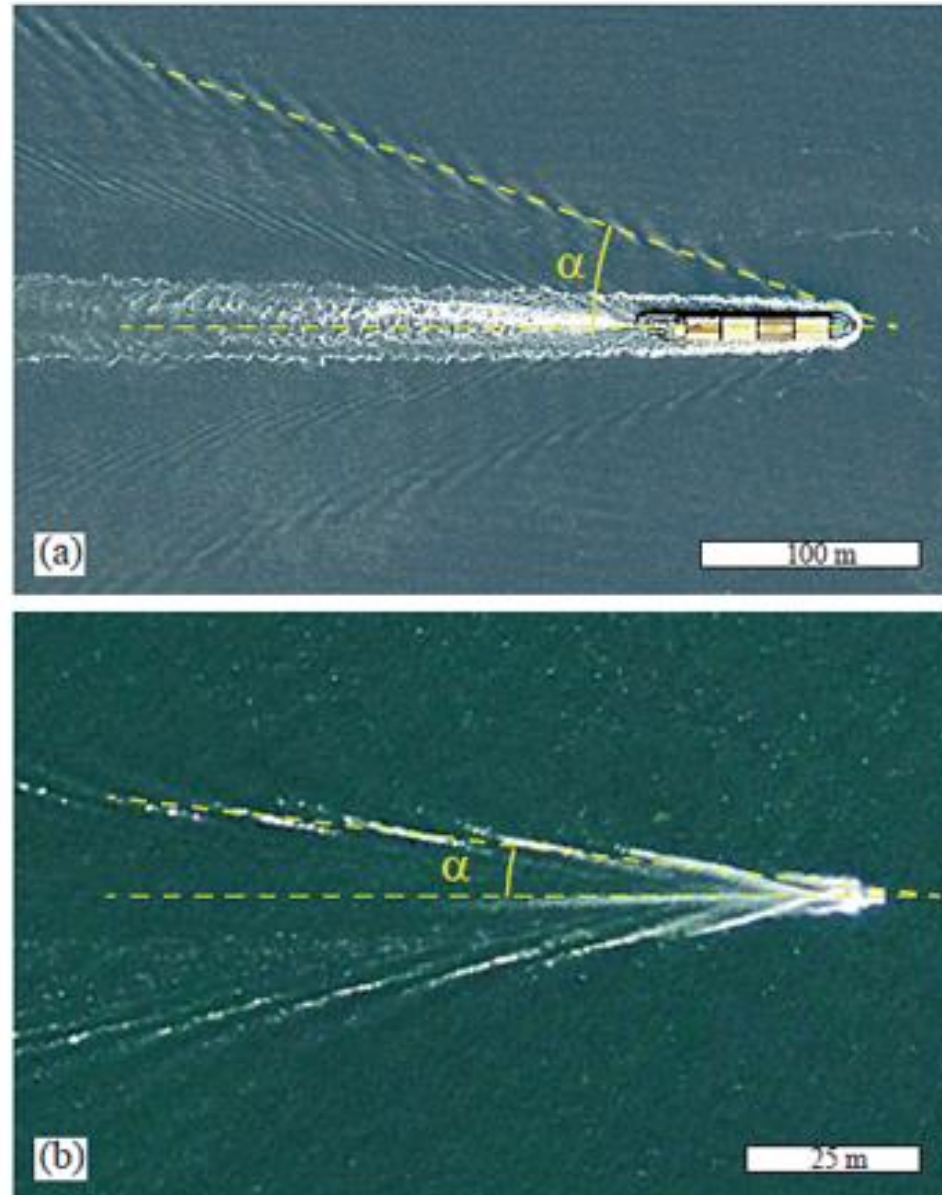
Waves created by a ship



Careful google-earth data analysis by Moisy and Rabaud (2013)



Careful google-earth data analysis by Moisy and Rabaud (2013)



Froude number

$$Fr = \frac{U}{c} \quad \text{c: celerity of wave}$$

$$Fr = \frac{U}{\sqrt{gd}} \quad \text{d: size of ship}$$

$$Fr = \frac{U}{\sqrt{gl}} \quad \text{shallow water}$$

Froude decomposition

$$C_T(F, Re) = C_D(Re) + C_W(F, Re)$$

Total drag Drag in Wave drag
 absence of
 free surface

Froude hypothesis

$$C_T(F, Re) = C_D(Re) + C_W(F, Re)$$

Total drag Drag in Wave drag
 absence of
 free surface

Alternative expression

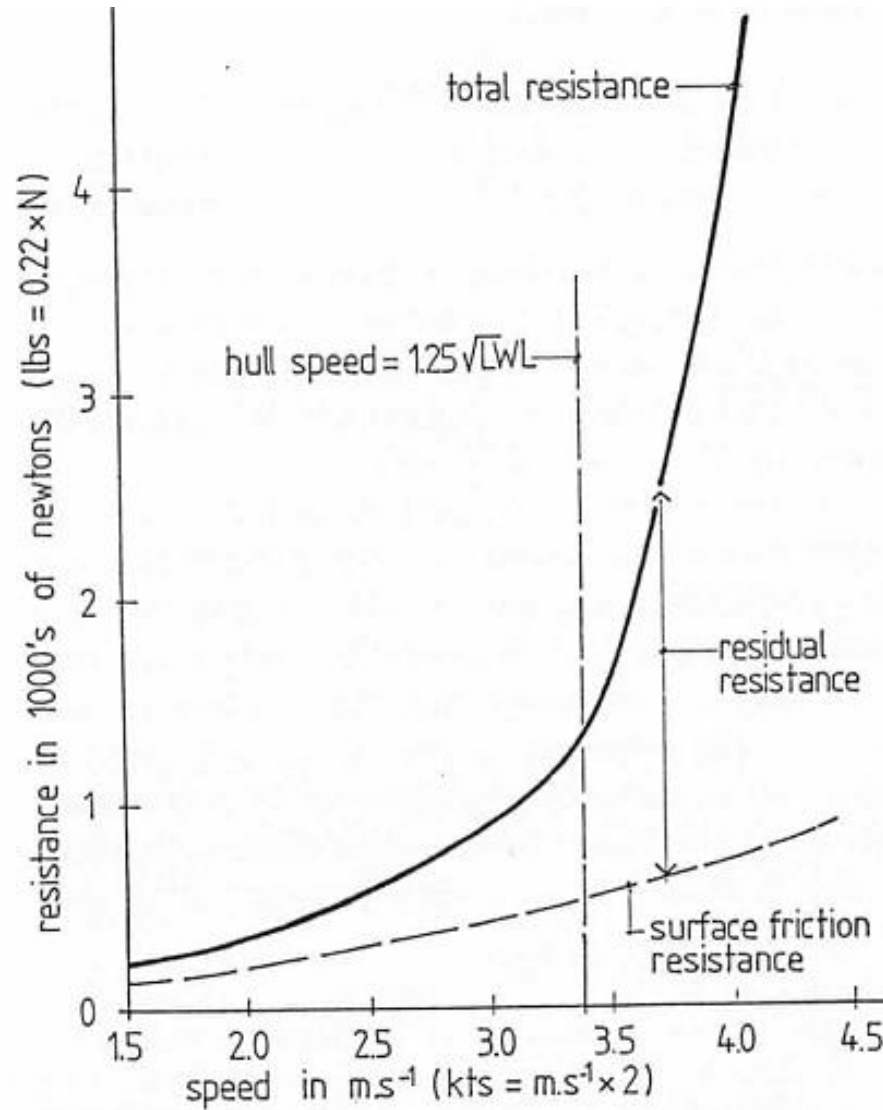
$$C_T(F, Re) = C_F(Re) + C_P(F, Re)$$

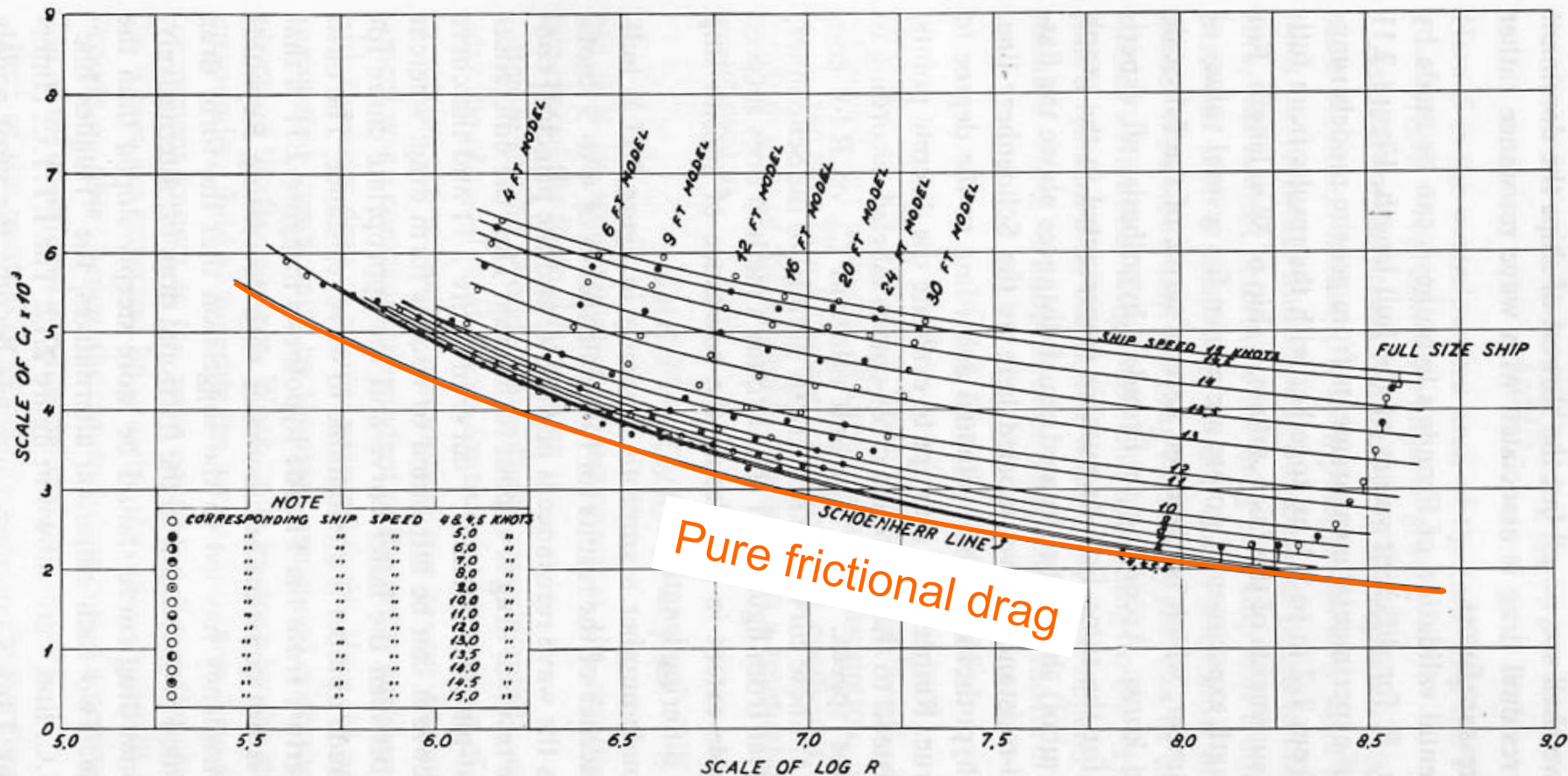
Total drag Friction drag Pressure drag

$$C_P(F, Re) \approx C_W(F)$$

Pressure drag Wave drag
 (Residual drag)

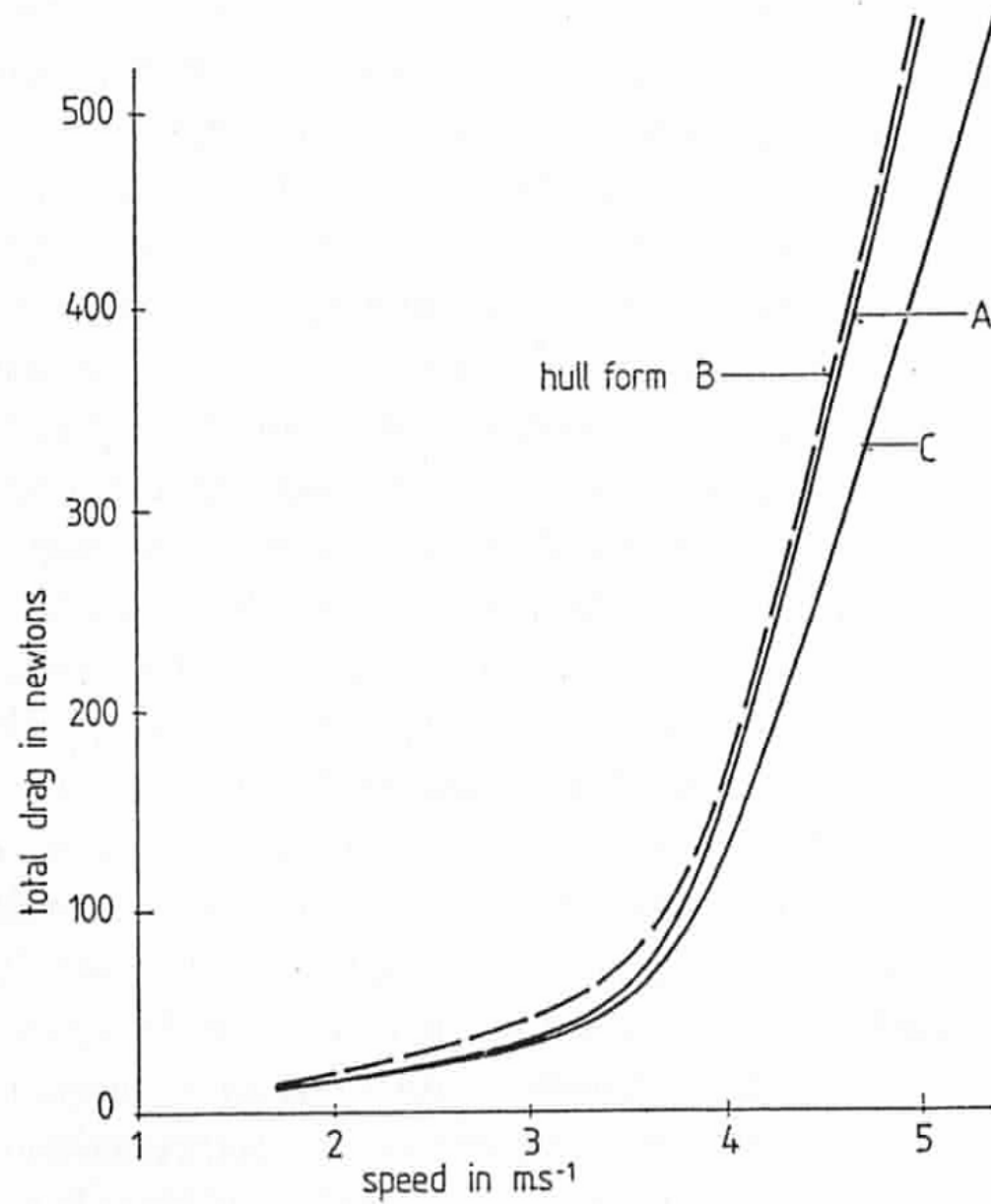
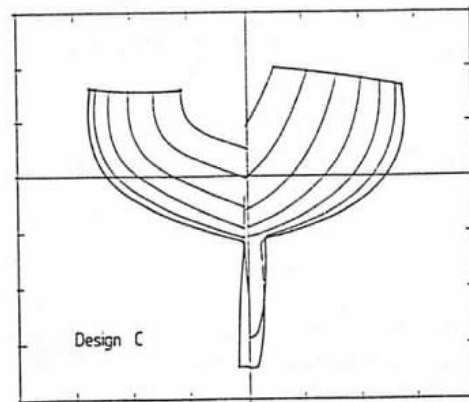
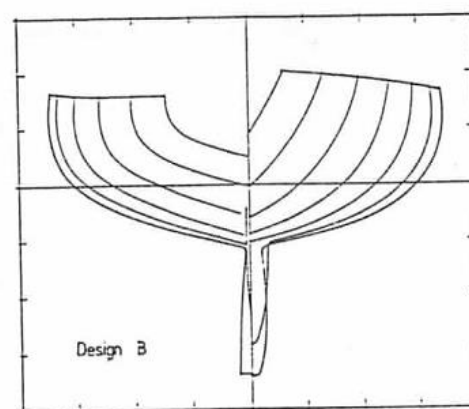
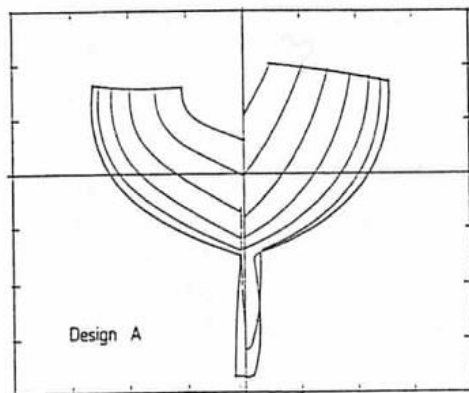
Friction resistance and residual resistance





2.11

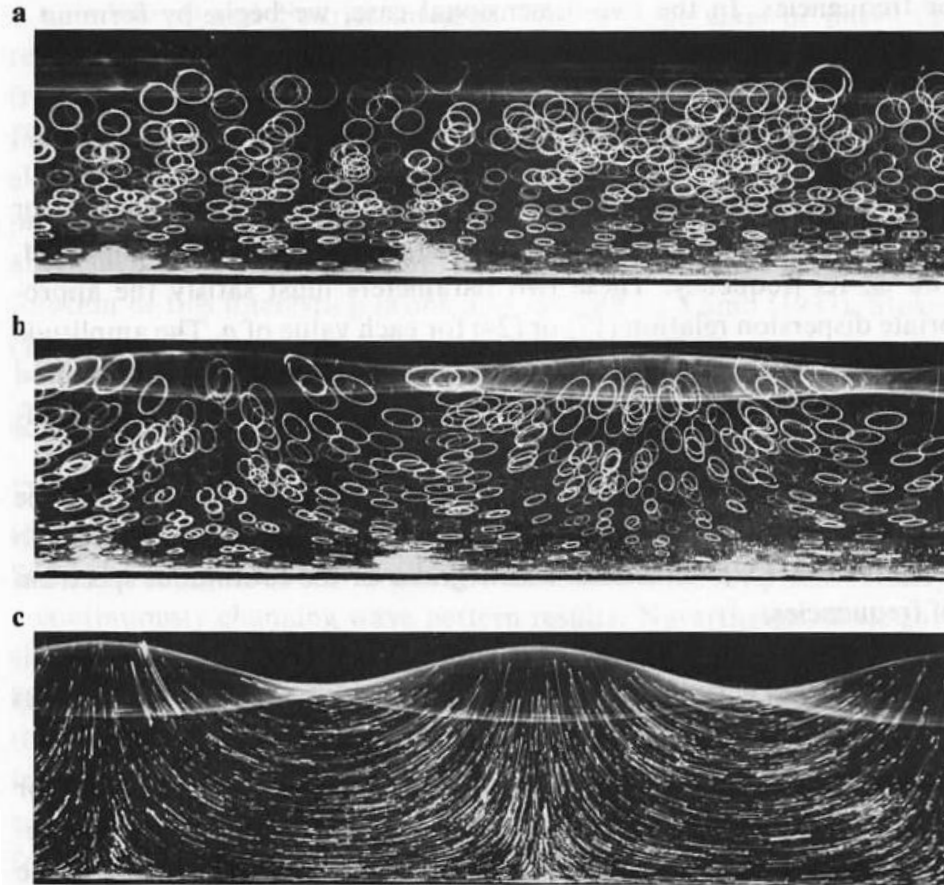
Total drag coefficients of the *Lucy Ashton* and several geosim models of the same vessel (from Troost and Zakay 1951). The faired curves represent constant values of the Froude number and, if Froude's hypothesis were strictly valid, these would be parallel with spacing independent of the Reynolds number. Note that, even for this small full-scale vessel (58 m long), there is a large gap between the largest model results and the full-scale results.



Standing waves

$$\begin{aligned}\eta &= A \cos(kx - \omega t) + A \cos(kx + \omega t) \\ &= 2A \cos kx \cos \omega t,\end{aligned}$$

Reflected waves



6.7

Particle trajectories in a plane progressive wave (a), a partial reflected wave (b), and a standing wave (c). These correspond respectively to a reflection coefficient of 0, 0.38, and 1.0 in equation (56). Note that the reflection coefficient can be measured from the maximum and minimum of the envelope, using (56). These photographs are based on time exposures, and are reproduced from a more extensive series of observations made by Ruellan and Wallet (1950).

Reflected waves

$$\eta = A \operatorname{Re}[e^{-ikx+i\omega t} + R e^{ikx+i\omega t}]$$

R: complex reflection coefficient

$$\eta = A \operatorname{Re}[e^{-ikx+i\omega t}(1 + R e^{2ikx})]$$

Wave energy

$$KE + PE = \rho \iiint_{\mathcal{V}} \left(\frac{1}{2} V^2 + gy \right) d\mathcal{V}$$

$$E = \rho \int_{-h}^{\eta} \left(\frac{1}{2} V^2 + gy \right) dy = \frac{1}{2} \rho \int_{-h}^{\eta} V^2 dy + \frac{1}{2} \rho g \left(\eta^2 - \cancel{h^2} \right).$$

Neglect potential energy of fluid at rest

Wave energy

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Neglect potential energy of fluid at rest

$$E = \frac{\rho \omega^2 A^2}{4k} e^{2k\eta} + \frac{1}{2} \rho g \eta^2.$$

Wave energy

$$KE + PE = \rho \iiint_{\mathcal{V}} \left(\frac{1}{2} V^2 + gy \right) d\mathcal{V}$$

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Neglect potential energy of fluid at rest

$$E = \frac{\rho \omega^2 A^2}{4k} e^{2k\eta} + \frac{1}{2} \rho g \eta^2.$$

$$E = \frac{1}{4} \rho g A^2 + \frac{1}{2} \rho g A^2 \cos^2(kx - \omega t)$$

Wave energy

$$KE + PE = \rho \iiint_{\mathcal{V}} \left(\frac{1}{2} V^2 + gy \right) d\mathcal{V}$$

$$E = \rho \int_{-h}^{\eta} \left(\frac{1}{2} V^2 + gy \right) dy = \frac{1}{2} \rho \int_{-h}^{\eta} V^2 dy + \frac{1}{2} \rho g \left(\eta^2 - \cancel{h^2} \right).$$

Neglect potential energy of fluid at rest

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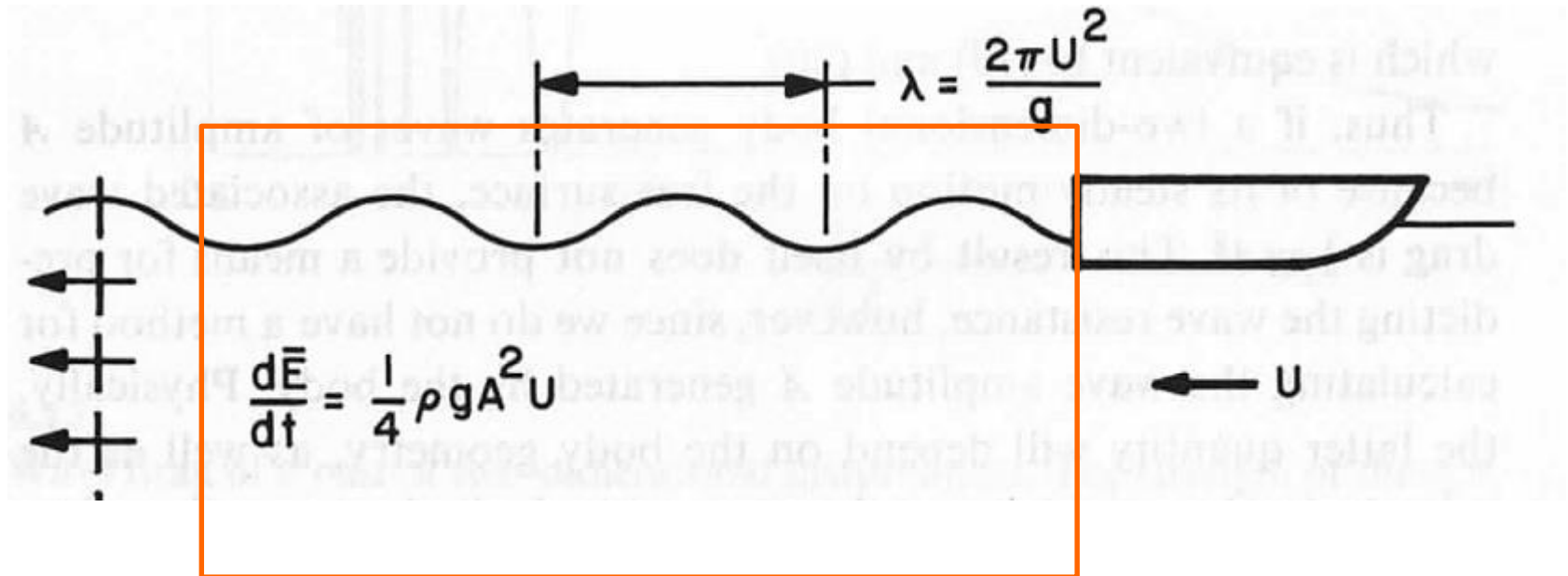
On average:

$$\bar{E} = \frac{1}{2} \rho g A^2$$

Wave energy flux

One can show that the energy travels at the group velocity...

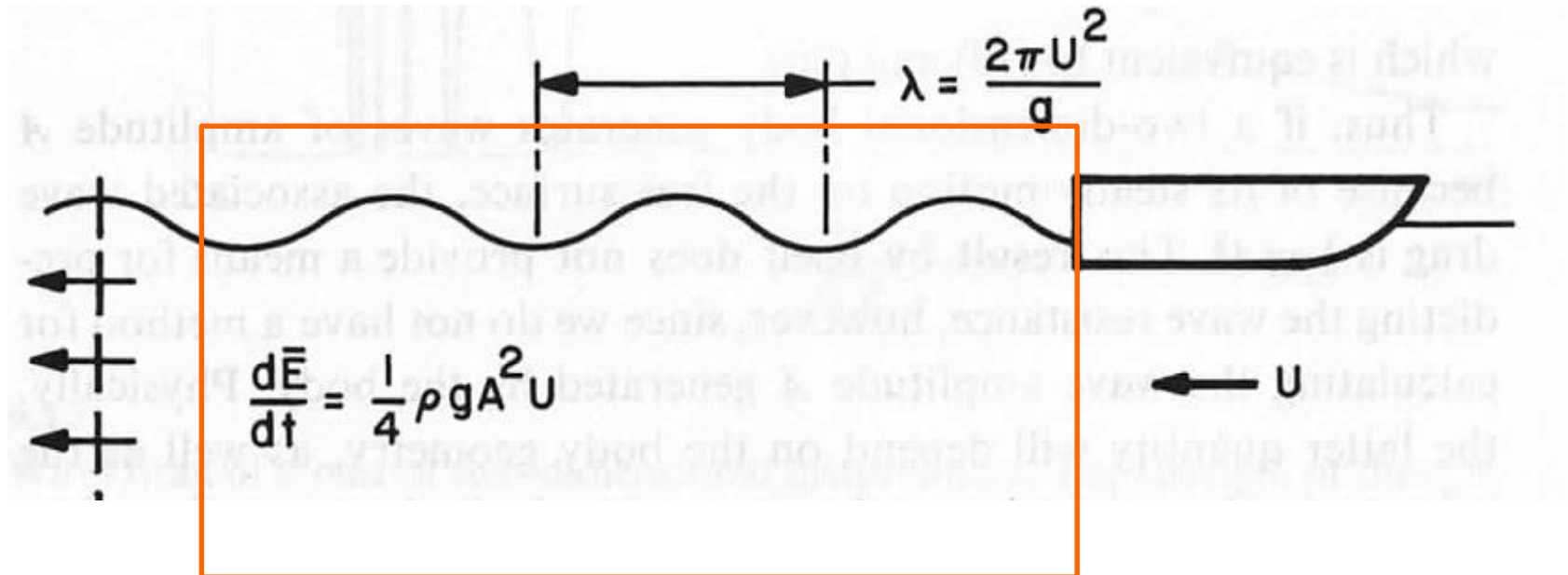
Energy balance (in ship's frame)



Energy input : -work of drag= -D.U

Energy output: (Vg-U)E

Energy balance (in ship's frame)



Energy input : -work of drag= -D.U

Energy output: (Vg-U)E

Deep water: Vg=U/2

$$D=E/2$$

$$D = \frac{1}{4} \rho g A^2$$

Wave production from bow

displacement

$$\eta = a \cos(kx + \varepsilon)$$

dispersion
relation

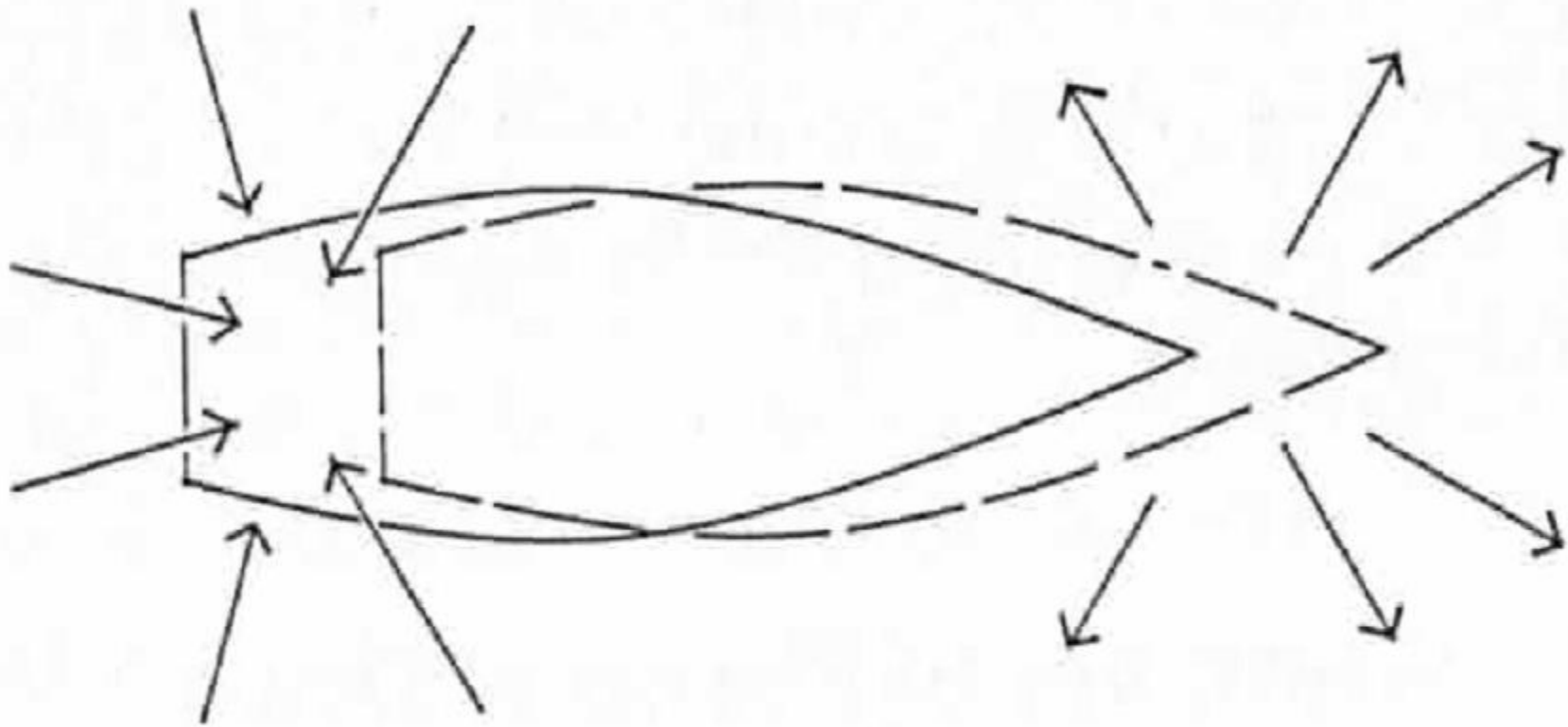
$$k = g/U^2$$

wave amplitude

$$a$$

wave energy

$$\frac{1}{4} \rho g a^2$$

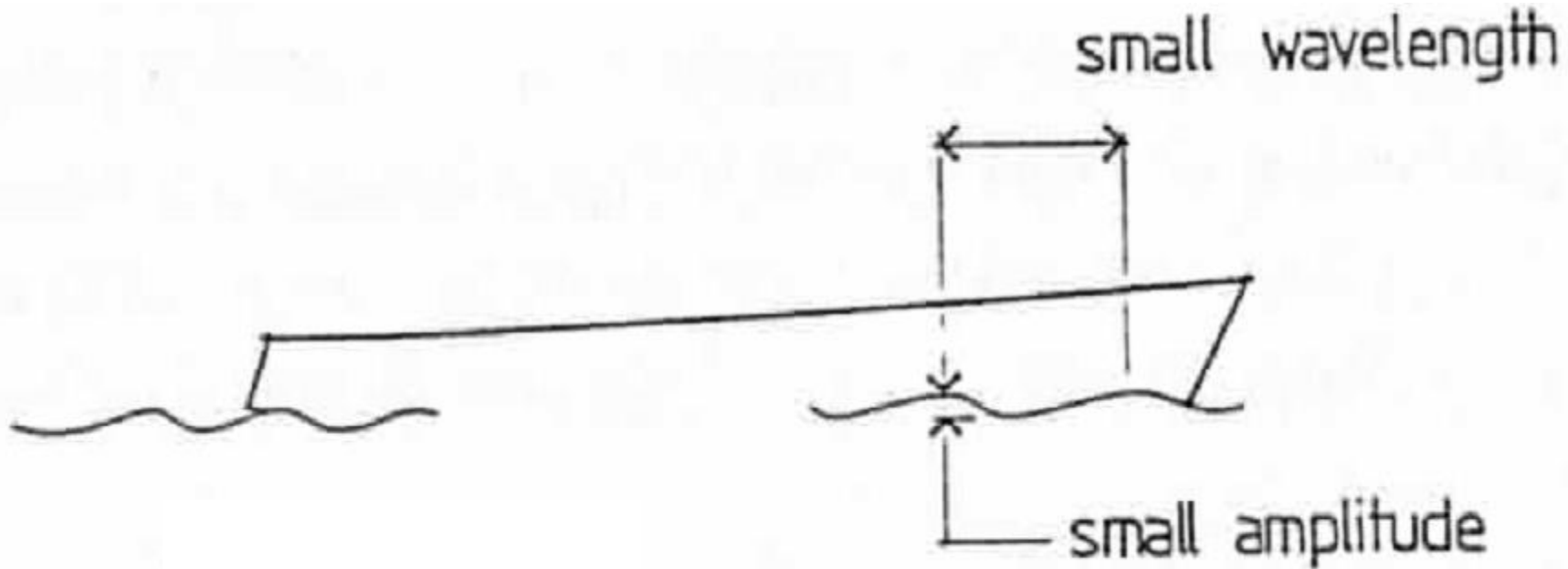


Stern
“Hole”

Bow
“Bump”

Wavelength selection as a function of velocity

Which of these two ships has the highest velocity?



Superimpose wave of opposite amplitude
from stern located at a distance l

displacement

$$\begin{aligned}\eta &= a \cos(kx + \varepsilon) - a \cos(kx + \varepsilon + kl) \\ &= \operatorname{Re}\{ae^{i(kx+\varepsilon)}(1 - e^{ikl})\}.\end{aligned}$$

dispersion
relation

$$k = g/U^2$$

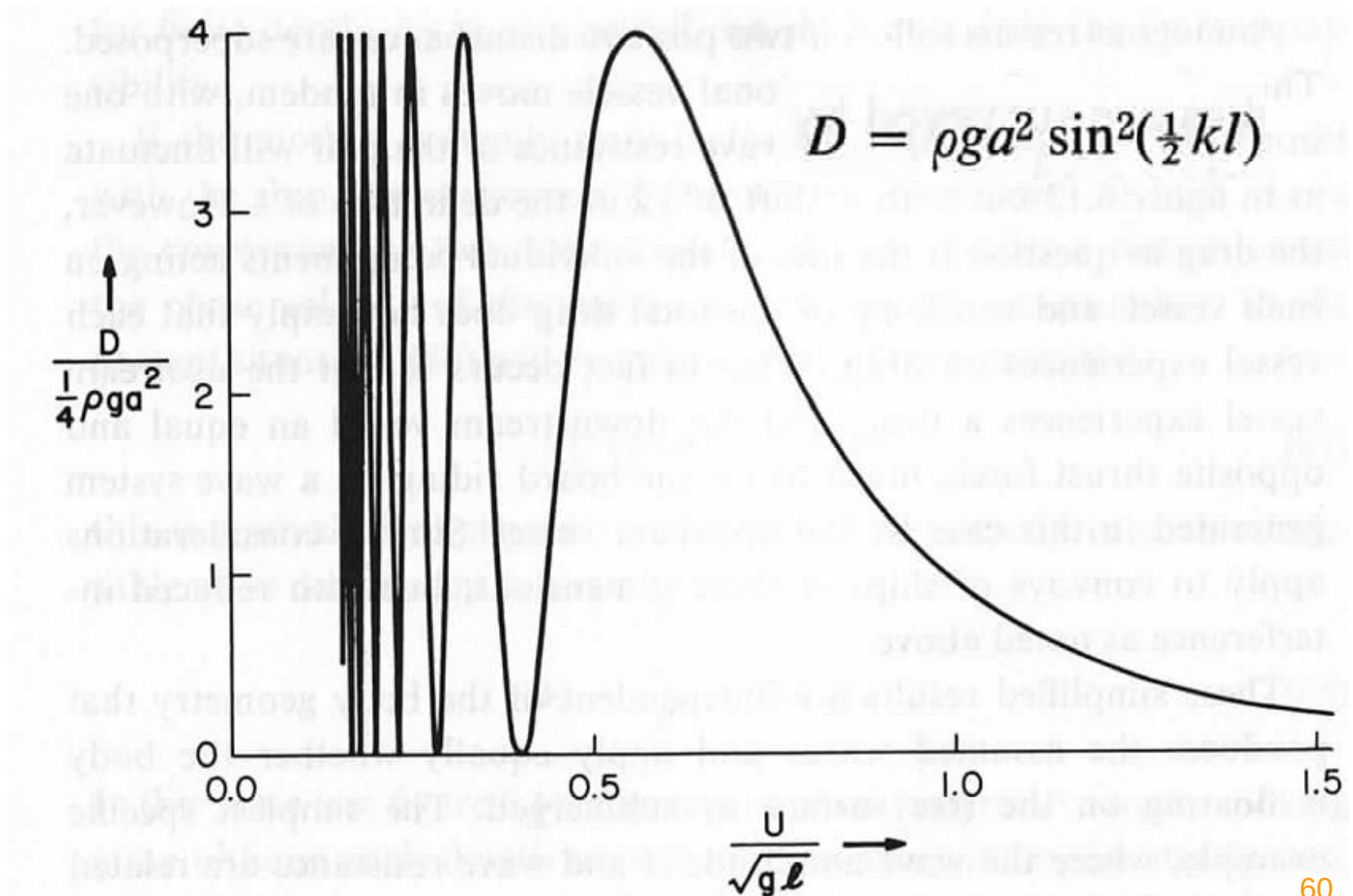
wave amplitude

$$A = a|1 - e^{ikl}| = 2a|\sin(\tfrac{1}{2}kl)|$$

wave energy

$$D = \rho g a^2 \sin^2(\tfrac{1}{2}kl)$$

Superimpose wave of opposite amplitude
from stern located at a distance l



The missing ingredient is now to determine
the wave amplitude!

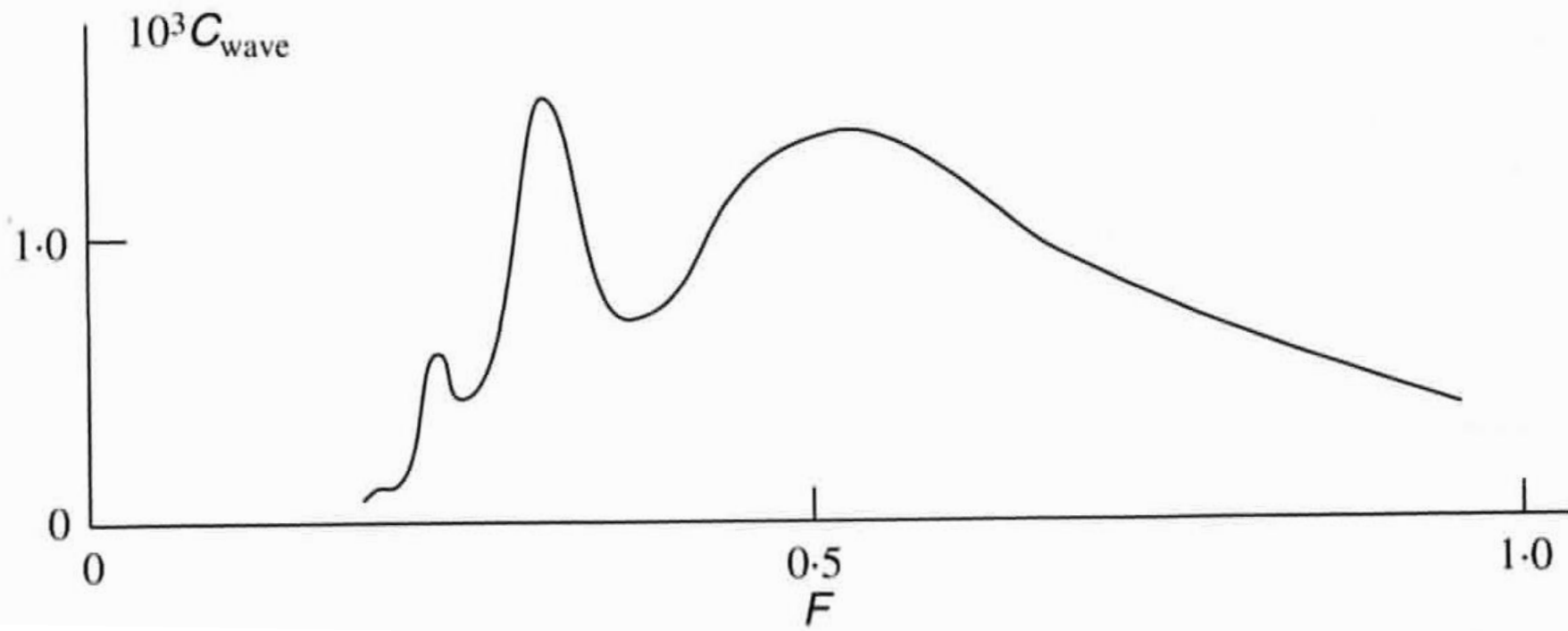
Combination of theoretical and empirical
results.

Generalisation to 3D

$$\frac{dE}{dt} = \frac{1}{2} \rho g \int_{-\infty}^{\infty} A^2 (V_g \cos \theta - U) dz.$$

$$D = \frac{1}{2} \pi \rho U^2 \int_{-\pi/2}^{\pi/2} |A(\theta)|^2 \cos^3 \theta d\theta.$$

Less interference



Which boat has less wave drag?

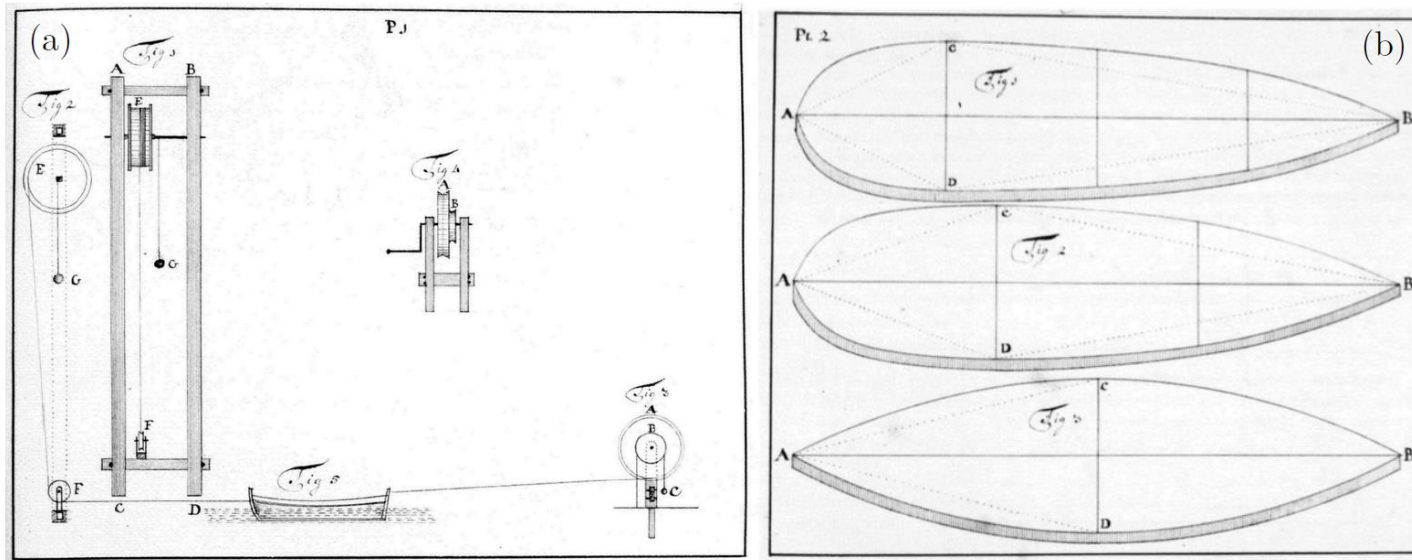


Figure 2: (a) Apparatus and (b) hull shapes used by Pieter van Zwijndregt in his towing experiments around 1757. The model hulls were towed over a 50 m distance. Reprinted from [6].

Submerged vessels feel less drag

Submarine name	Speed at the surface (km/h)	Speed when submerged (km/h)
Daphné (FR)	28	22
Agosta (FR)	37	23
Classe Franklin (US)	39	30
Classe Ohio (US)	37	22
Typhoon (USSR)	50	40
Oscar II (USSR)	59	28

Table 1.2: Speeds of different submarines at the surface and when submerged.

Michell's theory[1]

Boundary conditions. The condition that water particles do not cross the hull boundary is written:

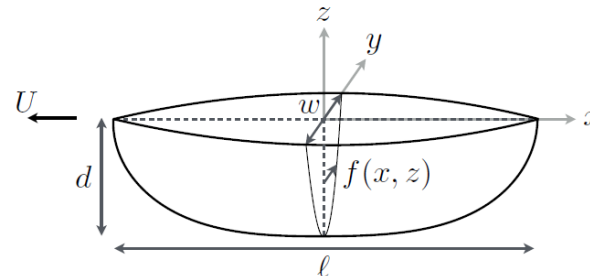
$$\Phi_y = \Phi_x f_x + \Phi_z f_z . \quad (1.33)$$

Similarly, ensuring that water particles do not cross the air-water interface yields the kinematic condition . The Bernoulli equation at the air-water interface is written:

$$p_0 + \frac{1}{2}\rho \mathbf{u}^2 + \rho g \xi = p_0 + \frac{1}{2}\rho U^2 , \quad (1.34)$$

with p_0 the atmospheric pressure. It is then straight-forward to obtain the dynamic condition:

$$2g\xi + \Phi_x^2 + \Phi_y^2 + \Phi_z^2 = U^2 . \quad (1.35)$$



Michell's theory[2]

For an infinitely deep fluid, we further have $\lim_{z \rightarrow -\infty} \Phi_z = 0$ (for a fluid of finite depth h , this condition is replaced by $\Phi_z(z = -h) = 0$). The last condition to ensure is the radiation condition given in Eq. (1.26).

Linearisation. The main assumption of Michell's model is to consider that the hull is thin ($w \ll \ell$, corresponding to small longitudinal slopes f_x , $f_z \ll 1$).

The linearisation of the kinematic condition on the hull boundary (1.33) leads to:

$$\Phi_y(x, \pm 0, z) = \mp U f_x . \quad (1.36)$$

The combination of the linearised kinematic and dynamic conditions (1.23, 1.35) gives on $z = 0$:

$$\Phi_z + \frac{U^2}{g} \Phi_{xx} = 0 . \quad (1.37)$$

Michell solves the Laplace equation (1.22) with the boundary conditions given in Eqs. (1.25, 1.26) and Eqs. (1.36, 1.37) using Fourier-transform methods.

Michell's theory[3]

In Michell's theory [7, 28], the wave drag for an infinitely deep fluid R_w can be written as:

$$R_w(f) = \frac{4\rho U^2}{\pi \ell^4 \text{Fr}^8} \int_1^{+\infty} |\mathcal{I}_f(\lambda, \text{Fr})|^2 \frac{\lambda^4}{\sqrt{\lambda^2 - 1}} d\lambda , \quad (1.38)$$

where:

$$\mathcal{I}_f(\lambda, \text{Fr}) = \int_{-d}^0 dz \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} f(x, z) e^{\lambda^2 z / (\ell \text{Fr}^2)} e^{i\lambda x / (\ell \text{Fr}^2)} dx . \quad (1.39)$$

Effect of finite depth

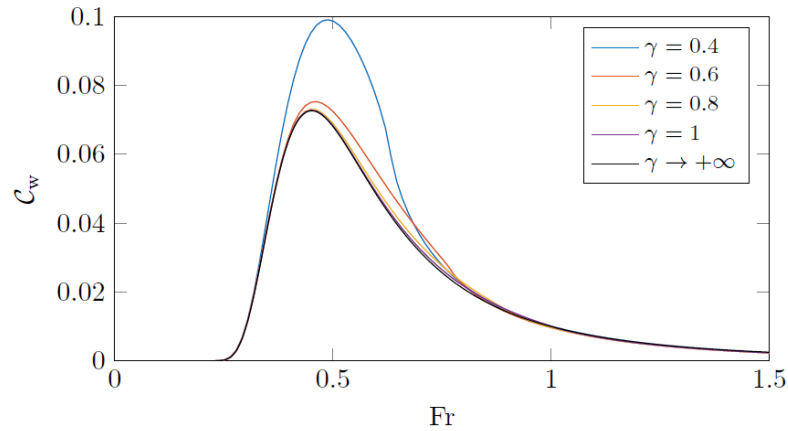


Figure 1.14: Wave drag coefficient $\mathcal{C}_w$ as a function of the Froude number Fr predicted by Michell's model for increasing water depth $\gamma = h/\ell$ (for $\alpha = 6.7$, $\beta = 2.3$ and $\tilde{f} = 1/2 \exp(-16\tilde{x}^2)$). The black curve corresponds to the infinite depth case.

Comparison with experiments

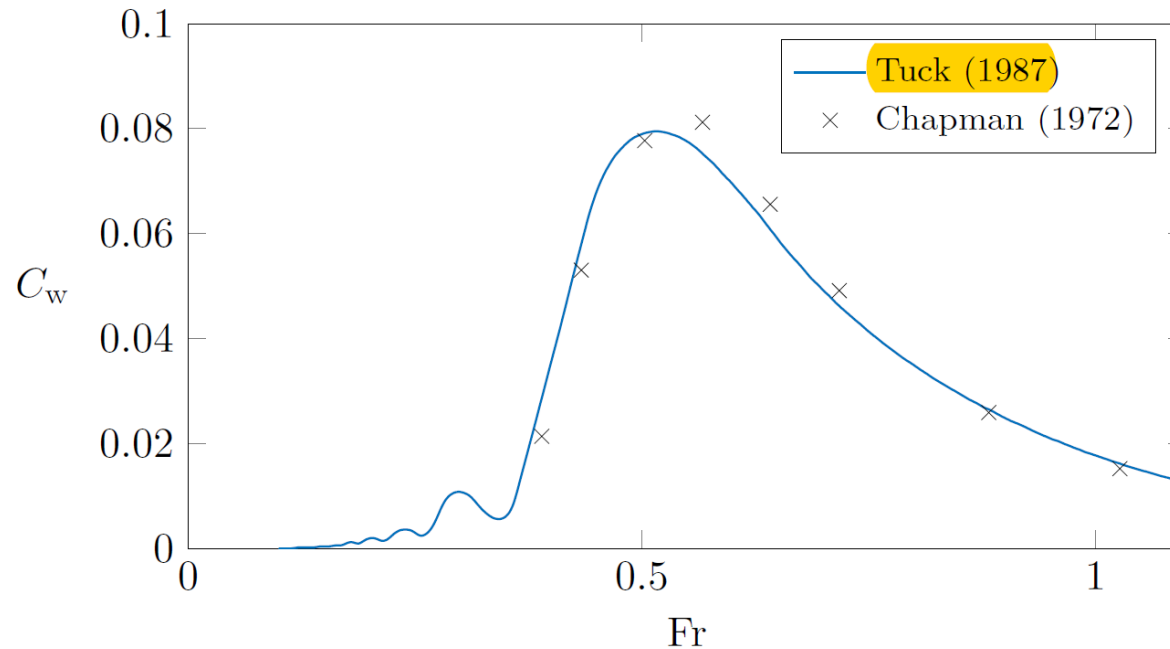


Figure 1.13: Wave-drag coefficient C_w as a function of the Froude number Fr predicted theoretically by [32] together with experimental data points from [31] (black crosses). The hull has a parabolic shape with $\alpha = 6.7$ and $\beta = 2.3$.

Role of hull shape

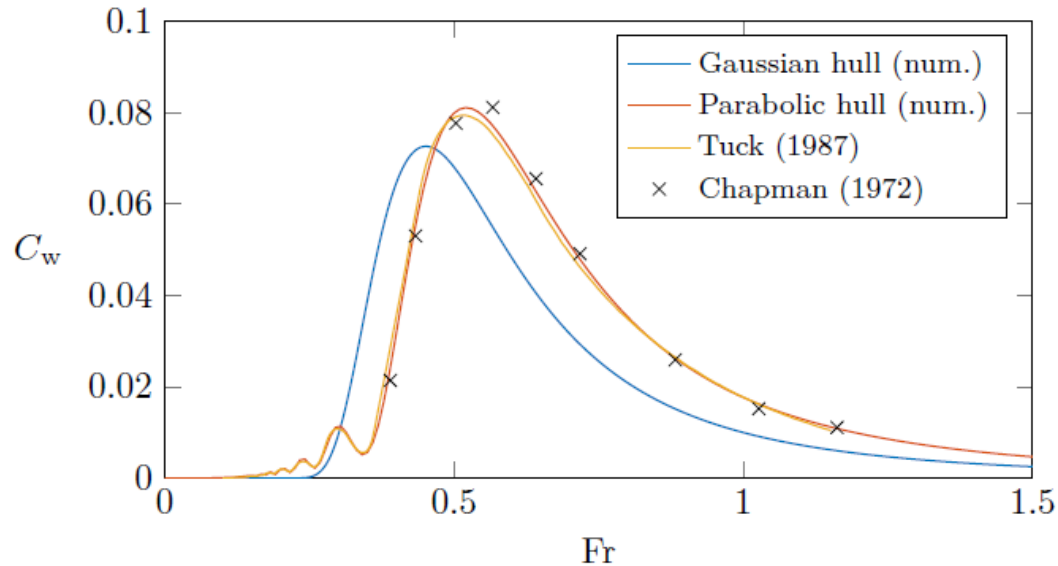


Figure 3.4: Wave-drag coefficient C_w as function of the Froude number Fr for a gaussian hull and a parabolic hull for $\alpha = 6.7$ and $\beta = 2.3$. These results are compared to the theoretical curve from [32] and experimental data points from [31] (black crosses).

Role of hull shape asymmetry in air

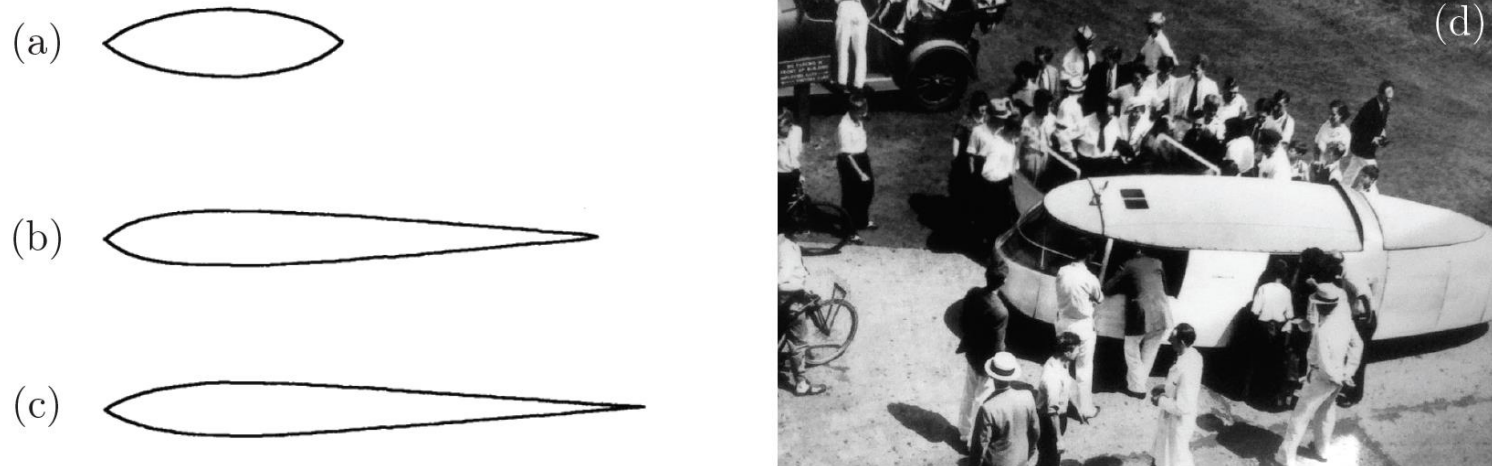


Figure 4.1: Two dimensional shapes of minimum drag found by [83] for (a) $Re = 20$, (b) $Re = 5\,000$ and (c) $Re = 20\,000$. (d) Picture of the *Dymaxion* car n°1, designed by the American inventor and architect Buckminster Fuller in 1933. This car had an aerodynamic bodywork to increase its fuel efficiency and reach top speed [85].

Role of hull shape asymmetry on the water surface

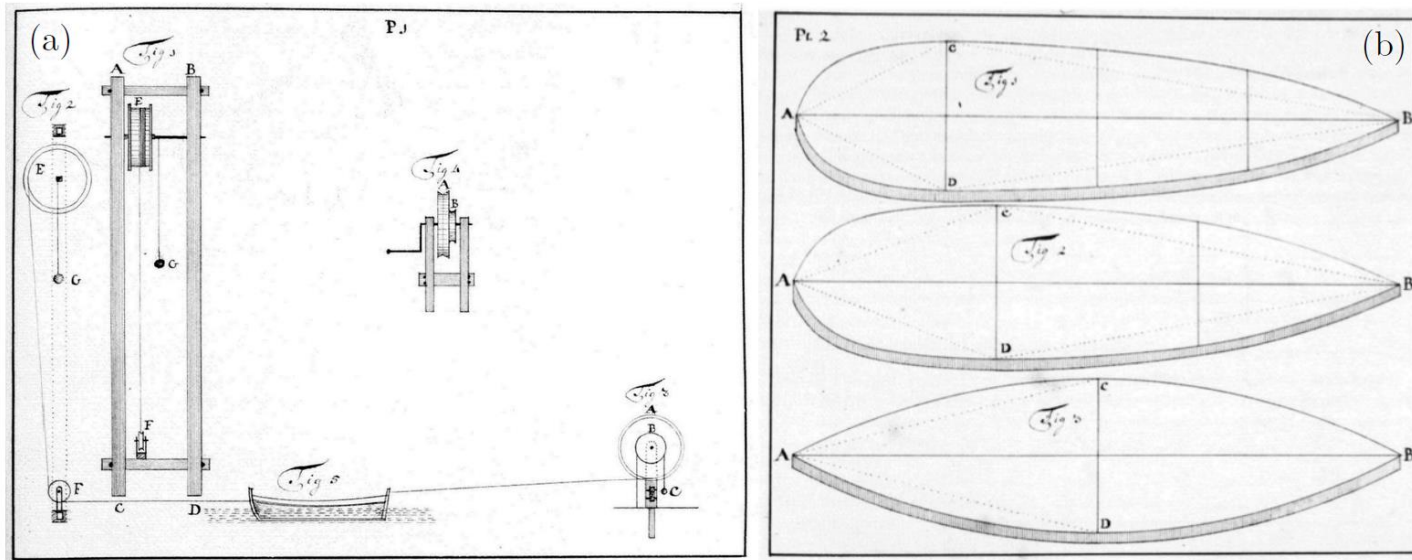


Figure 2: (a) Apparatus and (b) hull shapes used by Pieter van Zwijndregt in his towing experiments around 1757. The model hulls were towed over a 50 m distance. Reprinted from [6].

Role of hull shape asymmetry on the water surface [JP Boucher's thesis]



Figure 4.4: Schematics illustrating the two different configurations $\epsilon < 0$ and $\epsilon > 0$

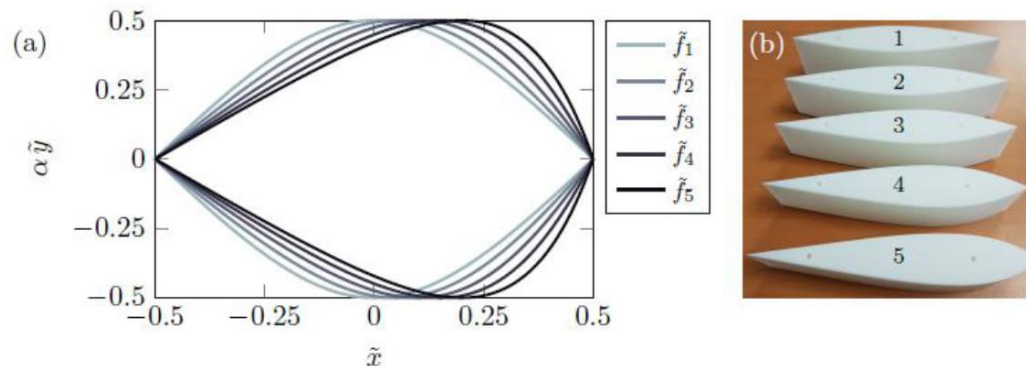


Figure 4.5: (a) Profile $\alpha\tilde{y} = \pm\tilde{f}(\tilde{x})$ for five different sets of the parameters (a, b, α, β) in Eq. (4.3), with increasing asymmetry. (b) Picture of the five 3d-printed model hulls defined by the functions $\tilde{f}_1$ to $\tilde{f}_5$. They are 18 cm-long, 3 cm-wide and 5 cm-high.

Role of hull shape asymmetry on the water surface [JP Boucher's thesis]

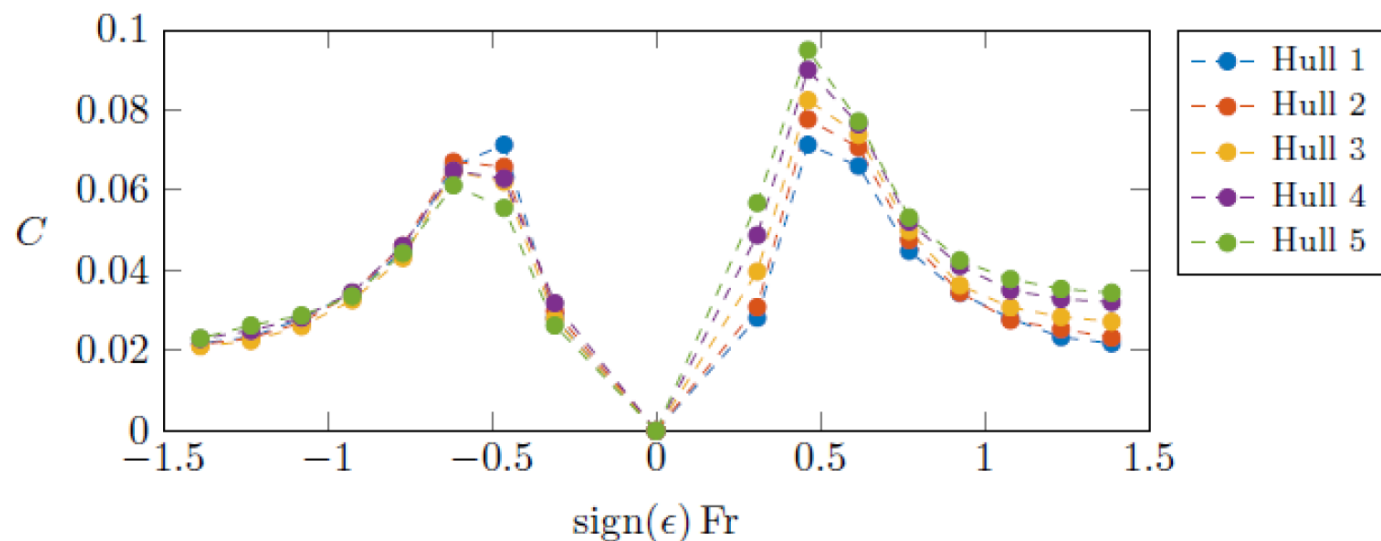


Figure 4.8: Total drag coefficient as a function of the signed Froude number $\text{sign}(\epsilon) \text{Fr}$ for the five hulls for $d/D = 0.75$. The use of the parameter $\text{sign}(\epsilon) \text{Fr}$ allows us to compare a given hull moving with the rounded part first ($\epsilon > 0$, right part of the plot) or with the pointed part first ($\epsilon < 0$, left part of the plot) (see Fig. 4.4).