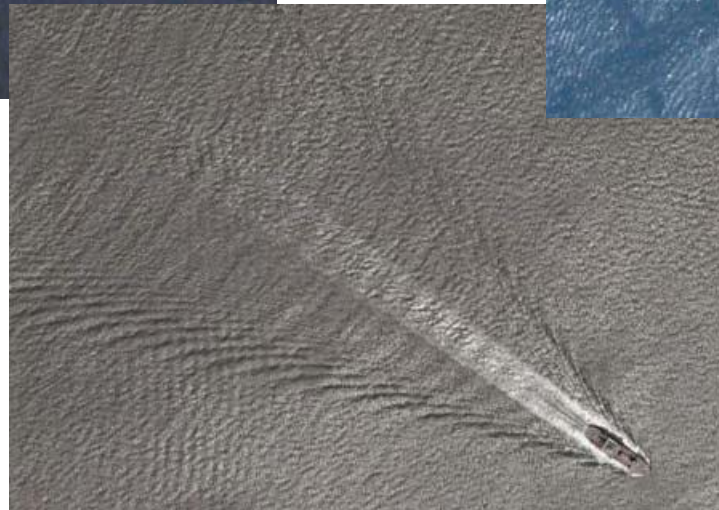
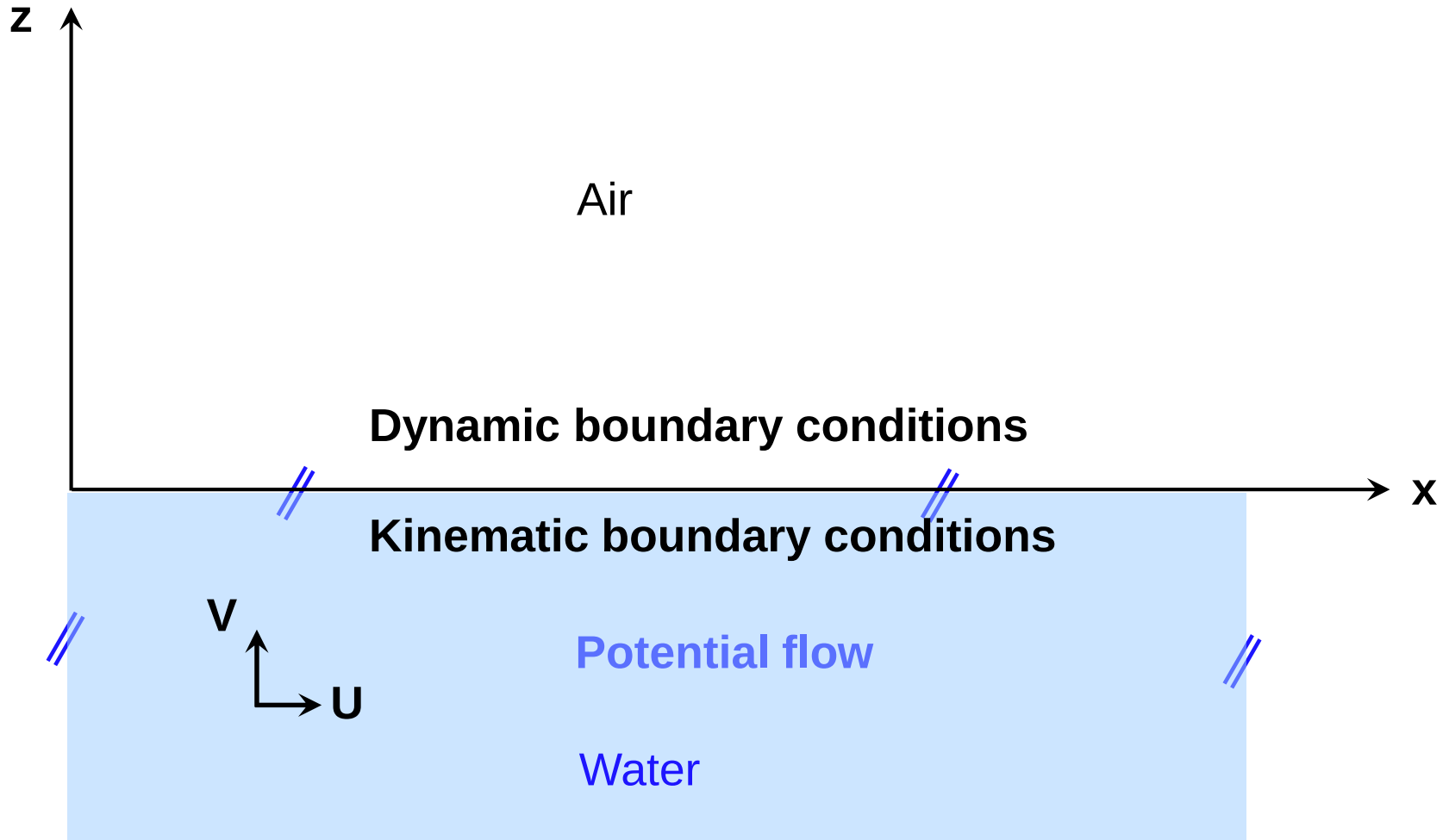


Hydrodynamics 14

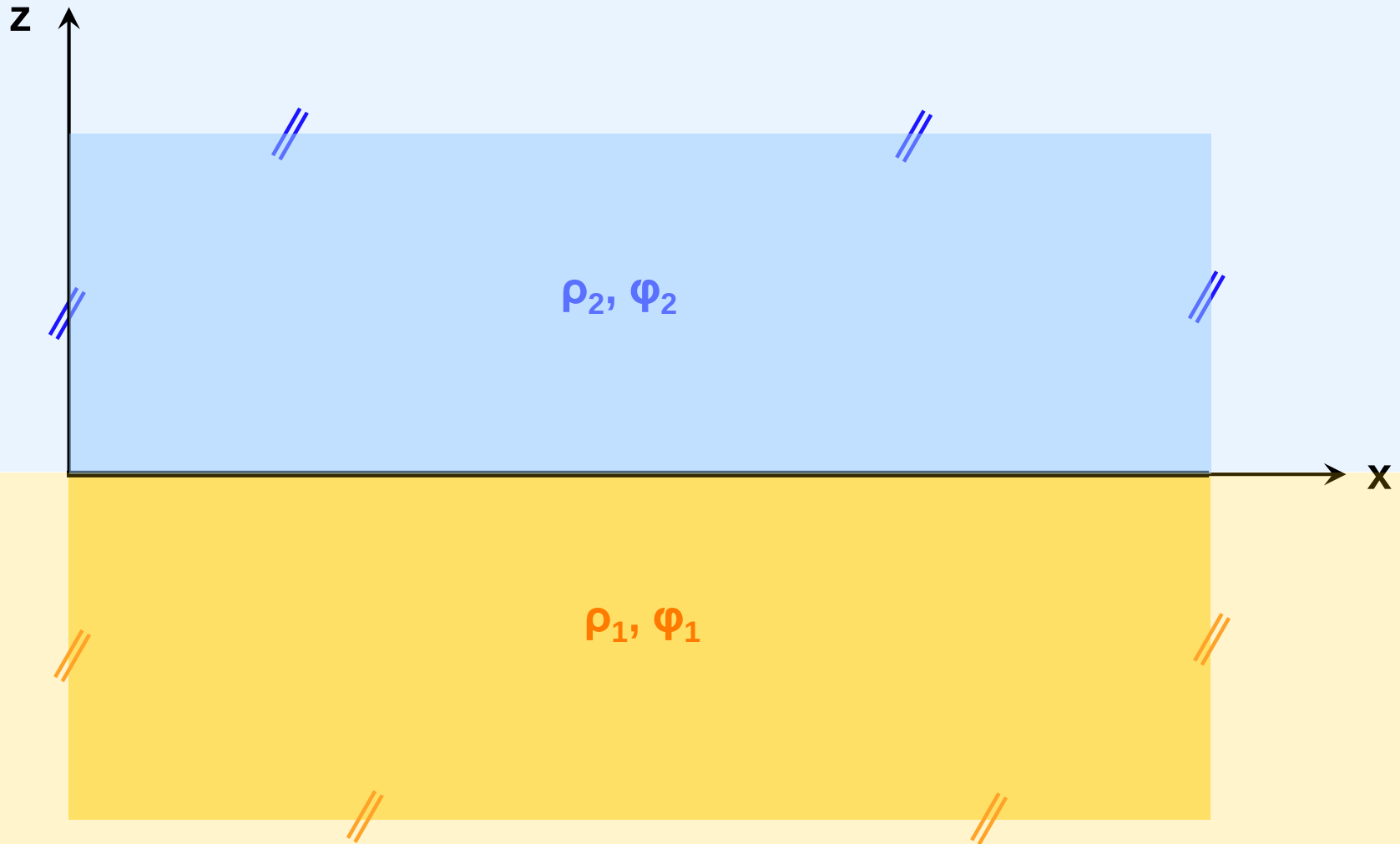
Waves



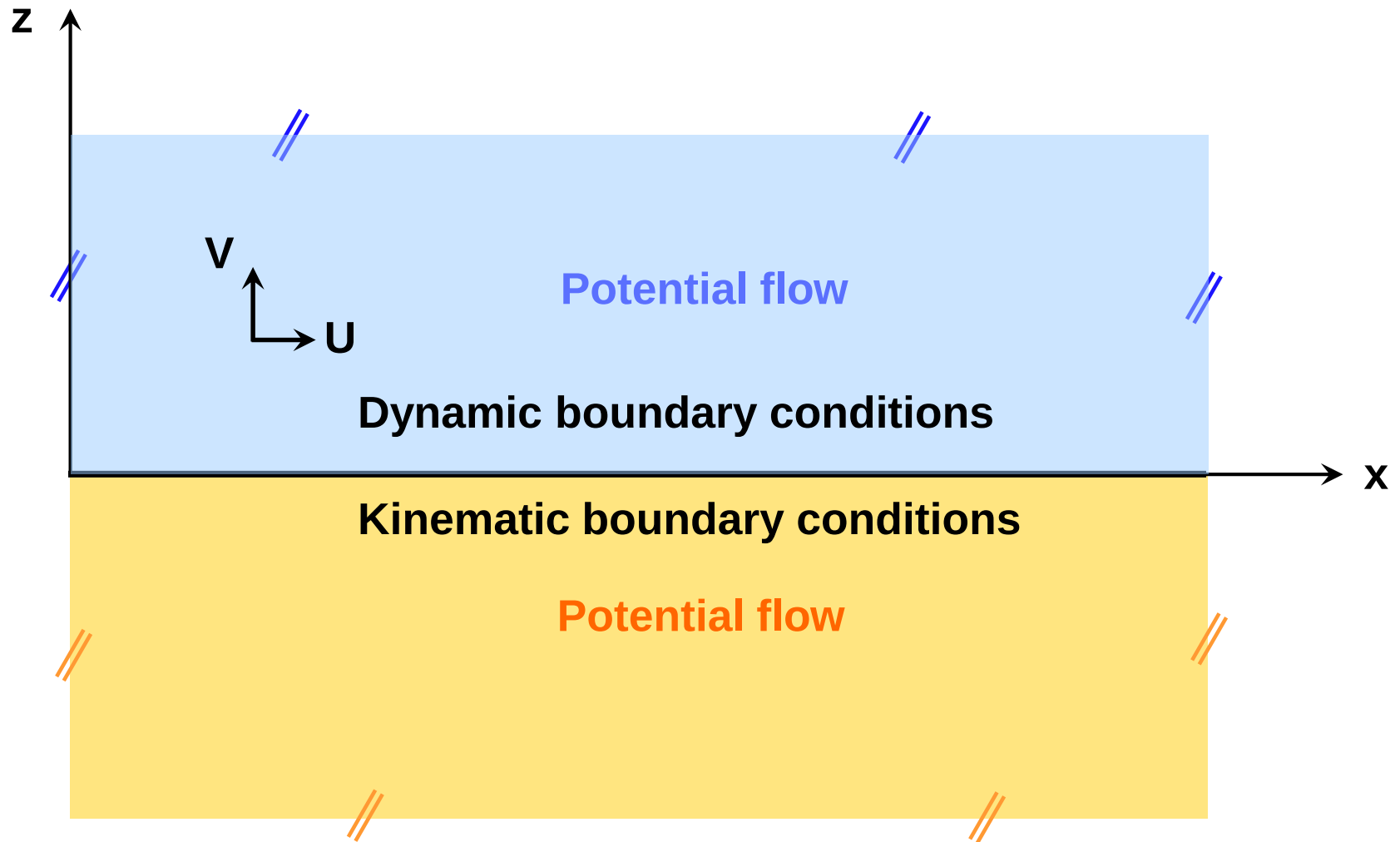
Waves



General case: two fluids



General case: two fluids



Linear waves dispersion relation

1. Equations and boundary conditions
2. Base state
3. Linearized equations
4. Normal mode expansion
5. Dispersion relation
6. Analysis of the dispersion relation

1. Equations

$$\begin{array}{l} \Delta \Phi_1 = 0 \\ \Delta \Phi_2 = 0 \end{array}$$

Potential flow

$$\begin{array}{ll} U_1 = \frac{\partial \Phi_1}{\partial x}, & V_1 = \frac{\partial \Phi_1}{\partial z} \\ U_2 = \frac{\partial \Phi_2}{\partial x}, & V_2 = \frac{\partial \Phi_2}{\partial z} \end{array}$$

Velocity field

1. Boundary conditions

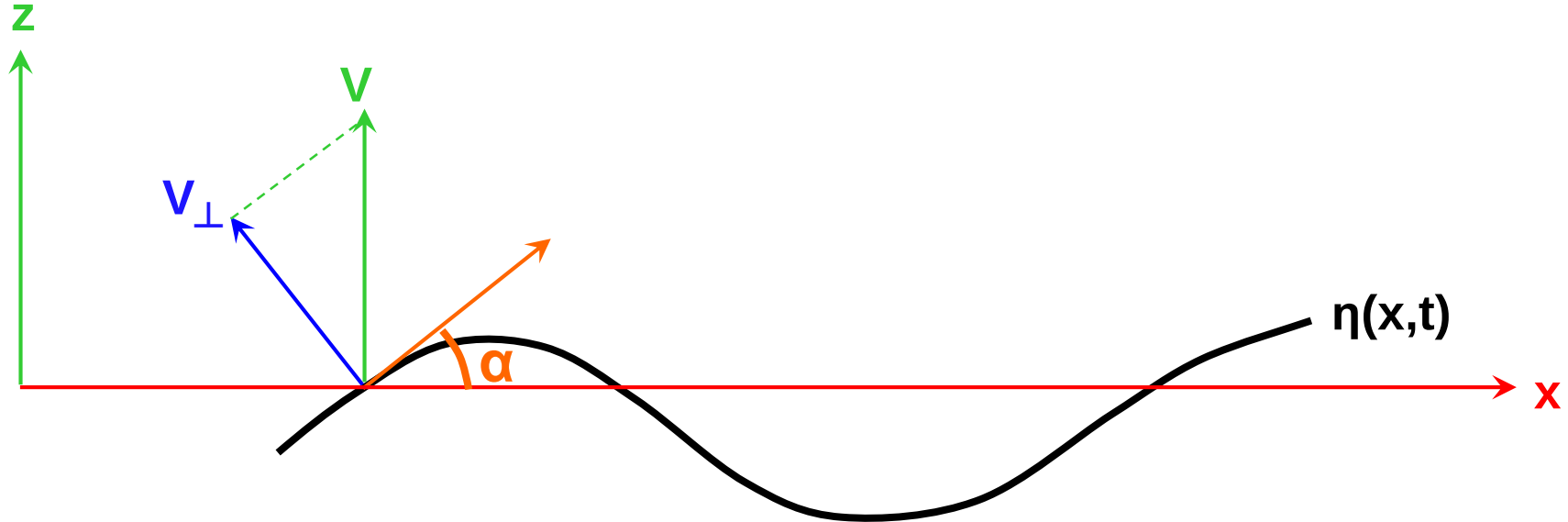
$$\Phi_1 = 0 \text{ at } z = -\infty$$

$$\Phi_2 = 0 \text{ at } z = +\infty$$

far-field

at $z = \eta$?

1. Kinematic boundary condition

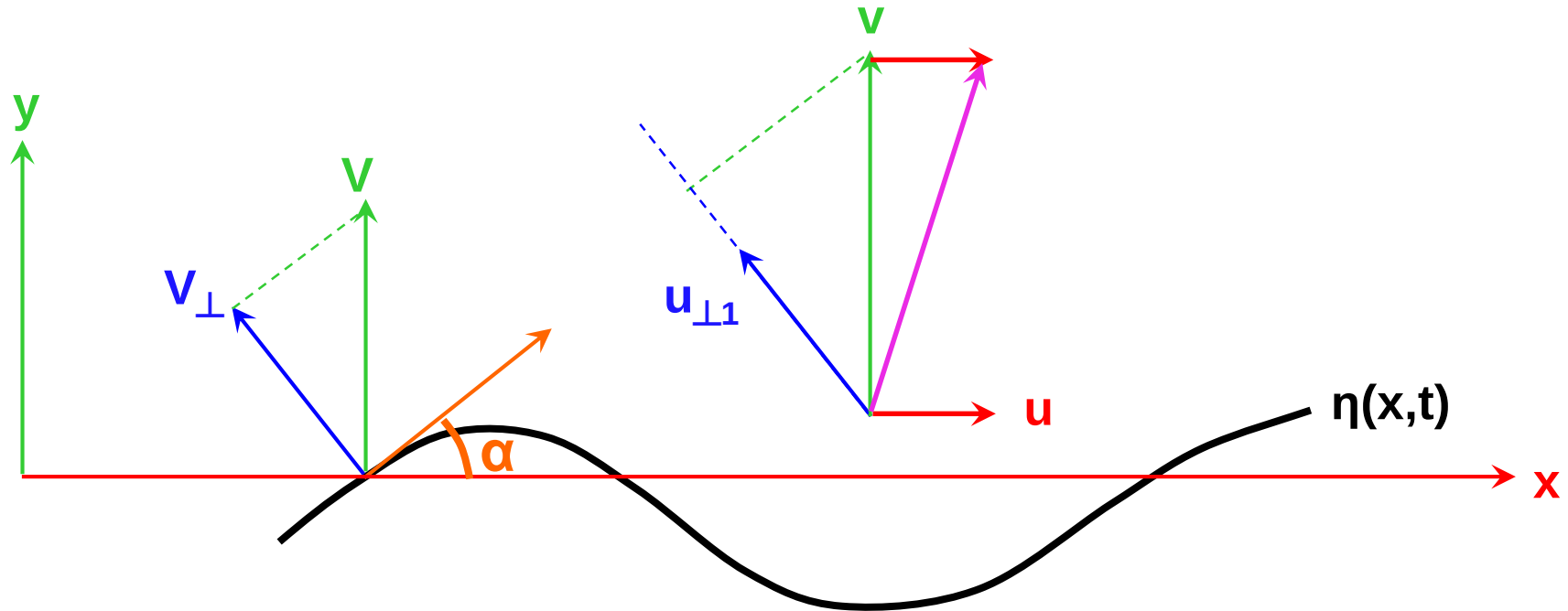


Kinematic condition : impermeability (no penetration)

No fluid particles going across the interface through the normal direction

$$V_{\perp} = \partial\eta/\partial t \cos(\alpha)$$

1. Kinematic boundary condition



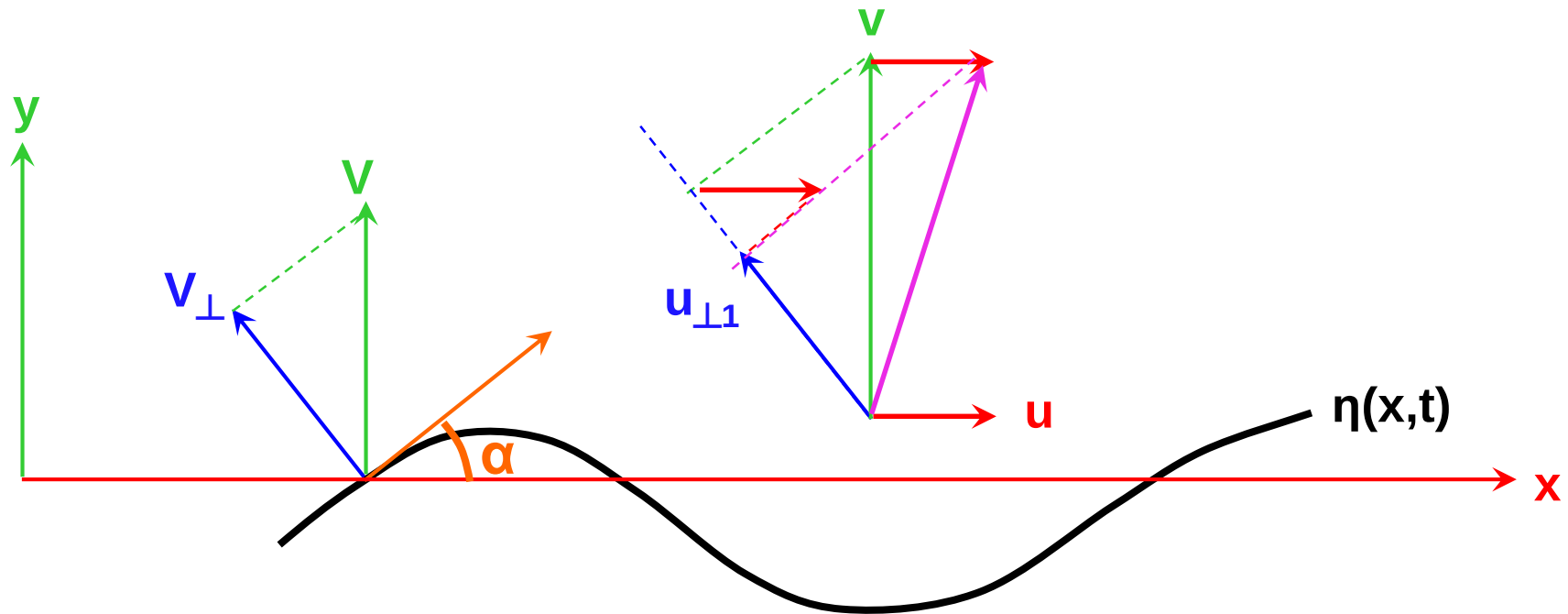
Kinematic condition : impermeability (no penetration)

No fluid particles going across the interface through the normal direction

$$v_{\perp} = \frac{\partial \eta}{\partial t} \cos(\alpha)$$

$$u_{\perp 1} = v_1 \cos(\alpha) +$$

1. Kinematic boundary condition



Kinematic condition : impermeability (no penetration)

No fluid particles going across the interface through the normal direction

$$\left. \begin{aligned} v_{\perp} &= \frac{\partial \eta}{\partial t} \cos(\alpha) \\ u_{\perp 1} &= v_1 \cos(\alpha) - u_1 \sin(\alpha) \end{aligned} \right\} \frac{\partial \eta}{\partial t} = v_1 - u_1 \tan(\alpha) \Rightarrow \boxed{\frac{\partial \eta}{\partial t} = v_1 - u_1 \frac{\partial \eta}{\partial x}}$$

1. Kinematic boundary conditions

$$\Phi_1 = 0 \text{ at } z = -\infty$$

$$\Phi_2 = 0 \text{ at } z = +\infty$$

far-field

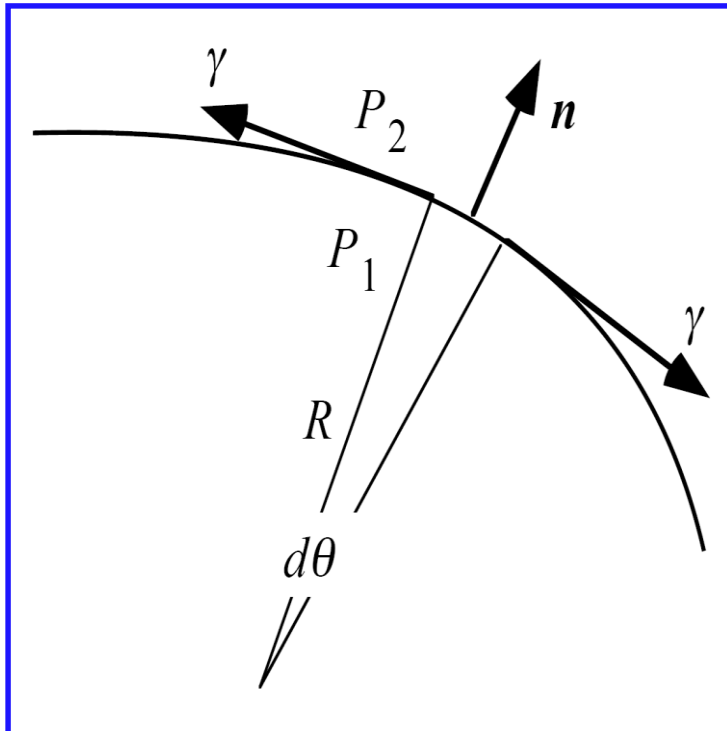
$$U_1 \frac{\partial \eta}{\partial x} - V_1 = - \frac{\partial \eta}{\partial t}$$

$$U_2 \frac{\partial \eta}{\partial x} - V_2 = - \frac{\partial \eta}{\partial t}$$

at $z = \eta$

1. Dynamic boundary conditions

$$P_1 - P_2 = -\gamma \frac{\frac{\partial^2 \eta}{\partial x^2}}{\left(1 + \frac{\partial \eta}{\partial x}\right)^{3/2}} \text{ at } z = \eta$$



$$\mathbf{n} = \frac{(-\partial_x \eta, 1)}{\sqrt{1 + \partial_x^2 \eta}}$$

$$\mathcal{C} = \nabla \cdot \mathbf{n}$$

1. More equations

$$\begin{aligned} \frac{\partial \Phi_1}{\partial t} + \frac{U_1^2 + V_1^2}{2} + \frac{P_1}{\rho_1} + gz &= C_1(t) = 0 \\ \frac{\partial \Phi_2}{\partial t} + \frac{U_2^2 + V_2^2}{2} + \frac{P_2}{\rho_2} + gz &= C_2(t) = 0 \end{aligned}$$

2nd Bernouilli relations

2. Base state

$$\Phi_1 = 0,$$

$$\Phi_2 = 0$$

$$\eta = 0$$

$$P_1 = -\rho_1 g z$$

3. Perturb and linearize perturbation expansion

Φ_1	$= 0$	$+\epsilon\phi_1$
Φ_2	$= 0$	$+\epsilon\phi_2$
U_1	$= 0$	$+\epsilon u_1$
V_1	$= 0$	$+\epsilon v_1$
U_2	$= 0$	$+\epsilon u_2$
V_2	$= 0$	$+\epsilon v_2$
P_1	$= -\rho_1 g z$	$+\epsilon p_1$
P_2	$= -\rho_2 g z$	$+\epsilon p_2$
η	$= 0$	$+\epsilon\sigma$

$\epsilon \ll 1$

Variables **Base state** **Small perturbation**

3. Linearized equations

$$\begin{array}{l} \Delta\phi_1 = 0 \\ \Delta\phi_2 = 0 \end{array}$$

perturbed potential flow

$$\begin{array}{ll} u_1 = \frac{\partial\phi_1}{\partial x}, & v_1 = \frac{\partial\phi_1}{\partial z} \\ u_2 = \frac{\partial\phi_2}{\partial x}, & v_2 = \frac{\partial\phi_2}{\partial z} \end{array}$$

3. Perturbed kinematic boundary conditions

$$\phi_1 = 0 \text{ at } z = -\infty$$

$$\phi_2 = 0 \text{ at } z = +\infty$$

$$-\epsilon^2 u_1 \frac{\partial \sigma}{\partial x} + \epsilon v_1 = \epsilon \frac{\partial \sigma}{\partial t} \text{ at } z = \epsilon \sigma$$

$$-\epsilon^2 u_2 \frac{\partial \sigma}{\partial x} + \epsilon v_2 = \epsilon \frac{\partial \sigma}{\partial t} \text{ at } z = \epsilon \sigma$$

3. Perturbed kinematic boundary conditions

$$\phi_1 = 0 \text{ at } z = -\infty$$

$$\phi_2 = 0 \text{ at } z = +\infty$$

~~$$-\epsilon^2 u_1 \frac{\partial \sigma}{\partial x} + \epsilon v_1 = \epsilon \frac{\partial \sigma}{\partial t} \text{ at } z = \epsilon \sigma$$
$$-\epsilon^2 u_2 \frac{\partial \sigma}{\partial x} + \epsilon v_2 = \epsilon \frac{\partial \sigma}{\partial t} \text{ at } z = \epsilon \sigma$$~~

$$v_1 = \frac{\partial \sigma}{\partial t} \text{ at } z = \epsilon \sigma$$
$$v_2 = \frac{\partial \sigma}{\partial t} \text{ at } z = \epsilon \sigma$$

3. Flattened kinematic boundary conditions

$$\begin{aligned}\frac{\partial \phi_1}{\partial z} &= \frac{\partial \sigma}{\partial t} \text{ at } z = \epsilon \sigma \\ \frac{\partial \phi_2}{\partial z} &= \frac{\partial \sigma}{\partial t} \text{ at } z = \epsilon \sigma\end{aligned}$$

Taylor expansion around 0: $\phi(\epsilon \sigma) = \phi(0) + (\epsilon \sigma) \frac{\partial \phi}{\partial z} \Big|_0$

$$\begin{aligned}\frac{\partial \phi_1}{\partial z} &= \frac{\partial \sigma}{\partial t} \text{ at } z = 0 \\ \frac{\partial \phi_2}{\partial z} &= \frac{\partial \sigma}{\partial t} \text{ at } z = 0\end{aligned}$$

⇒ transforms a b.c. at an unknown interface into a fixed place!

3. Perturbed dynamic boundary conditions

$$(P_1 + \epsilon p_1 - P_2 - \epsilon p_2)|_{\epsilon\sigma} = -\gamma\epsilon \frac{\partial^2 \sigma}{\partial x^2} \left(1 - 3/2\epsilon^2 \left(\frac{\partial \sigma}{\partial x} \right)^2 \right)$$

Replace $P_1 = -g\rho_1 z$, ...

and linearize

$$g(\rho_2 - \rho_1)\sigma + (p_1 - p_2)|_{\epsilon\sigma} = -\gamma \frac{\partial^2 \sigma}{\partial x^2}$$

flatten

$$(\rho_2 - \rho_1)g\sigma + (p_1 - p_2)|_0 = -\gamma \frac{\partial^2 \sigma}{\partial x^2}$$

3. Perturbed and linearized Bernoulli

Perturbed 2nd Bernoulli relations

$$\begin{aligned}\epsilon \frac{\partial \phi_1}{\partial t} + \cancel{\epsilon^2 \frac{u_1^2 + v_1^2}{2}} + \epsilon \frac{p_1}{\rho_1} &= 0 \\ \epsilon \frac{\partial \phi_2}{\partial t} + \cancel{\epsilon^2 \frac{u_2^2 + v_2^2}{2}} + \epsilon \frac{p_2}{\rho_2} &= 0\end{aligned}$$

Linearized 2nd Bernoulli relations

$$\begin{aligned}\frac{\partial \phi_1}{\partial t} + \frac{p_1}{\rho_1} &= 0 \\ \frac{\partial \phi_2}{\partial t} + \frac{p_2}{\rho_2} &= 0\end{aligned}$$

4. Normal mode expansion

Fourier transform in x and t

$$\begin{aligned}\phi_1 &= f_1(z)\exp(i(kx - \omega t)), \\ \phi_2 &= f_2(z)\exp(i(kx - \omega t)), \\ \sigma &= C\exp(i(kx - \omega t)),\end{aligned}$$

k is the wavenumber and ω the frequency (in rad/s)

$$\lambda = 2\pi/k$$

$$T = 2\pi/\omega$$

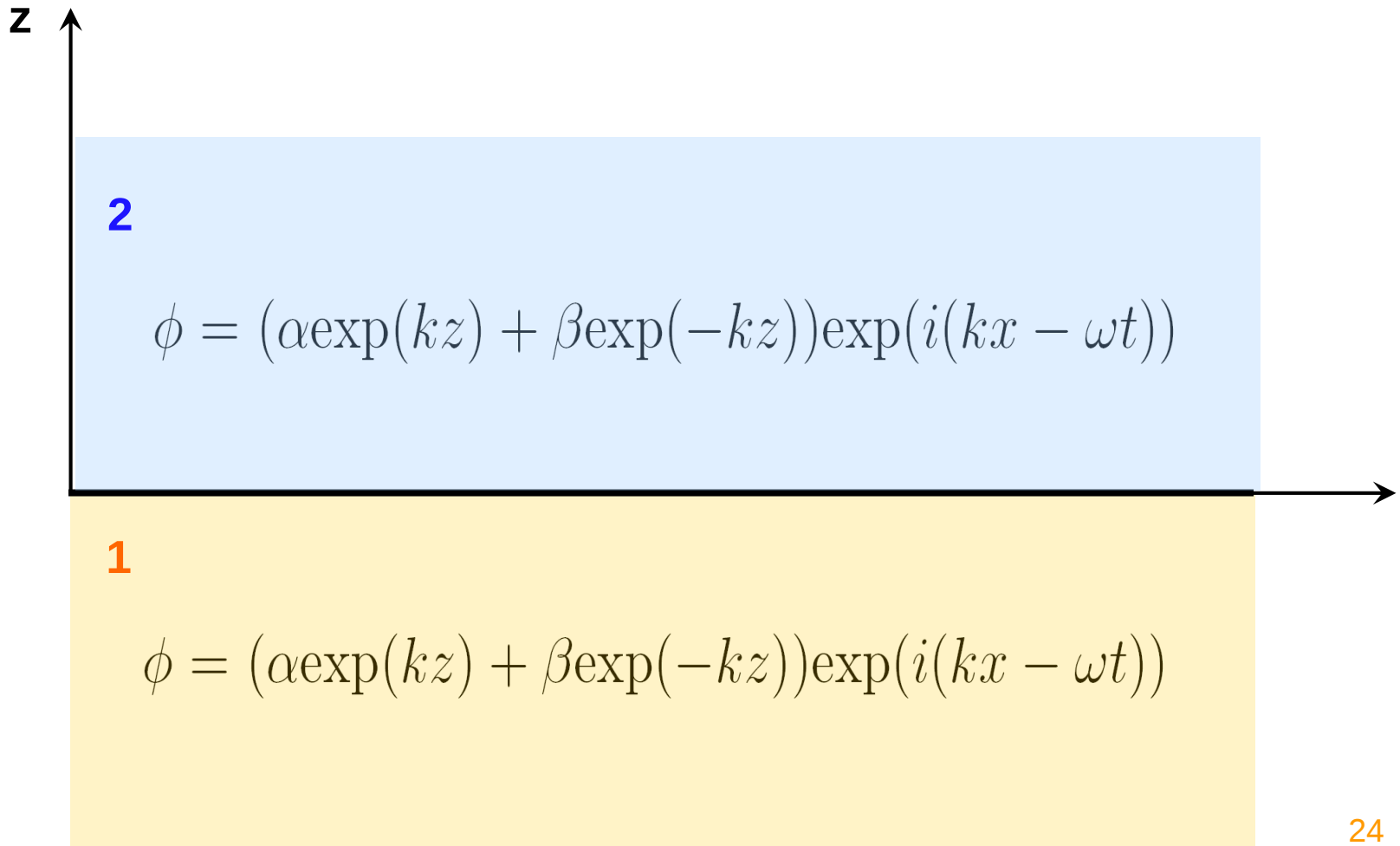
$$f = \omega/(2\pi)$$

4. Normal mode expansion

Solution to Laplace equation:

4. Normal mode expansion

Solution to Laplace equation:



4. Normal mode expansion

Solution to Laplace equation:

$$\begin{aligned}\phi_1 &= A \exp(kz) \exp(i(kx - \omega t)), \\ \phi_2 &= B \exp(-kz) \exp(i(kx - \omega t)), \\ \sigma &= C \exp(i(kx - \omega t)).\end{aligned}$$

4. Normal mode expansion

Replace in boundary conditions

$$\begin{aligned} g(\rho_2 - \rho_1)C + i\omega\rho_1 A - i\omega\rho_2 B &= \gamma k^2 C \\ kA &= -i\omega C \\ -kB &= -i\omega C \end{aligned}$$

This is an eigenvalue problem $i\omega X = MX$!

$$kg(\rho_2 - \rho_1)C + \omega^2\rho_1 C + \omega^2\rho_2 C = \gamma k^3 C,$$

5. Dispersion relation

$$\omega^2 = \frac{-kg(\rho_2 - \rho_1) + \gamma k^3}{\rho_1 + \rho_2}$$

•Unstable if there exists one ω , $\text{Im}(\omega) > 0$

$$\rho_2 > \rho_1$$

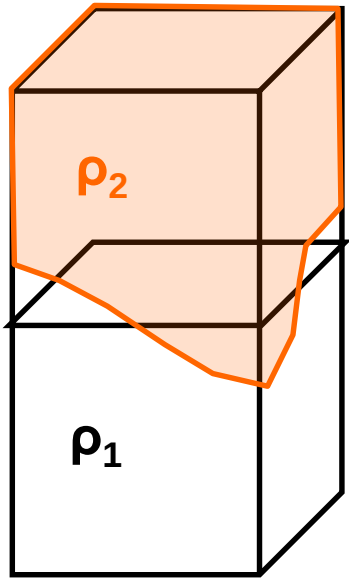
•Neutral if for all ω , $\text{Im}(\omega) = 0$:

$$\rho_1 > \rho_2$$

•Stable (or damped) if for all ω , $\text{Im}(\omega) < 0$:

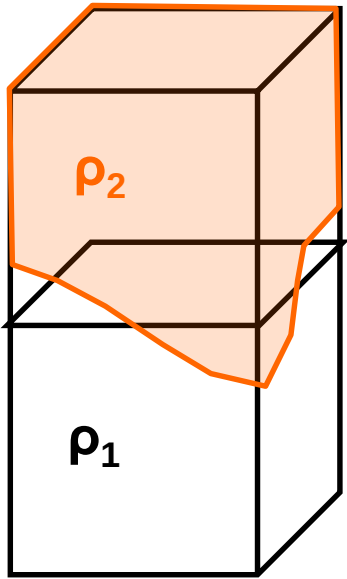
The flow considered is not damped, we have neglected dissipation by neglecting viscosity

Dispersion relation



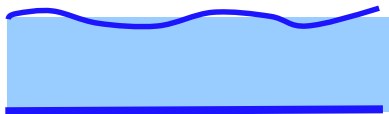
$$\omega^2 = \frac{-kg(\rho_2 - \rho_1) + \gamma k^3}{\rho_1 + \rho_2}$$

Dispersion relation



$$\omega^2 = \frac{-kg(\cancel{\rho_2} - \rho_1) + \gamma k^3}{\rho_1 + \cancel{\rho_2}}$$

$$\omega^2 = \tanh(kH) \left(\frac{\gamma k^3}{\rho} + gk \right)$$



Dispersion relation

$$\omega^2 = \tanh(kH) \left(\frac{\gamma k^3}{\rho} + gk \right)$$

Capillary wavenumber: $k_c = \sqrt{\rho g / \gamma}$

Length scale: $\tilde{k} = k / k_c$

Time scale: $\tilde{\omega} = \omega / \sqrt{g k_c}$

One single non-dimensional parameter $\tilde{H} = H k_c$

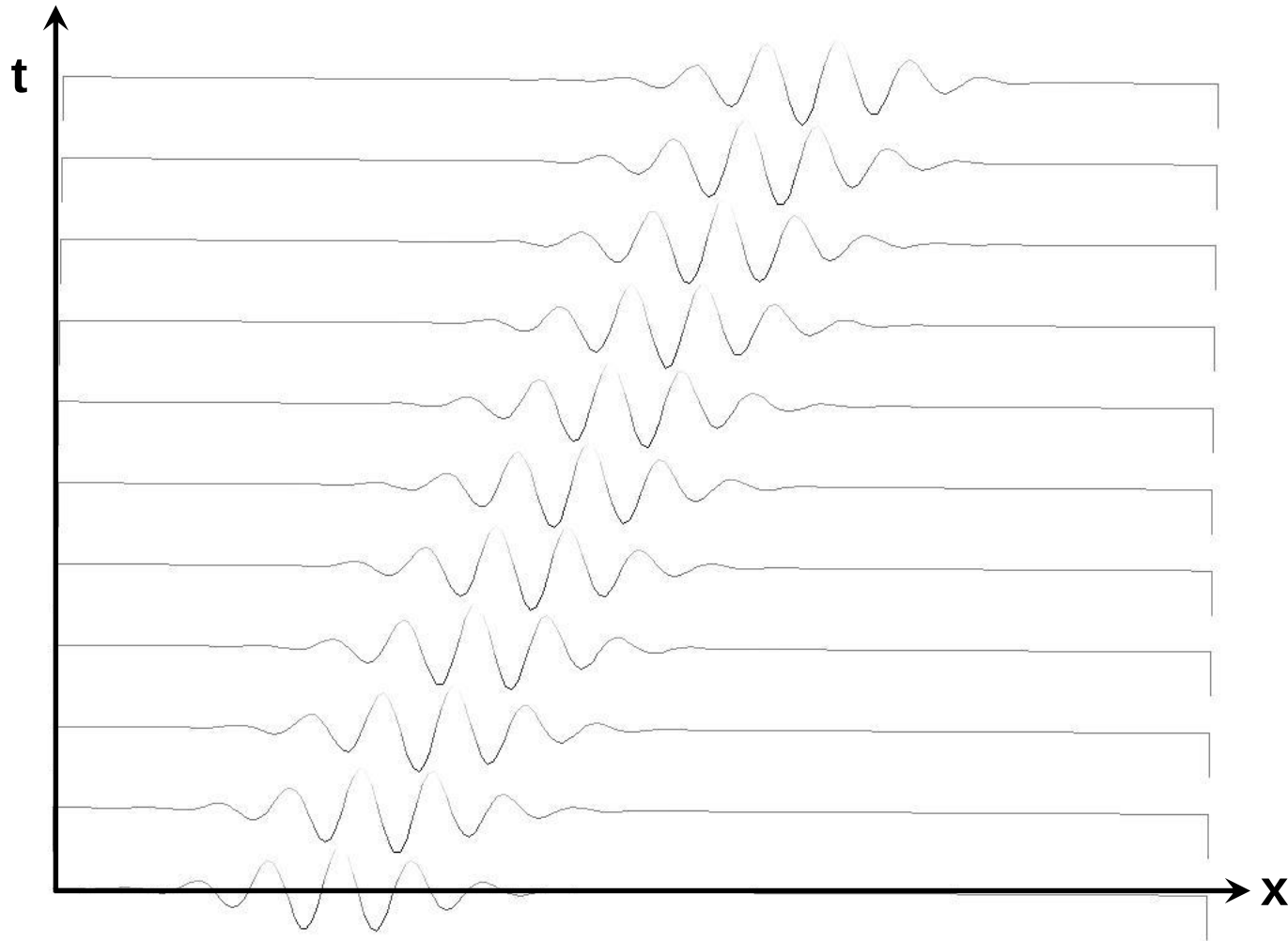
$$\tilde{\omega}^2 = \tanh(\tilde{k} \tilde{H}) \left(\tilde{k}^3 + \tilde{k} \right)$$

Dispersion relation

$$\tilde{\omega}^2 = \tanh(\tilde{k}\tilde{H}) \left(\tilde{k}^3 + \tilde{k} \right)$$

	gravity $\tilde{k} \ll 1$	capillary $\tilde{k} \gg 1$
shallow water $\tilde{k} \ll 1/\tilde{H}$	$\pm \tilde{k}$	$\pm \tilde{k}^2 \sqrt{\tilde{H}}$
Deep water $\tilde{k} \gg 1/\tilde{H}$	$\pm \sqrt{\tilde{k}}$	$\pm \tilde{k} \sqrt{\tilde{k}}$

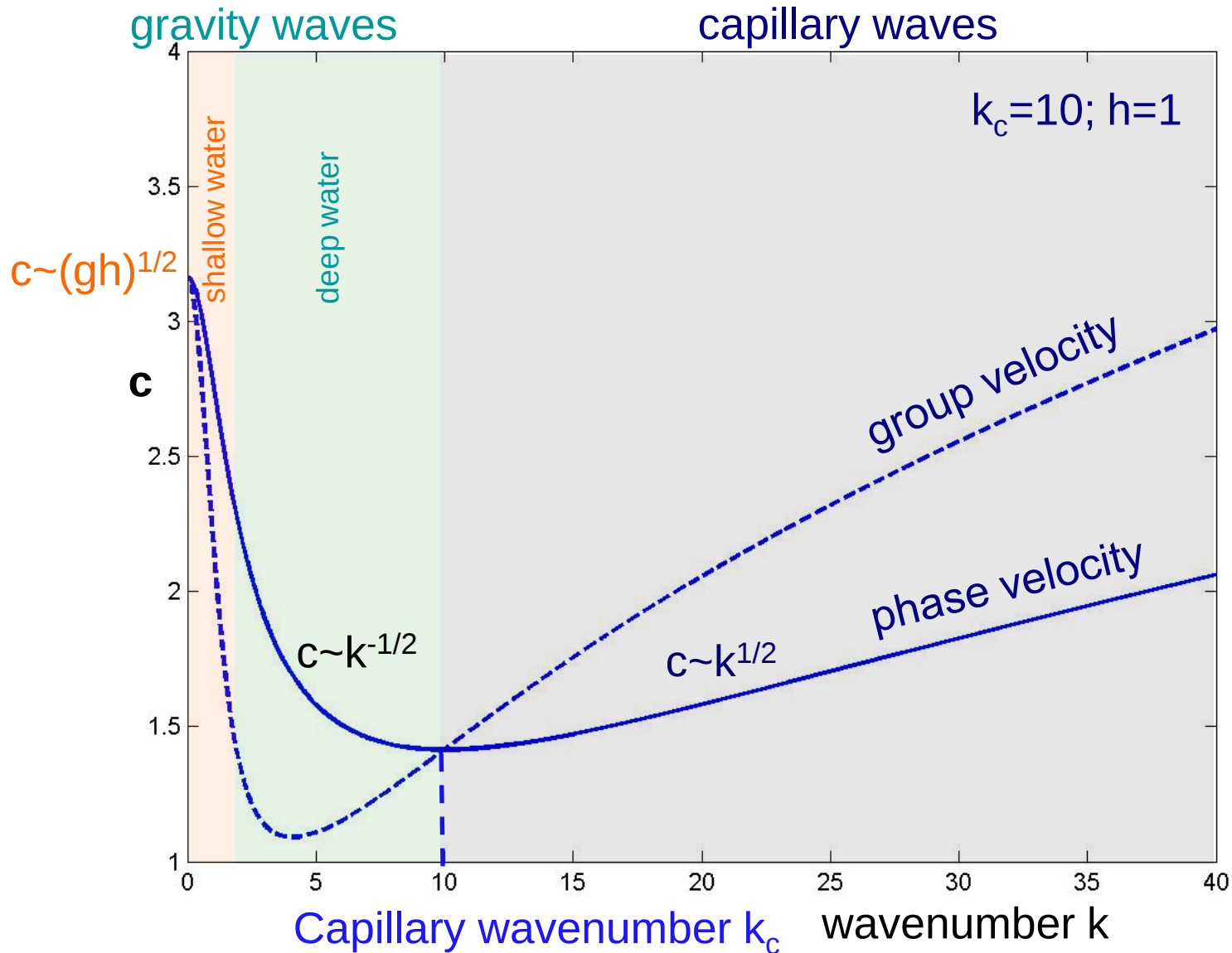
Difference between group velocity v and phase velocity c



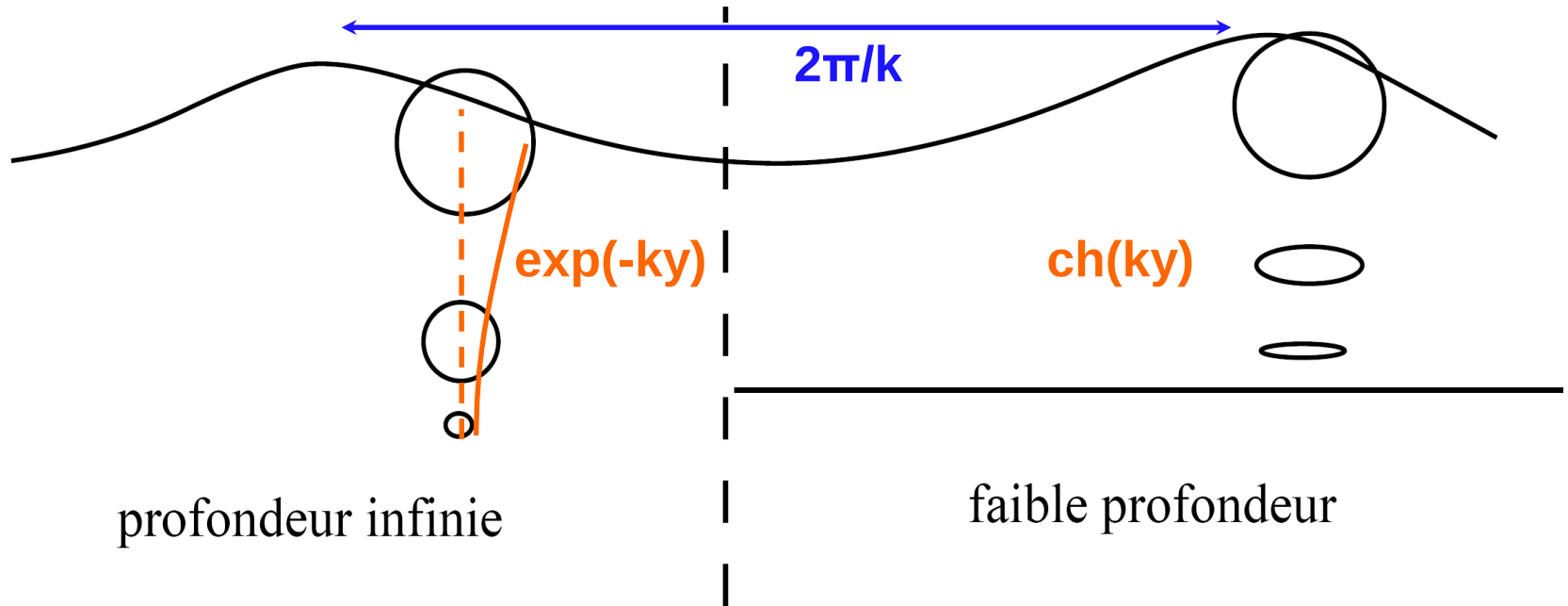
Dispersion relation

	gravity $\tilde{k} \ll 1$	capillary $\tilde{k} \gg 1$
shallow water $\tilde{k} \ll 1/\tilde{H}$	$\omega_{\text{shallow/gravity}} \sim \pm k \sqrt{gH}$ $c_{\text{shallow/gravity}} \sim \pm \sqrt{gH}$ $v_{\text{shallow/gravity}} \sim \pm \sqrt{gH}$	$\omega_{\text{shallow/capillary}} \sim \pm k^2 \sqrt{\gamma H / \rho}$ $c_{\text{shallow/capillary}} \sim \pm k \sqrt{\gamma H / \rho}$ $v_{\text{shallow/capillary}} \sim \pm 2k \sqrt{\gamma H / \rho}$
Deep water $\tilde{k} \gg 1/\tilde{H}$	$\omega_{\text{deep/gravity}} \sim \pm \sqrt{gk}$ $c_{\text{deep/gravity}} \sim \pm \sqrt{\frac{g}{k}}$ $v_{\text{deep/gravity}} \sim \pm \frac{1}{2} \sqrt{\frac{g}{k}}$	$\omega_{\text{deep/capillary}} \sim \pm k^{3/2} \sqrt{\gamma / \rho}$ $c_{\text{deep/capillary}} \sim \pm k^{1/2} \sqrt{\gamma / \rho}$ $v_{\text{deep/capillary}} \sim \pm 3/2 k^{1/2} \sqrt{\gamma / \rho}$

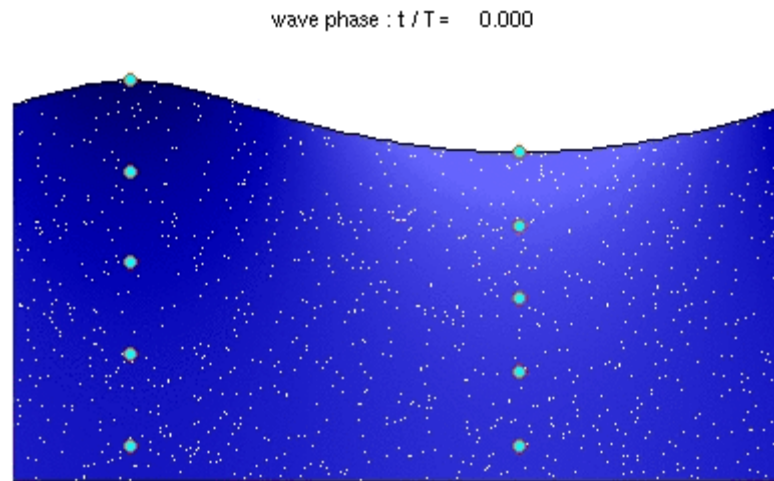
Dispersion relation



Trajectories below the waves



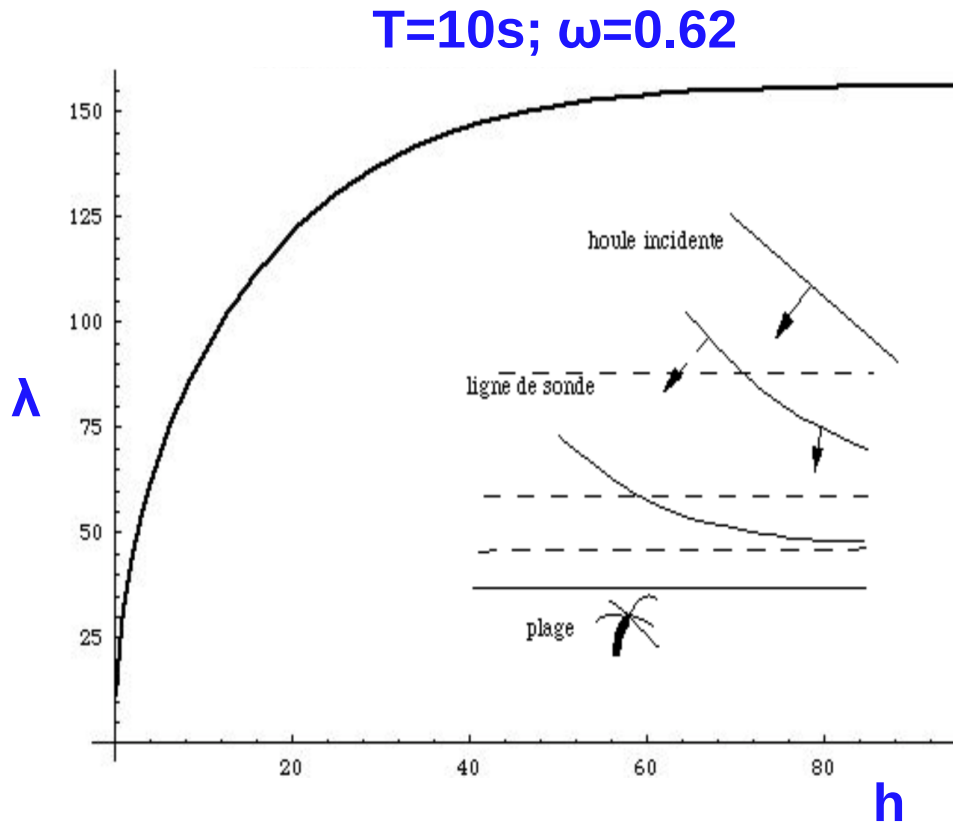
Stokes drift!



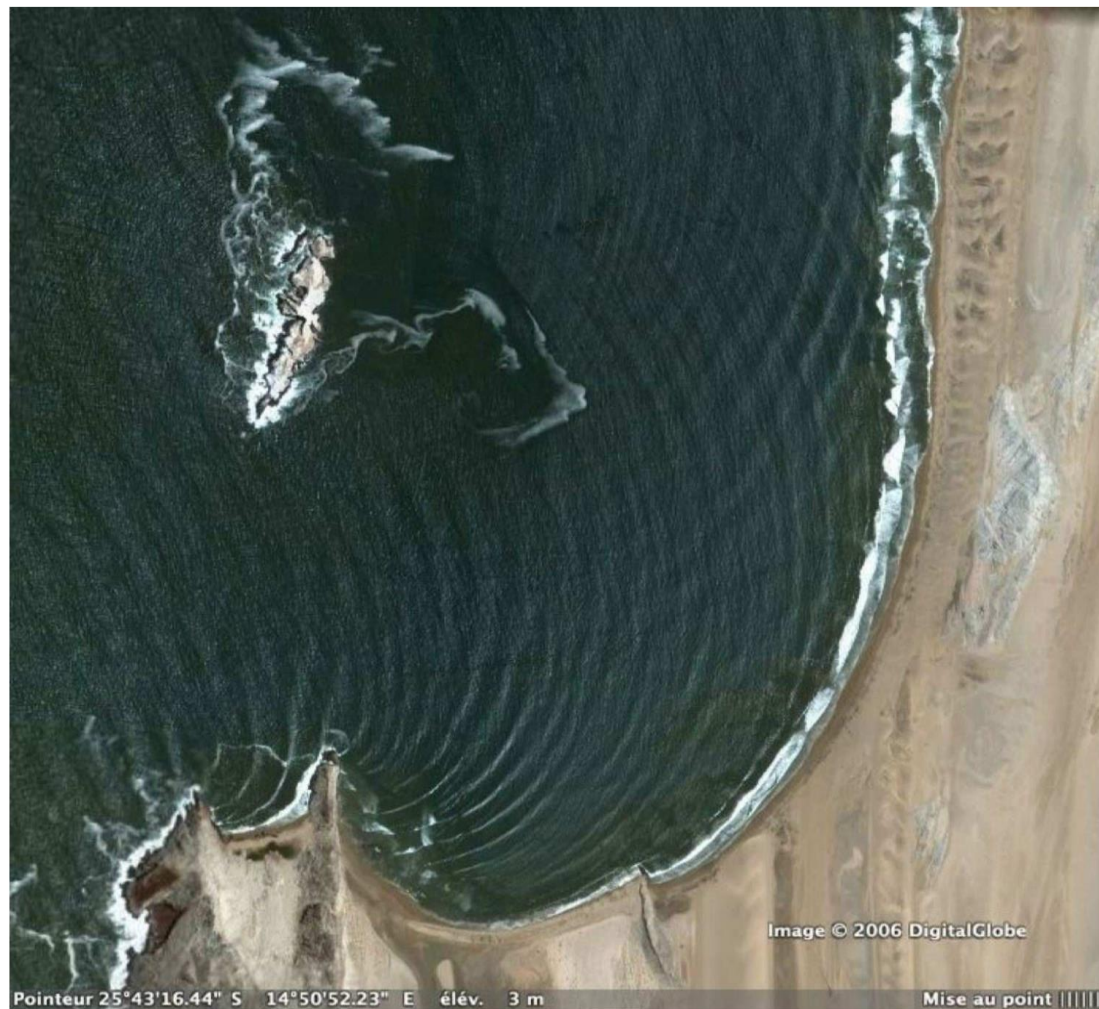
Why are the waves parallel to the shore?

$$c \sim (gh)^{1/2}$$

$$\lambda \sim T(gh)^{1/2}$$

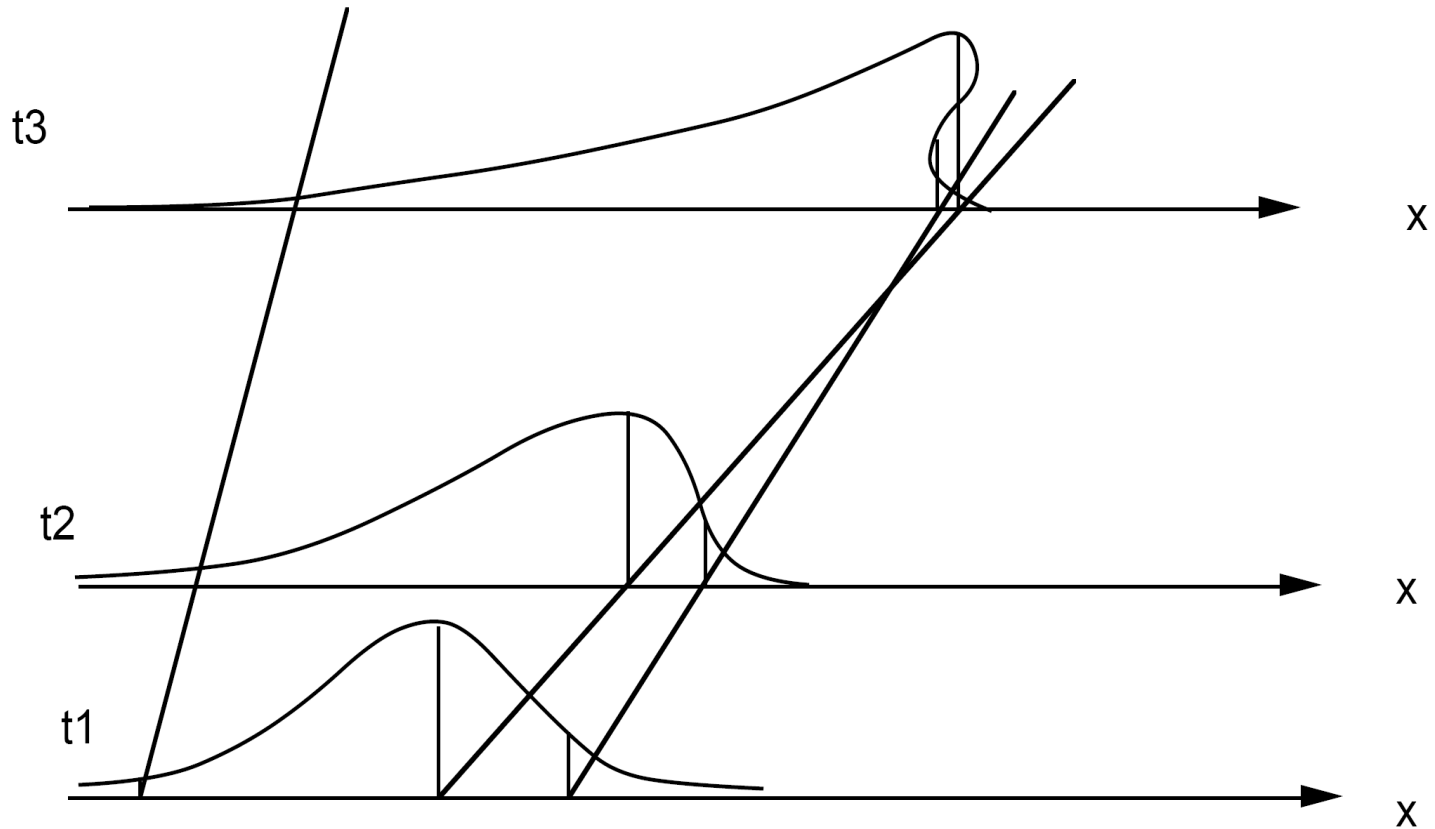


Refraction and diffraction of waves



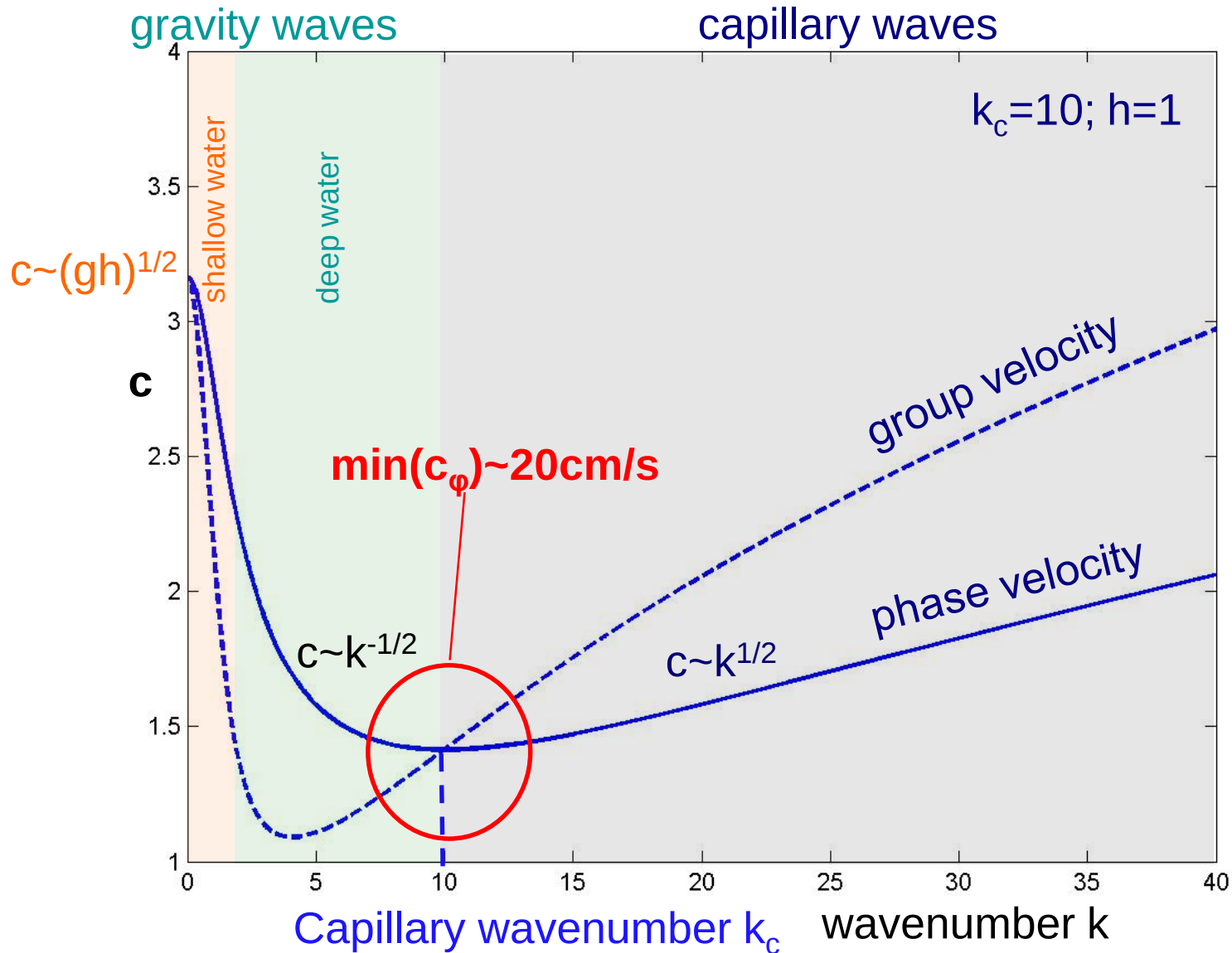
Satellite view Namibian coast

Nonlinear waves, wavebreaking



The celerity increases with the depth

Dispersion relation



Conditions for wave pattern formation?



$$V_{\text{duck}} \lesseqgtr c_{\text{min}} \quad ?$$

Dispersion relation

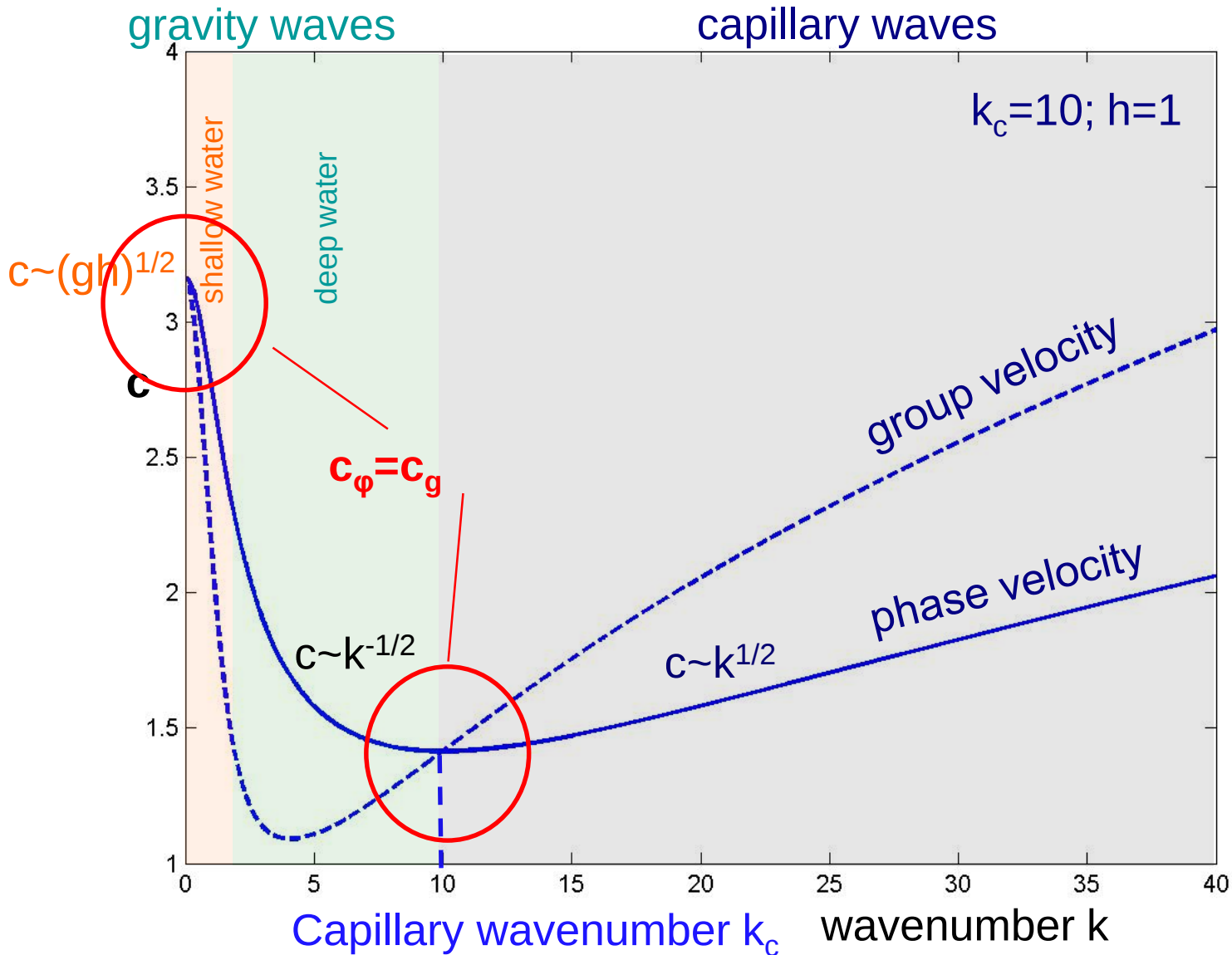


Diagramme spatio-temporel

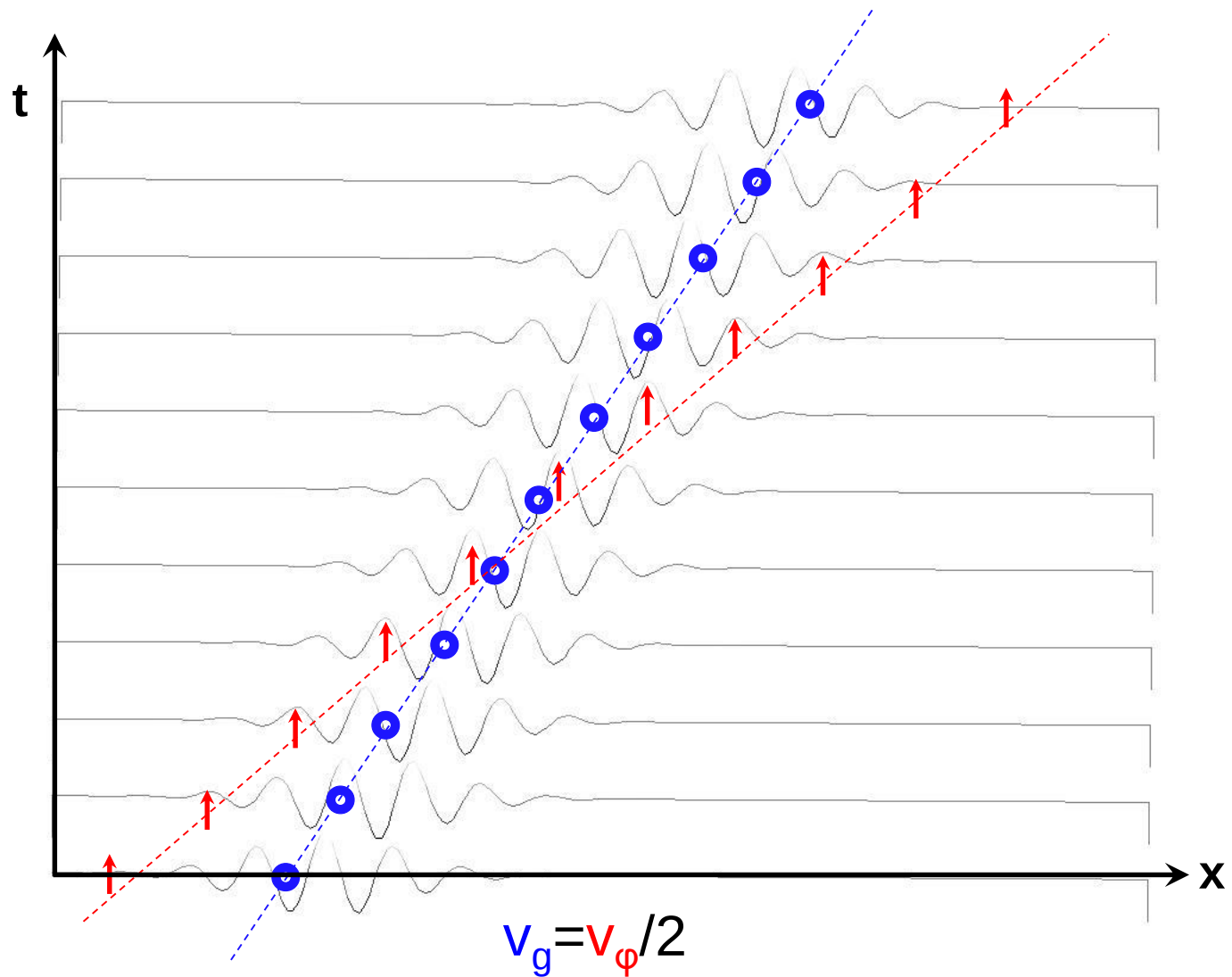


Diagramme spatio-temporel

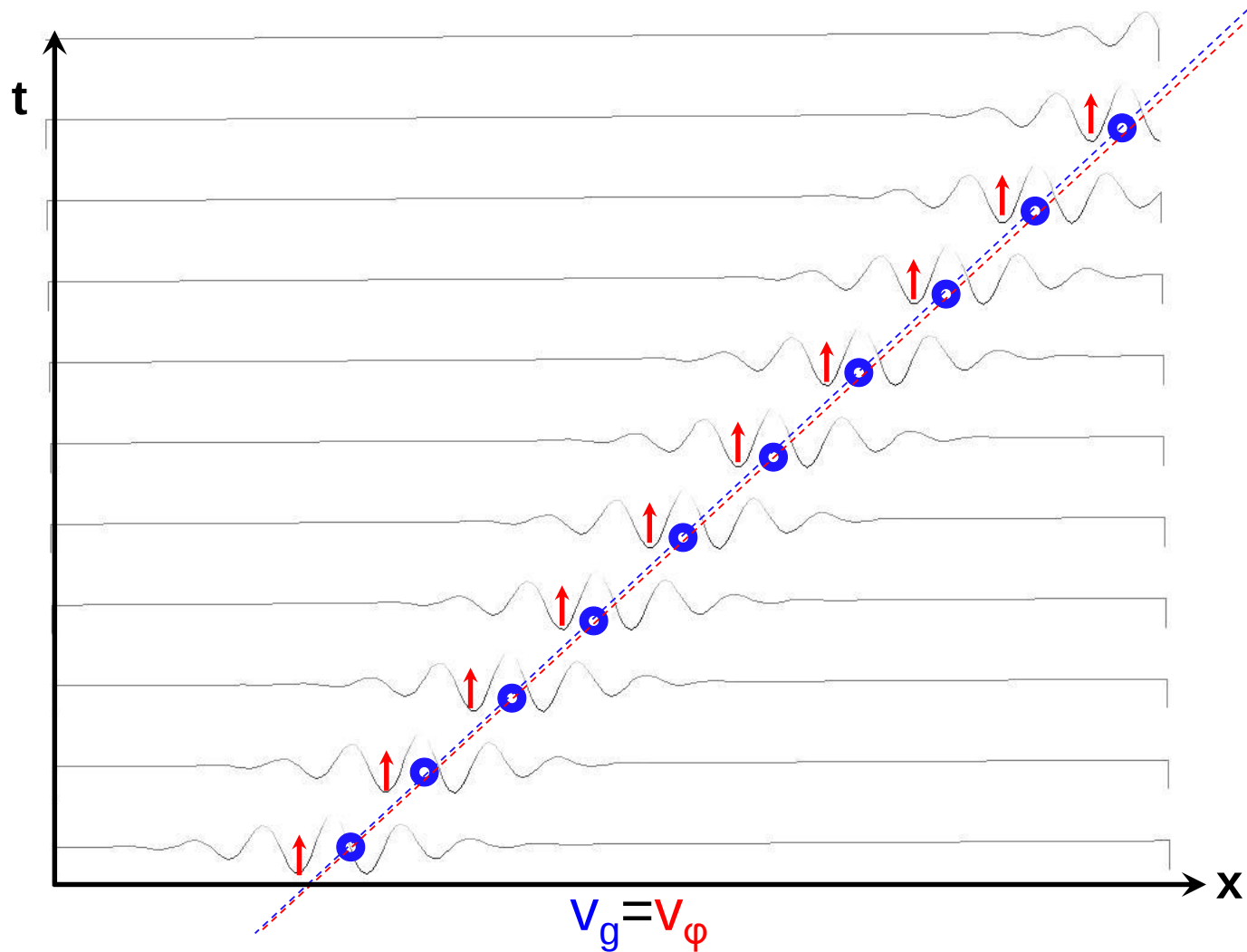
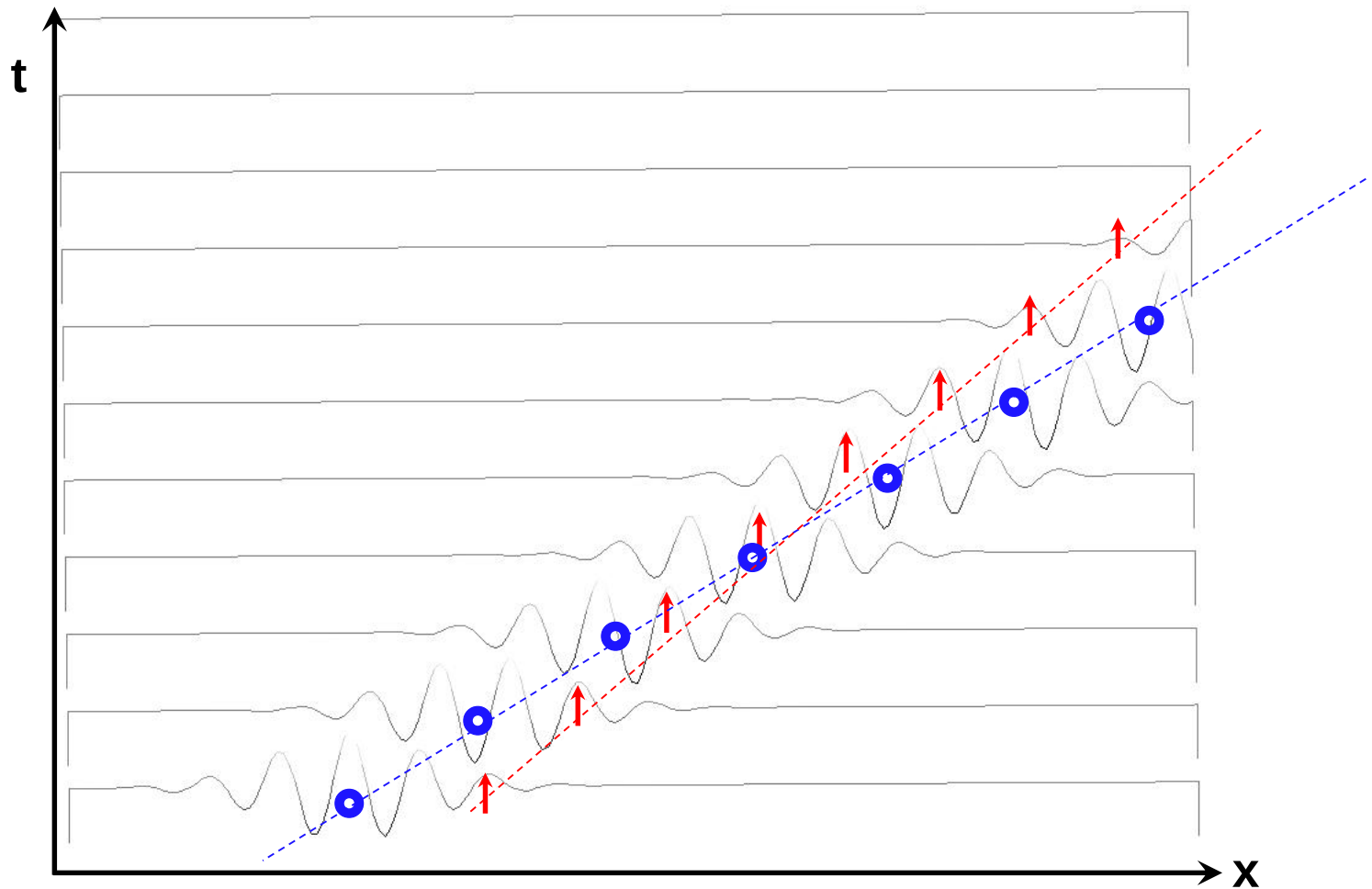


Diagramme spatio-temporel



$$v_g = \frac{3}{2} v_\phi$$

Spectral analysis

Fourier transform:

$$u(x, t) = \frac{1}{2} \int_0^{+\infty} \hat{u}(k) e^{i(kx - \omega(k)t)} dk + c.c.$$

$\hat{u}(k)$ is given by Fourier transform at time $t=0$

Carrier/enveloppe :

$$u(x, t) = \frac{1}{2} A(x, t) e^{i(k_0 x - \omega_0 t)} + c.c.$$

Enveloppe :

$$A(x, t) = \int_0^{\infty} \hat{u}(k) e^{i(k-k_0)x - i(\omega-\omega_0)t} dk.$$

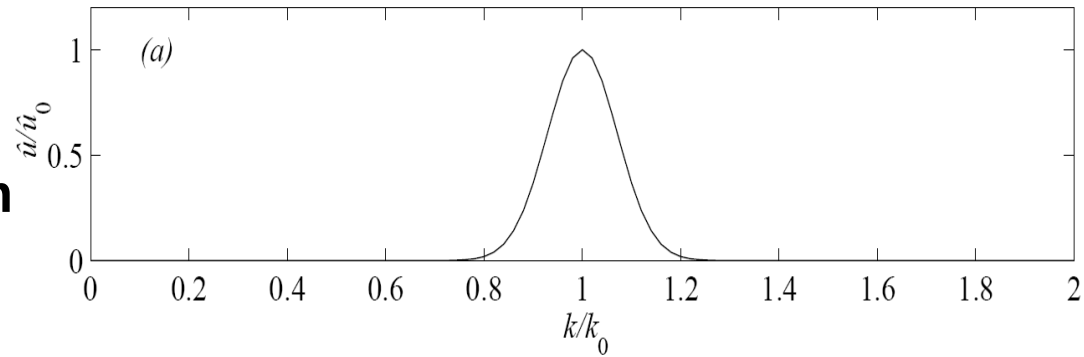
Spectral analysis

Gaussian spectrum: $\hat{u}(k) = u_0 e^{-\sigma^2(k-k_0)^2}$

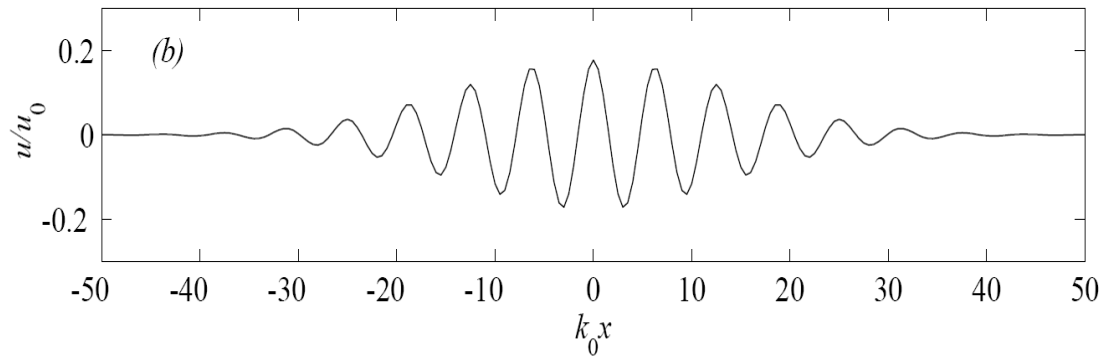
Initial envelope : $A(x, 0) = \frac{u_0 \sqrt{\pi}}{2\sigma} e^{-\frac{x^2}{4\sigma^2}}$

Gaussian spectrum

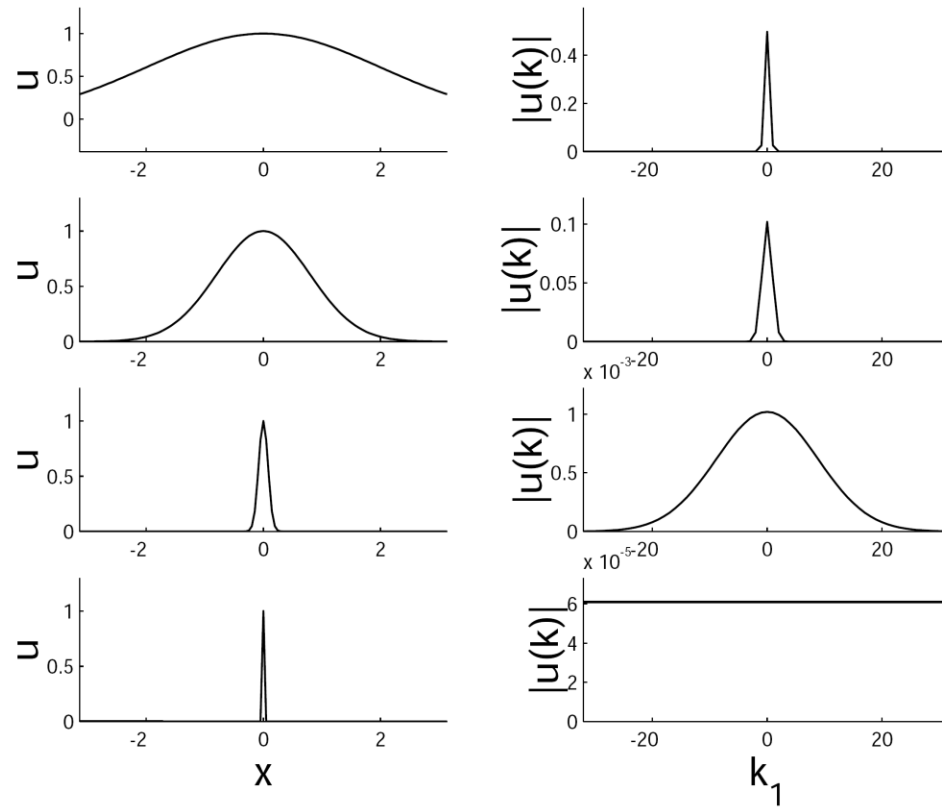
spectrum



wave

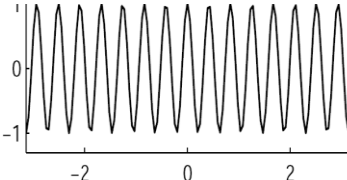
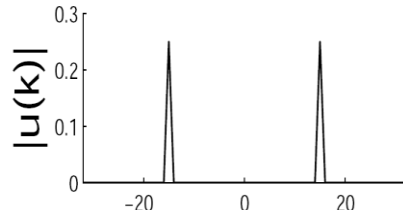
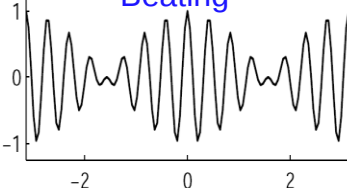
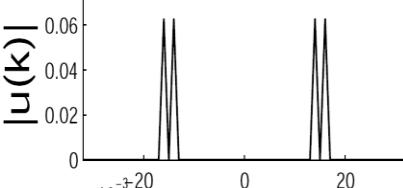
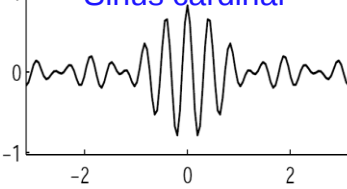
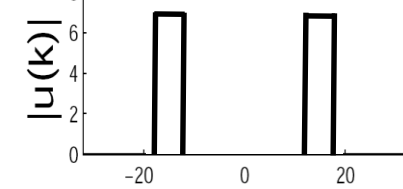
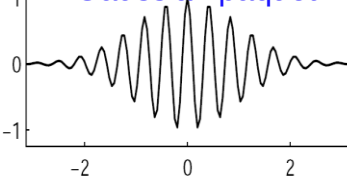
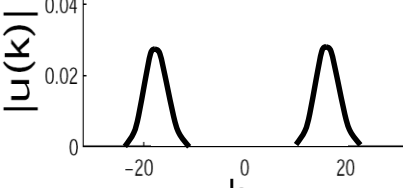


Paquets d'ondes gaussiens



$u_0(x)$	$\hat{u}_0(k_1)$
$\exp\left(-\frac{x^2}{2\sigma^2}\right)$	$\frac{\sigma}{\sqrt{2\pi}} \exp\left(-\frac{\sigma^2 k^2}{2}\right)$
$\delta(x)$	$\frac{1}{2\pi}$

Waves and spectrum

$2 \cos(k_0 x)$	Monochromatic wave  	$\delta(k_1 + k_0) + \delta(k_1 - k_0)$
$\cos[(k_0 - \kappa)x] + \cos[(k_0 + \kappa)x]$ $= 2 \cos(k_0 x) \cos(\kappa x)$	Beating  	$\delta(k_1 + k_0 + \kappa) + \delta(k_1 - k_0 - \kappa)$ $+ \delta(k_1 + k_0 - \kappa) + \delta(k_1 - k_0 + \kappa)$
$\int_{k_0 - \kappa}^{k_0 + \kappa} \cos(k_1 x) dk_1$ $= 4 \kappa \cos(k_0 x) \frac{\sin(\kappa x)}{\kappa x}$	Sinus cardinal  	$= 1 \text{ pour } k_1 \pm k_0 \leq \kappa$ $= 0 \text{ sinon}$
$2 \kappa \sqrt{2\pi} \cos(k_0 x) \exp\left(-\frac{\kappa^2 x^2}{2}\right)$	Gaussian paquet  	$\exp\left[-\frac{(k_1 + k_0)^2}{2 \kappa^2}\right] + \exp\left[-\frac{(k_1 - k_0)^2}{2 \kappa^2}\right]$

Spectral analysis

$$A(x, t) = \int_0^{\infty} \hat{u}(k) e^{i(k-k_0)x - i(\omega - \omega_0)t} dk.$$

Spectral analysis

$$A(x, t) = \int_0^\infty \hat{u}(k) e^{i(k-k_0)x - i(\omega - \omega_0)t} dk.$$

Definition of group velocity $\omega - \omega_0 = c_g(k - k_0), \quad c_g = \frac{\partial \omega}{\partial k}(k_0)$

Spectral analysis

$$A(x, t) = \int_0^{\infty} \hat{u}(k) e^{i(k-k_0)x - i(\omega - \omega_0)t} dk.$$

$$\hat{u}(k) = u_0 e^{-\sigma^2 (k-k_0)^2}$$

Spectral analysis

Gaussian spectrum: $\hat{u}(k) = u_0 e^{-\sigma^2(k-k_0)^2}$

Initial envelope : $A(x, 0) = \frac{u_0 \sqrt{\pi}}{2\sigma} e^{-\frac{x^2}{4\sigma^2}}$

Spectral analysis

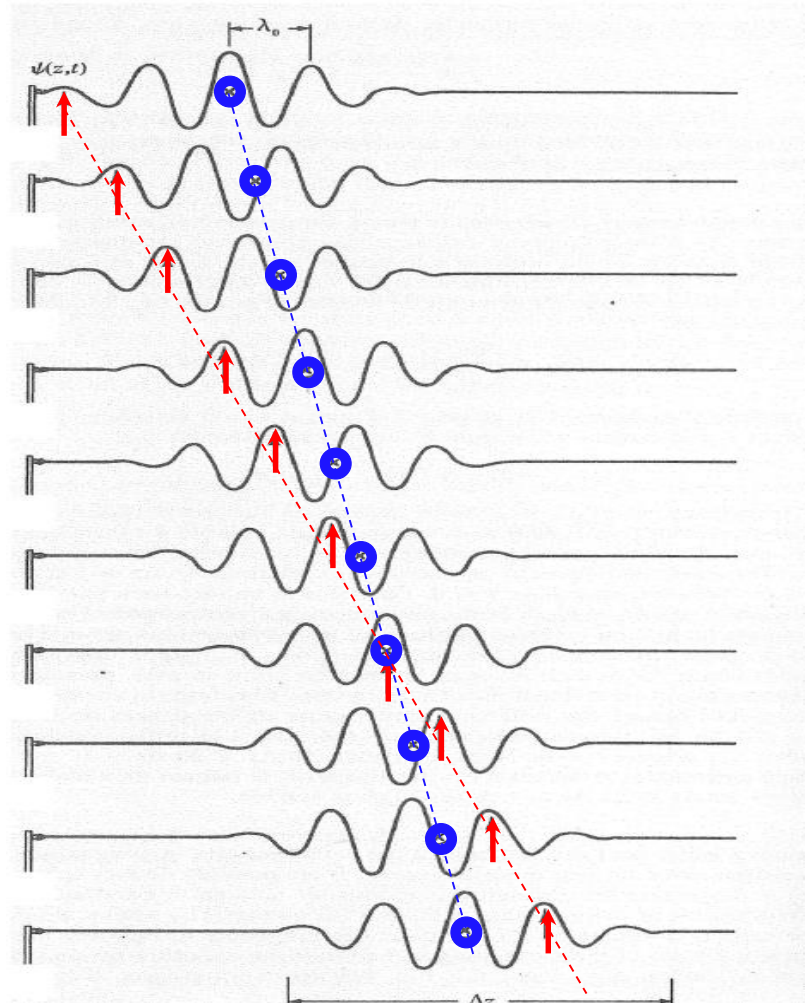
$$A(x, t) = \int_0^\infty \hat{u}(k) e^{i(k-k_0)x - i(\omega - \omega_0)t} dk.$$

Definition of group velocity $\omega - \omega_0 = c_g(k - k_0), \quad c_g = \frac{\partial \omega}{\partial k}(k_0)$

$$\hat{u}(k) = u_0 e^{-\sigma^2(k-k_0)^2}$$

$$A(x, t) = \frac{u_0 \sqrt{\pi}}{2\sigma} e^{-\frac{(x - c_g t)^2}{4\sigma^2}}$$

Group velocity



Wave packet

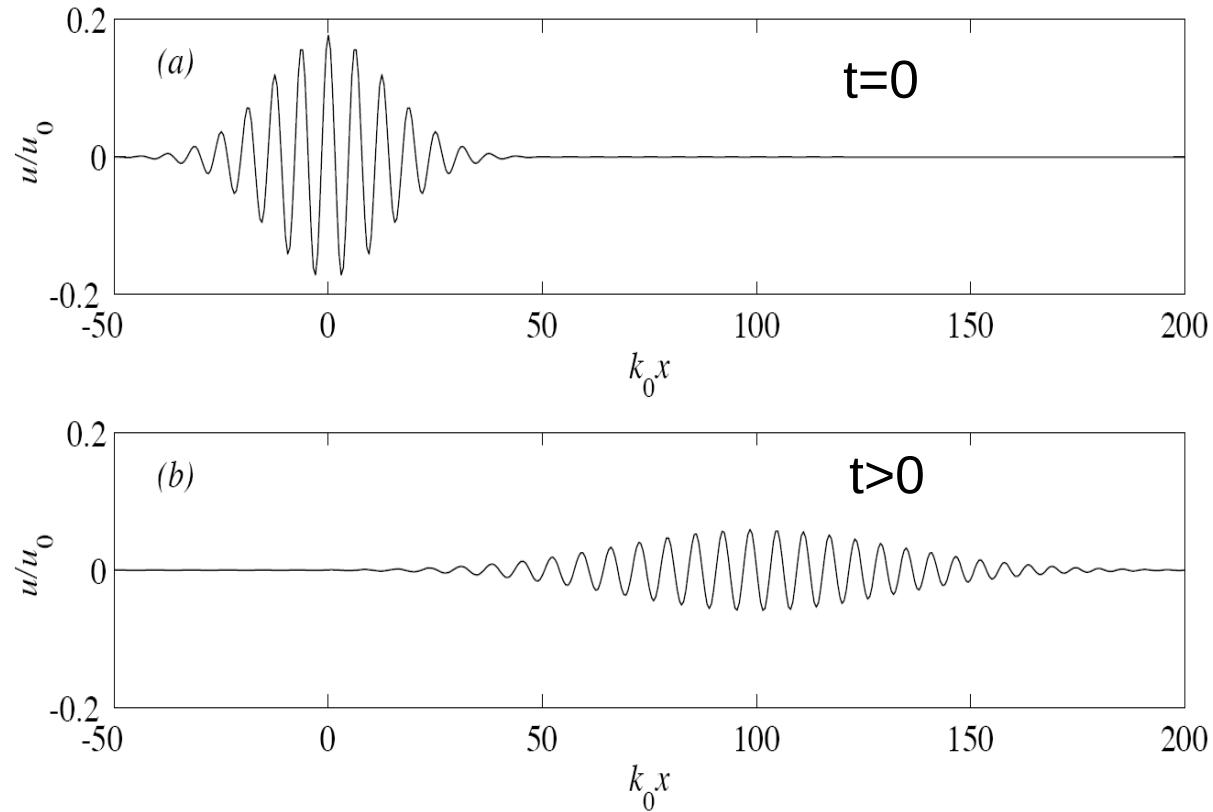
Spectral analysis

Higher order
development

$$\omega - \omega_0 = c_g(k - k_0) + \frac{\omega_0''}{2}(k - k_0)^2$$
$$c_g = \frac{\partial \omega}{\partial k}(k_0), \quad \omega_0'' = \frac{\partial^2 \omega}{\partial k^2}(k_0)$$

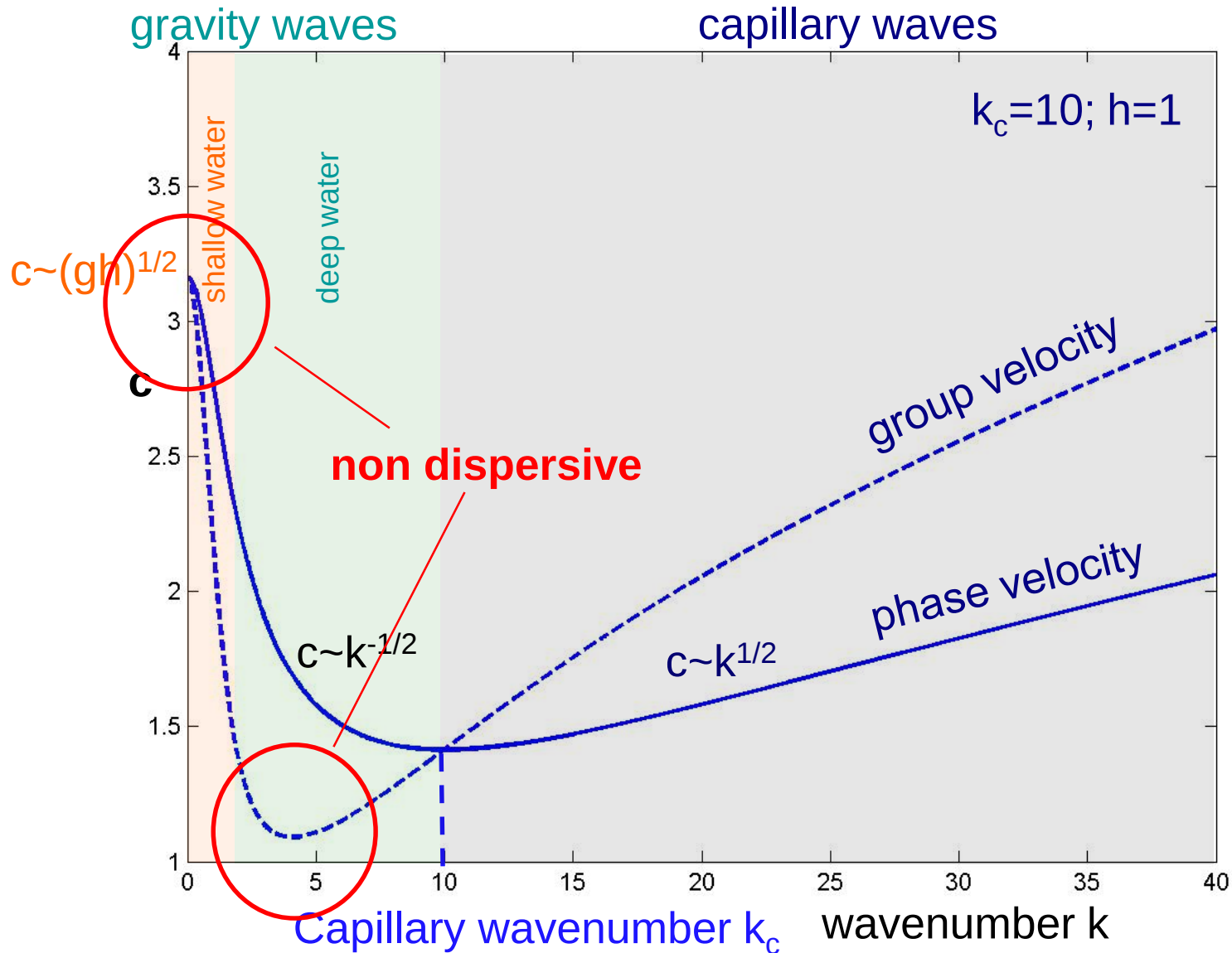
$$A(x, t) = \frac{u_0}{2} \sqrt{\frac{\pi}{\sigma^2 + \frac{1}{2}i\omega_0''t}} \exp\left(-\frac{(x - c_g t)^2}{4(\sigma^2 + \frac{1}{2}i\omega_0''t)}\right)$$

Wave packet dispersion

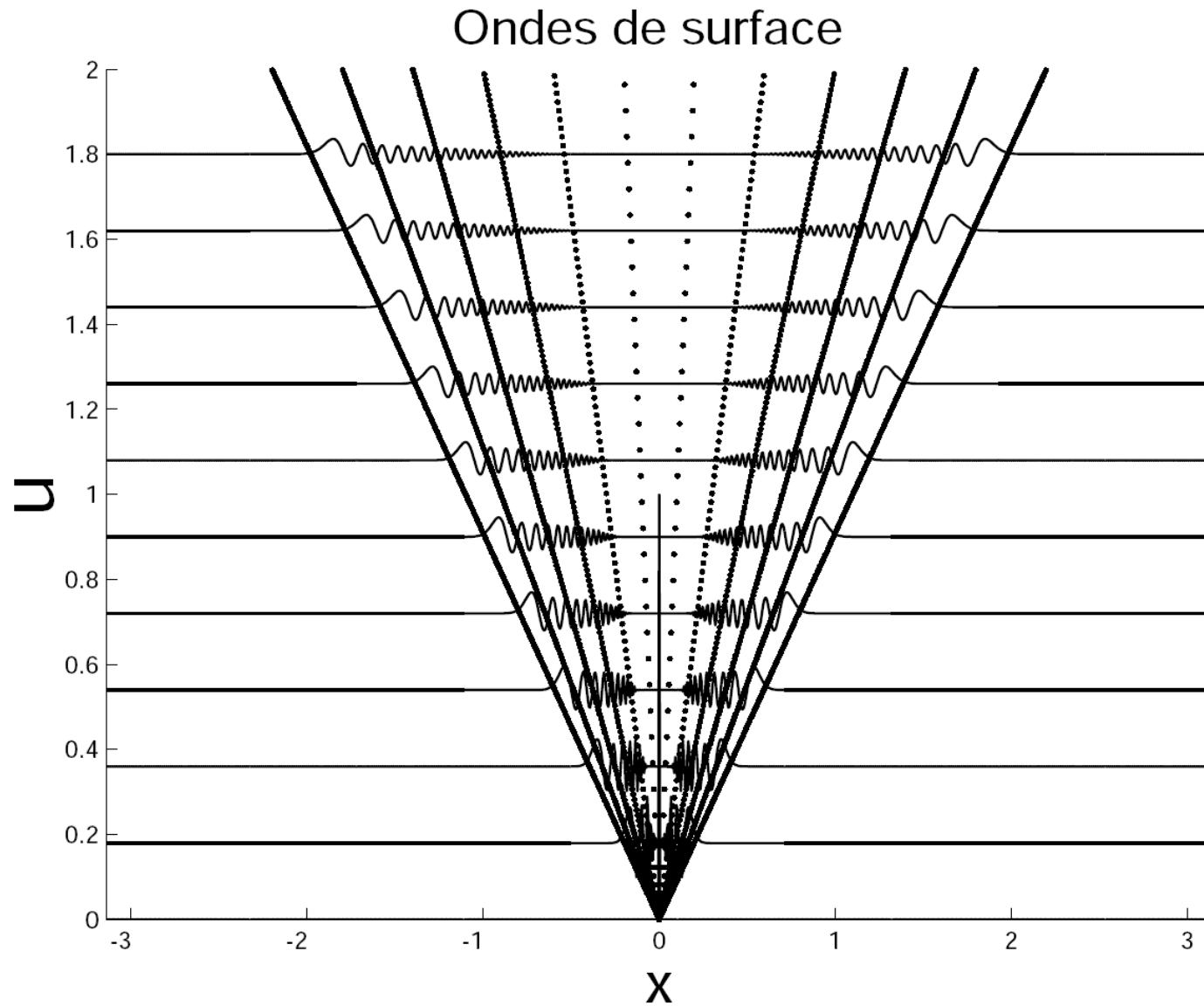


Onde correspondant à l'enveloppe $\hat{u}(k)$ pour $\sigma^{-1}k_0 = 0,1$ et $\omega_0'' = 4c_g/k_0$: (a), instant initial $t = 0$; (b), $c_g t = 100/k_0$.

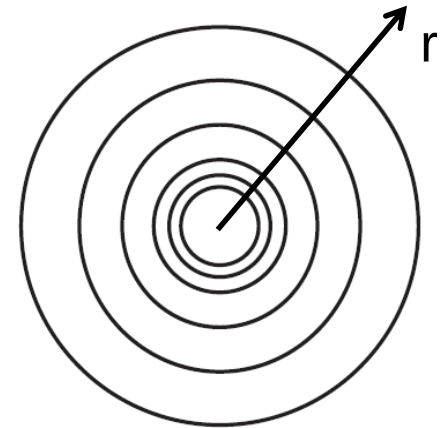
Dispersion relation



Dispersion



Dispersion



Waves with k reach r at time $t=r/v(k)$

For deep gravity waves: $v_{\text{deep/gravity}} \sim \pm \frac{1}{2} \sqrt{\frac{g}{k}}$

$$k = gt^2/4r^2$$

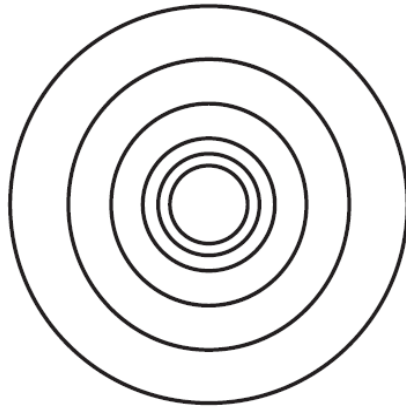
Since $\omega_{\text{deep/gravity}} \sim \pm \sqrt{gk}$

$$\omega = gt/2r$$

⇒ The frequency increases with time

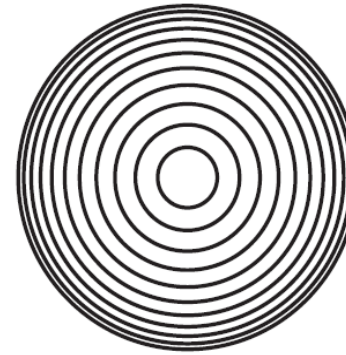
“Ronds dans l’eau”

Gravity
waves

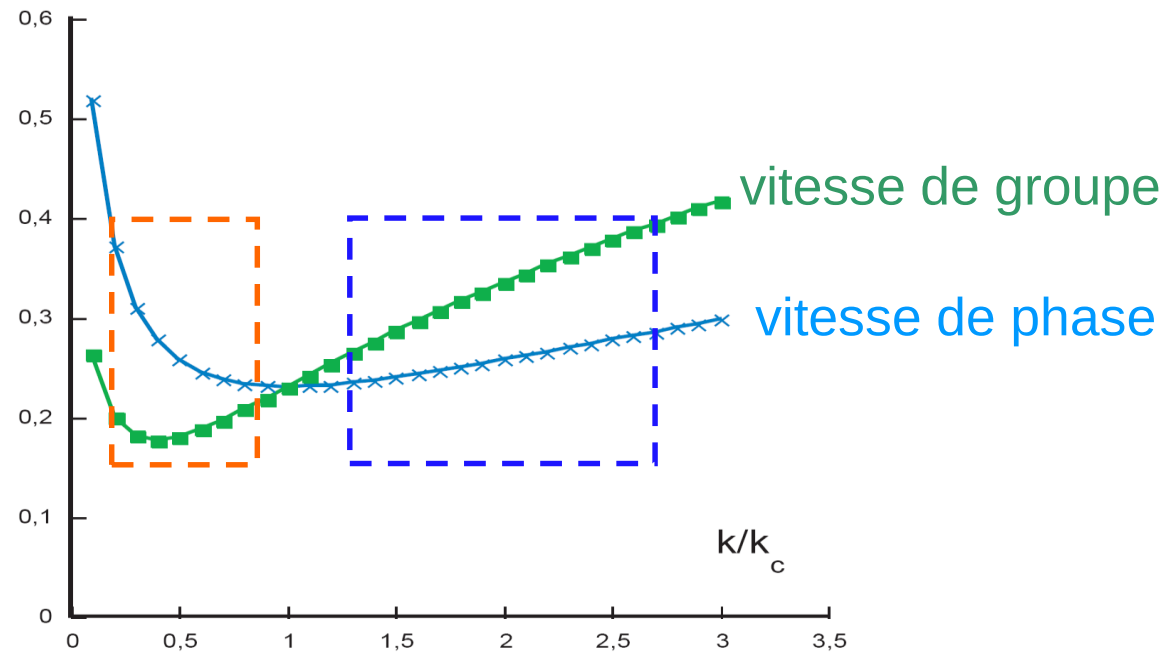


stone $> l_c$

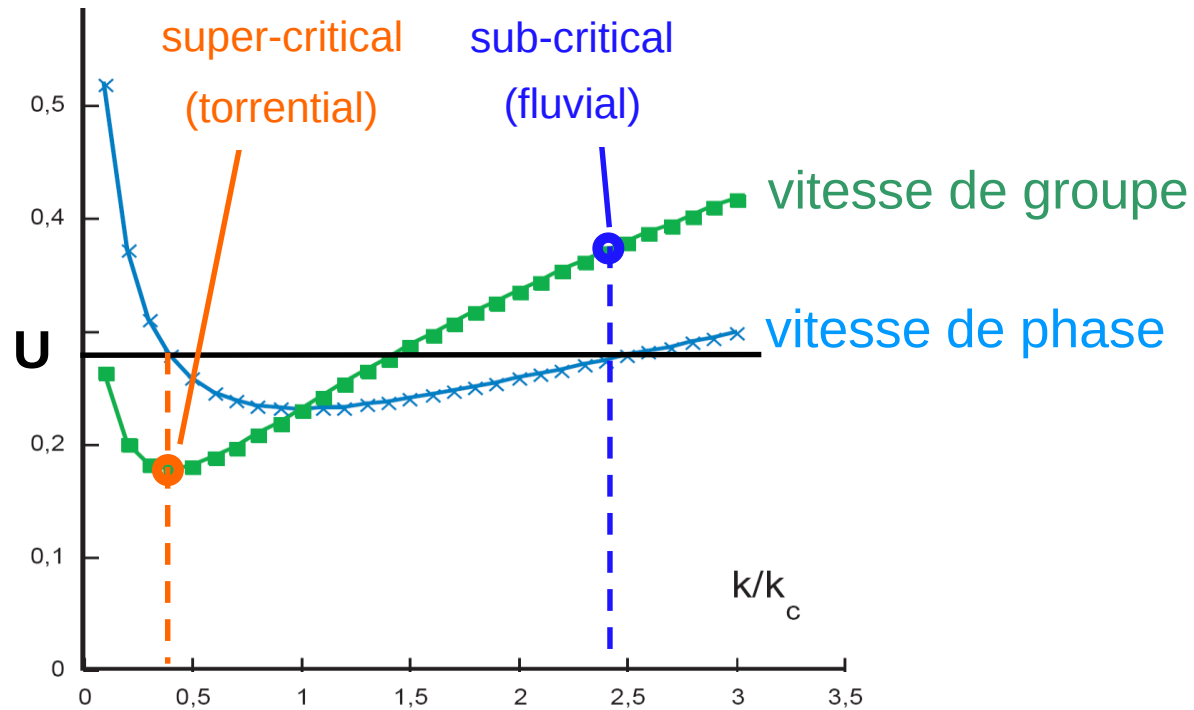
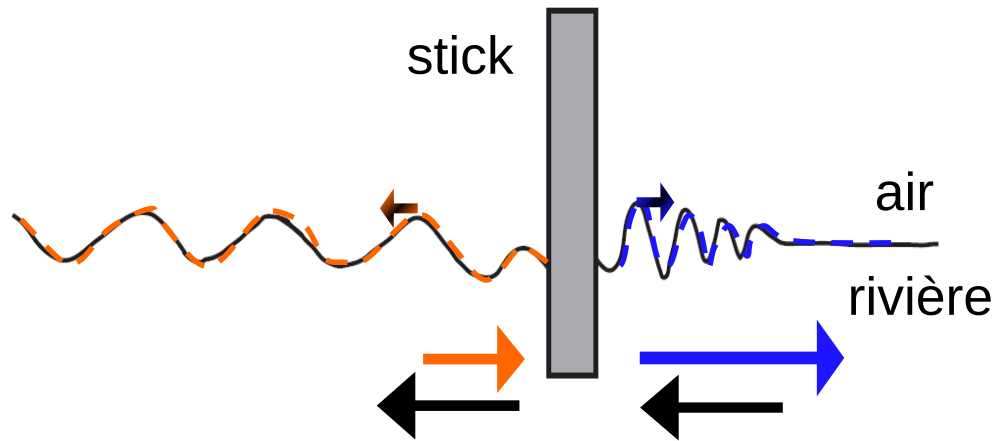
Capillary
waves



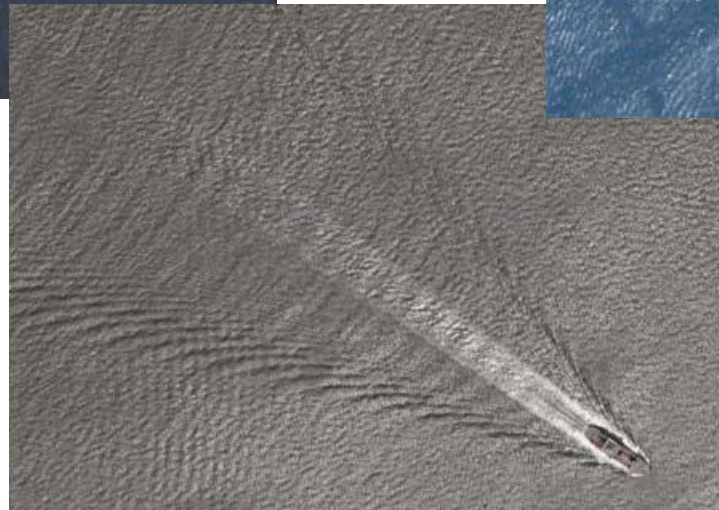
water drop $< l_c$



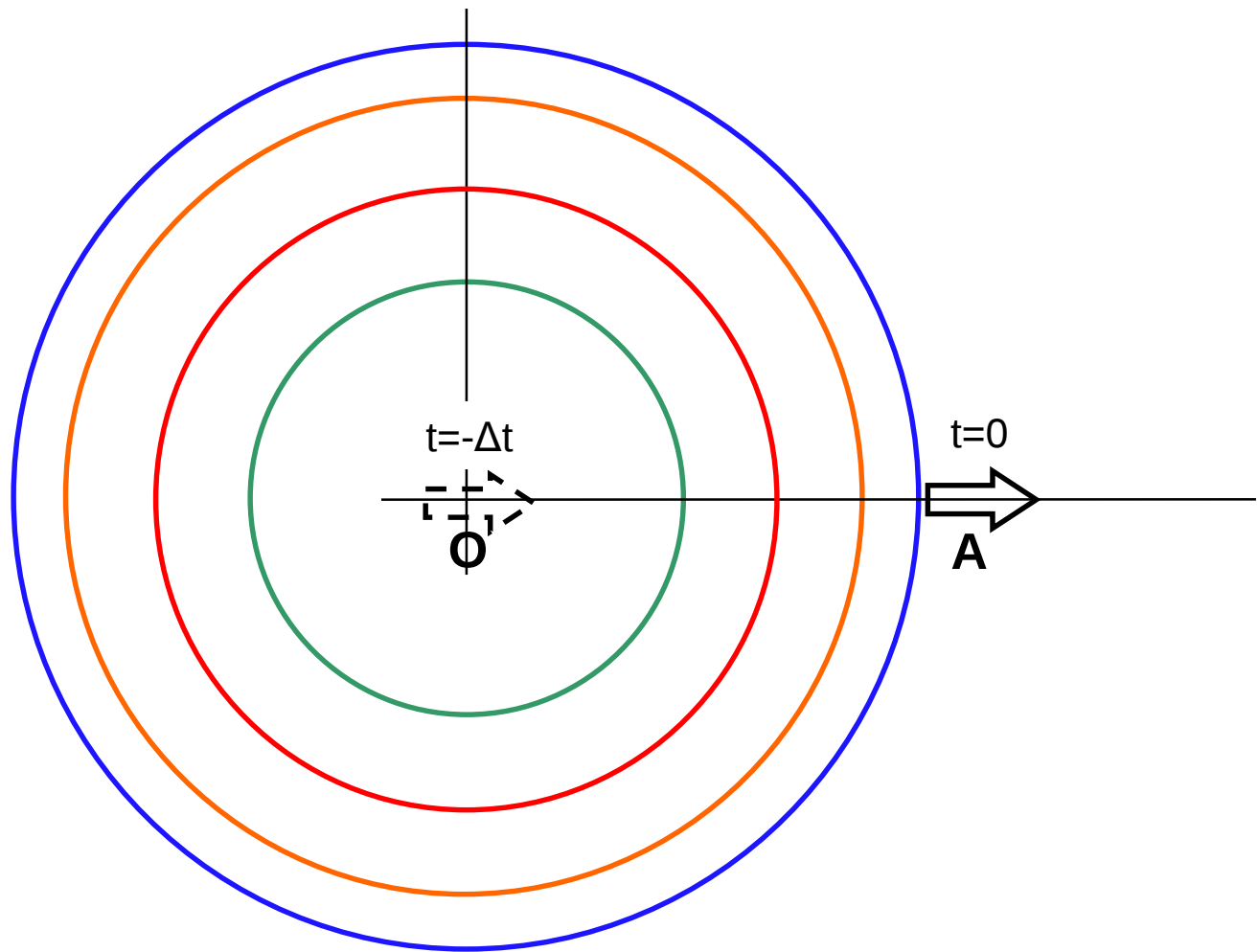
Waves created by an obstacle in a river



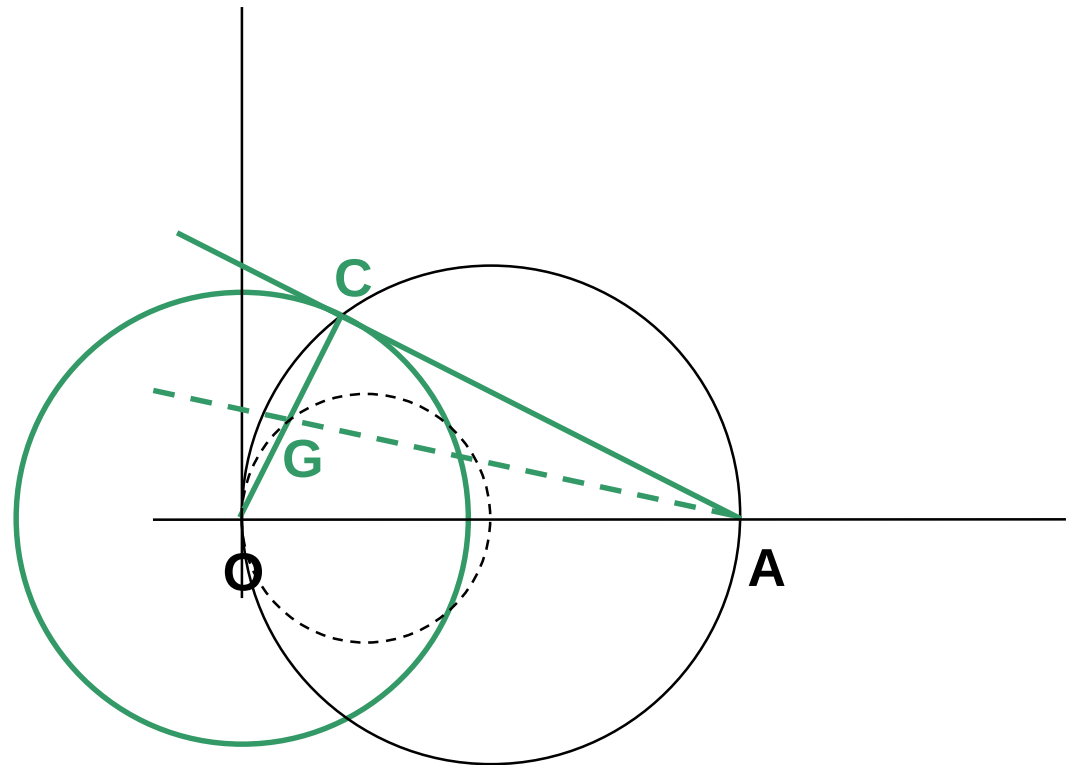
Kelvin wakes



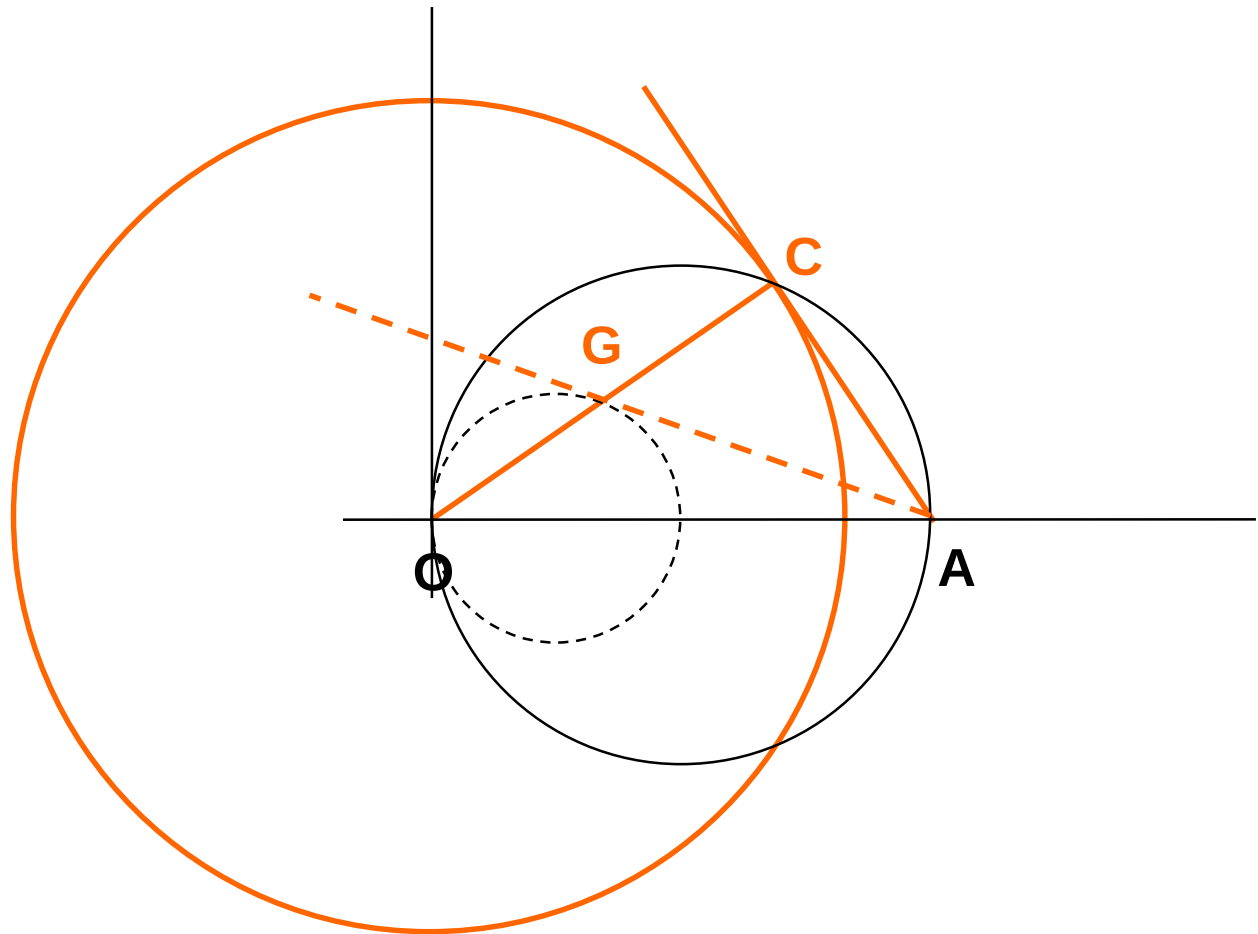
Kelvin wake



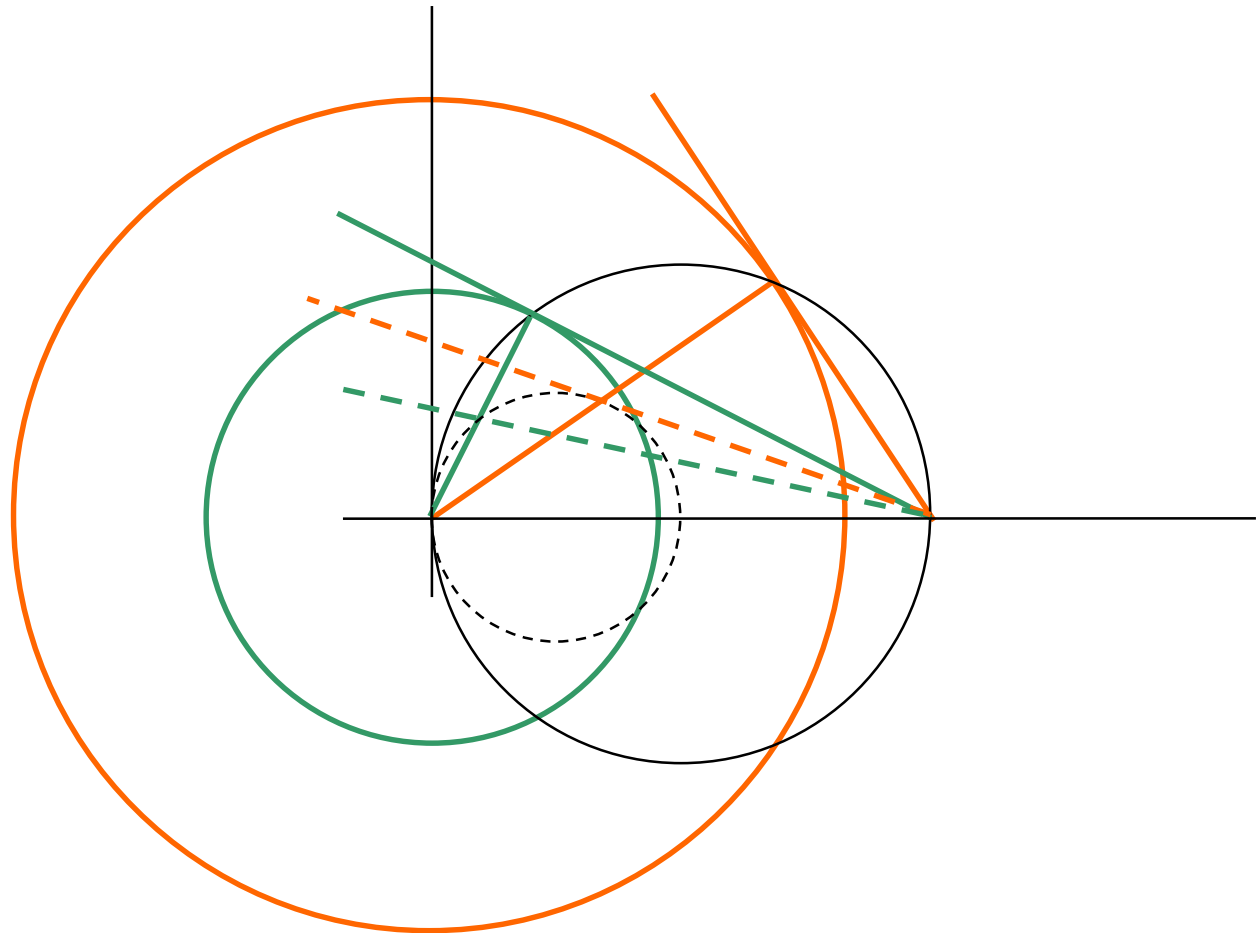
Waves created by a ship



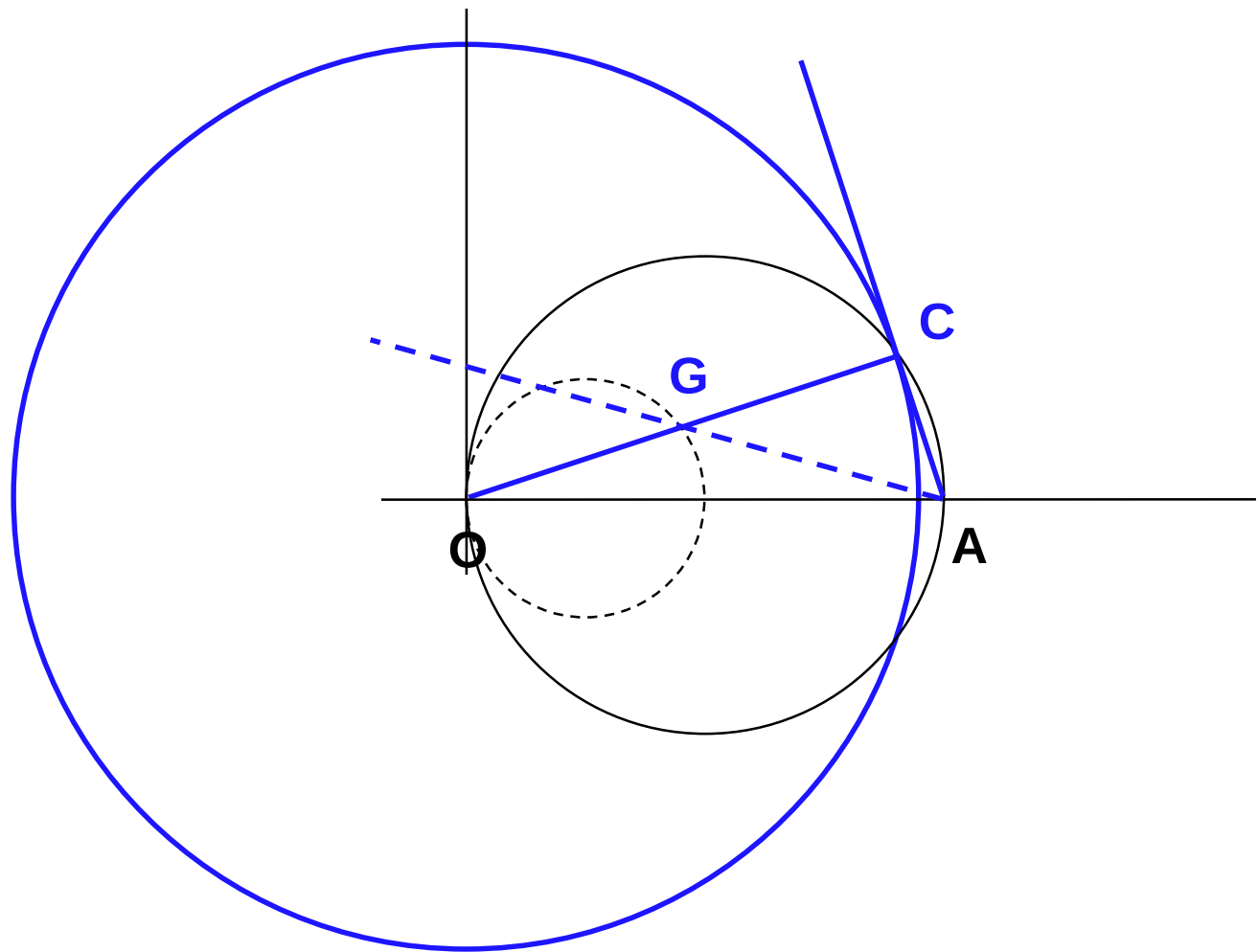
Waves created by a ship



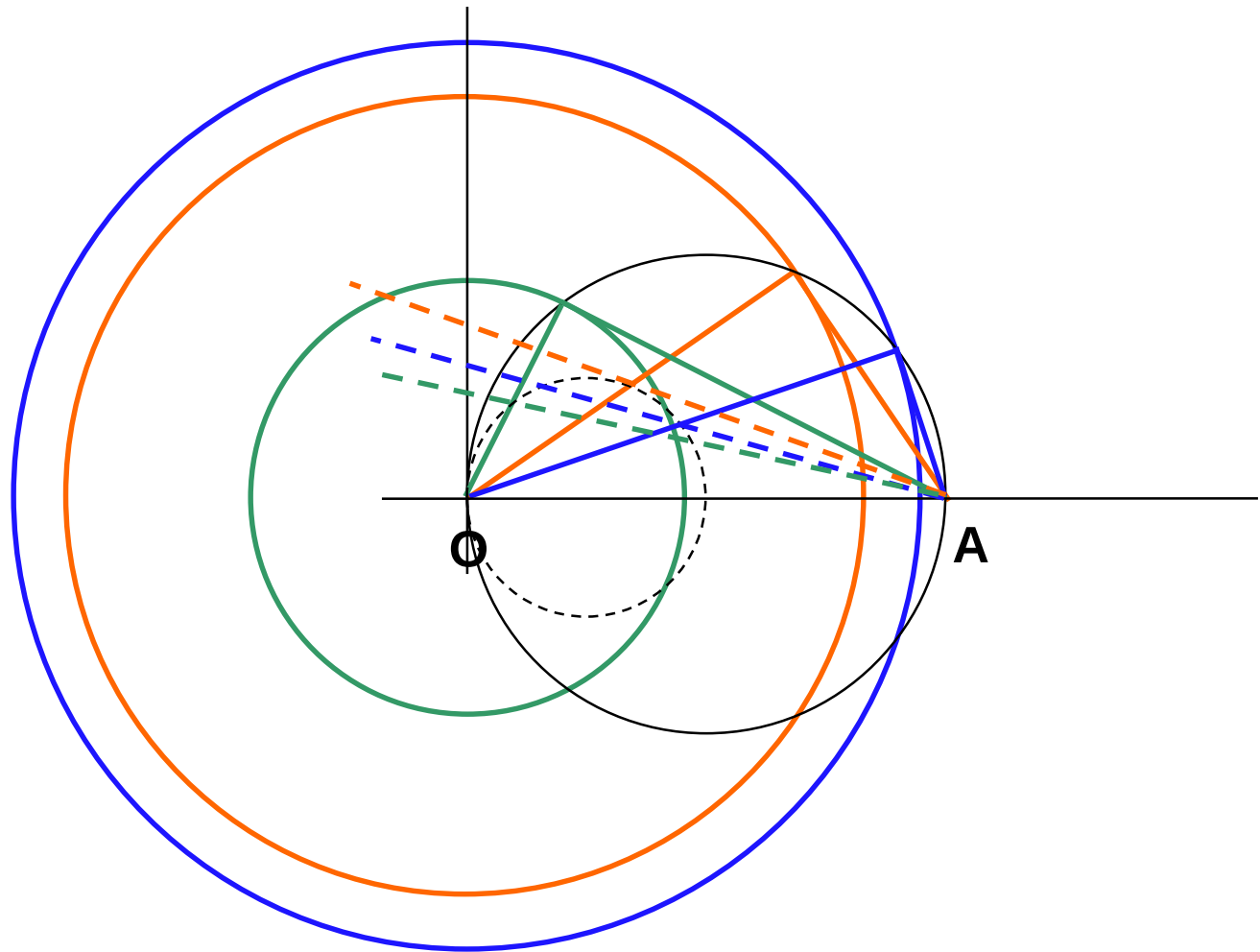
Waves created by a ship



Waves created by a ship



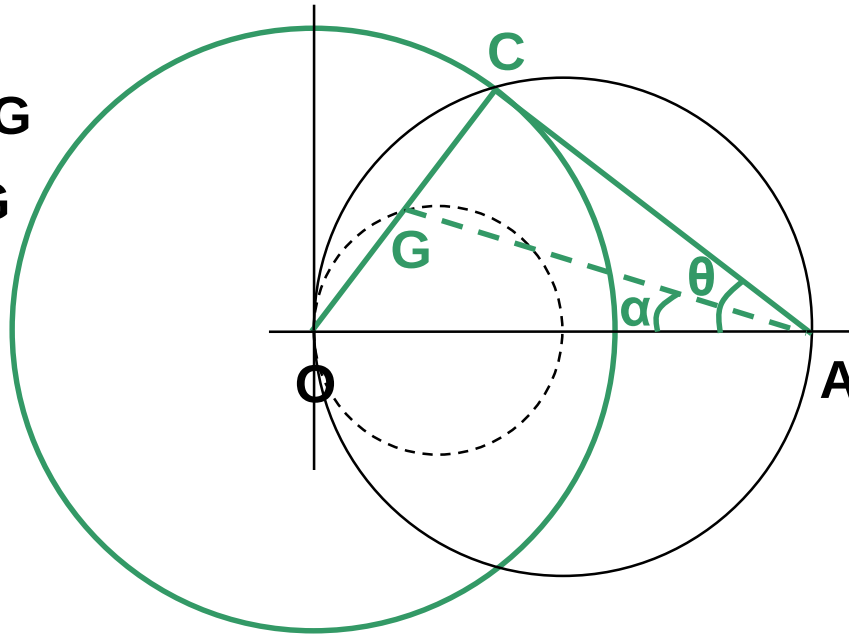
Waves created by a ship



Waves created by a ship

$$\sin(\alpha)/OG = \cos(\theta)/AG$$

$$\sin(\theta - \alpha)AG = GC = OG$$

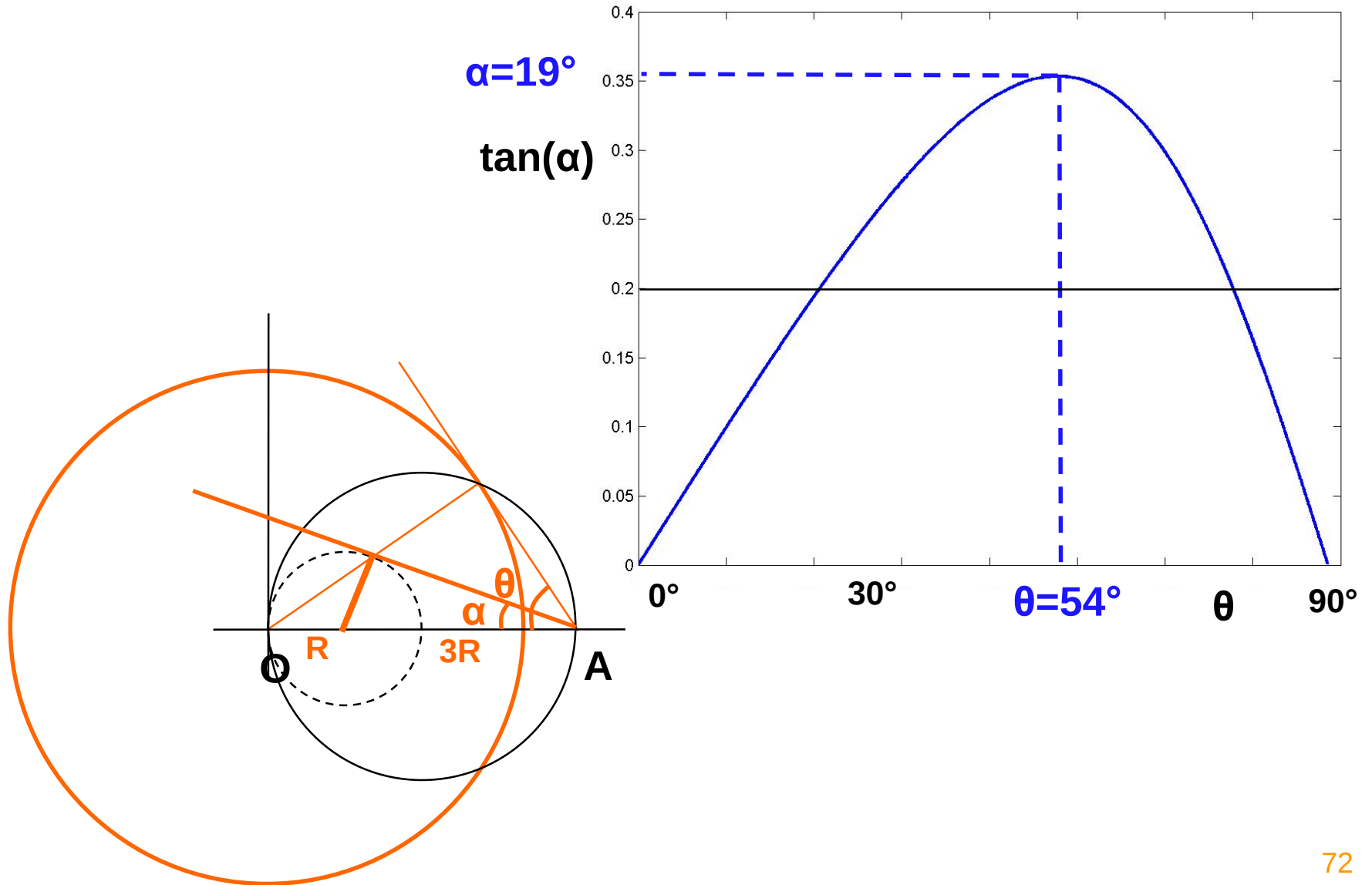


$$\Rightarrow \sin(\alpha) = \cos(\theta) \sin(\theta - \alpha)$$

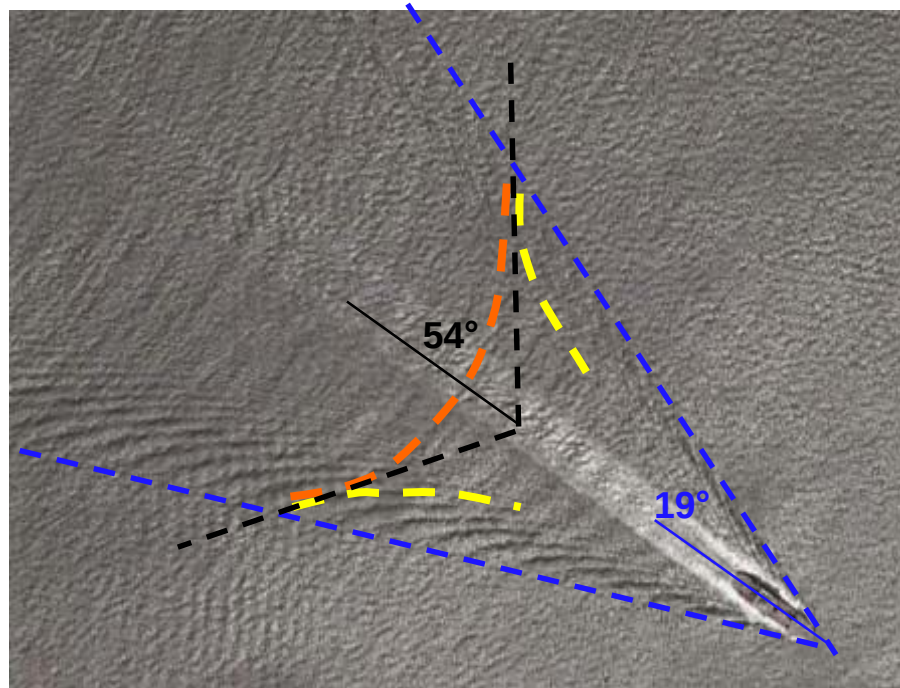
$$\Rightarrow \sin(\alpha) = \cos(\theta) (\sin(\theta) \cos(\alpha) + \cos(\theta) \sin(\alpha))$$

$$\Rightarrow \tan(\alpha) = \cos(\theta) \sin(\theta) / (1 + \cos^2(\theta))$$

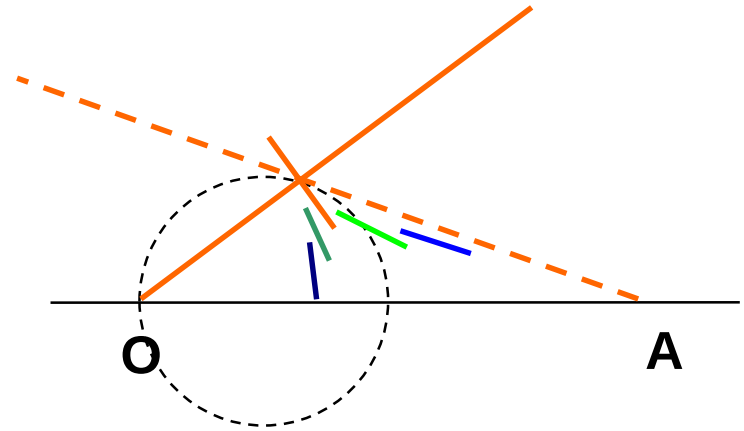
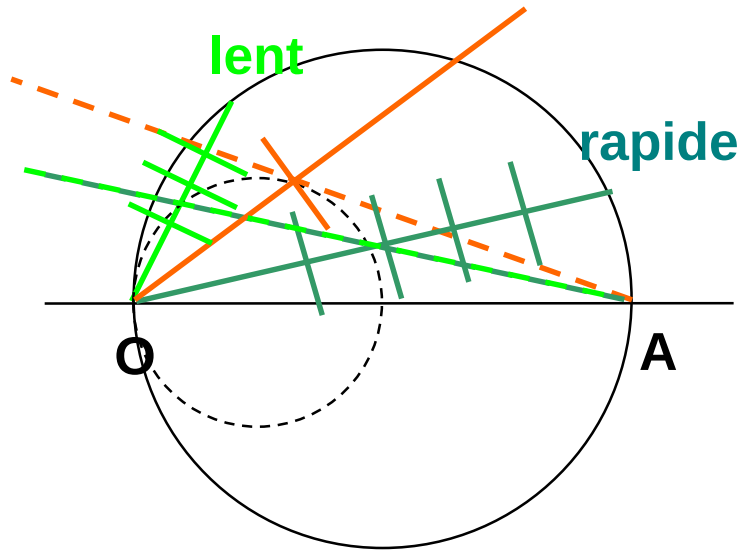
Waves created by a ship



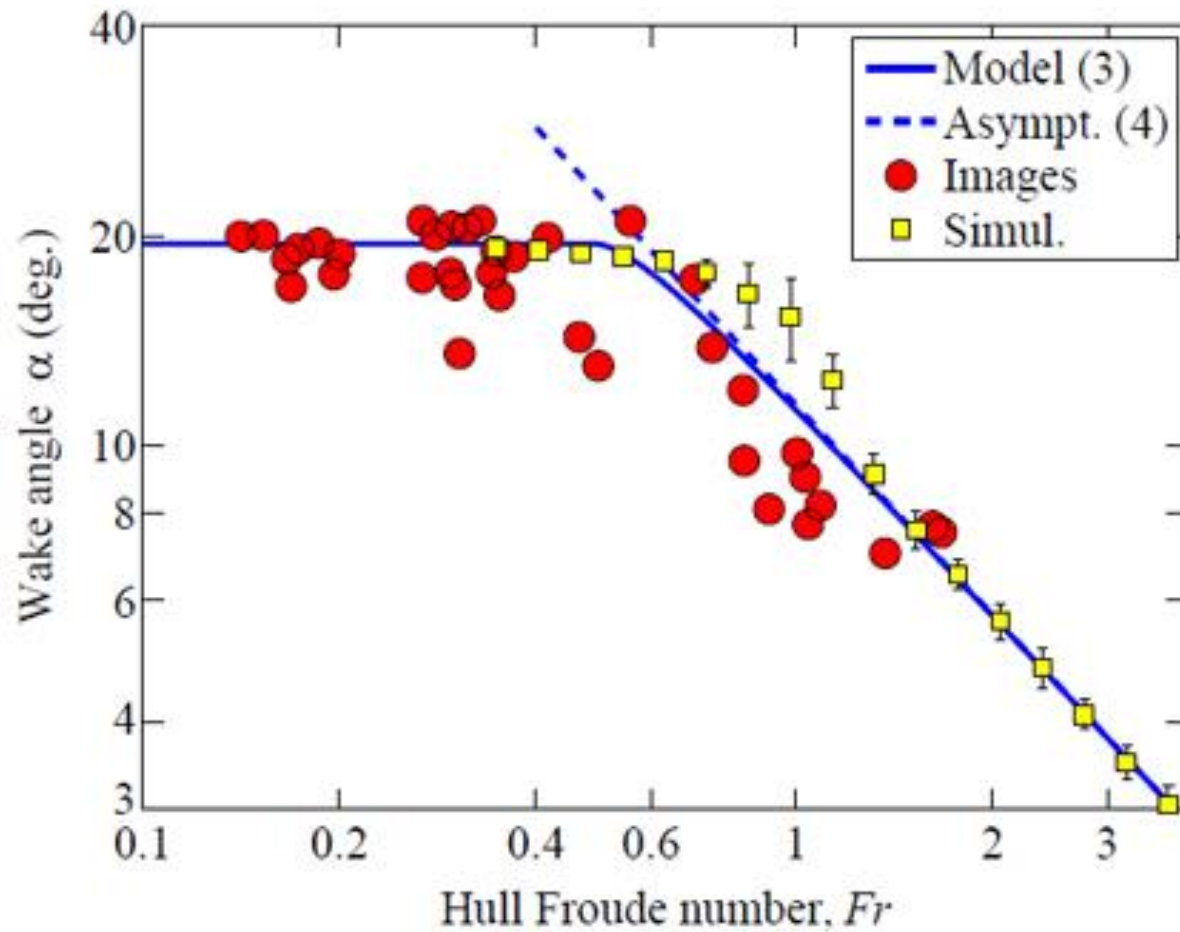
Waves created by a ship



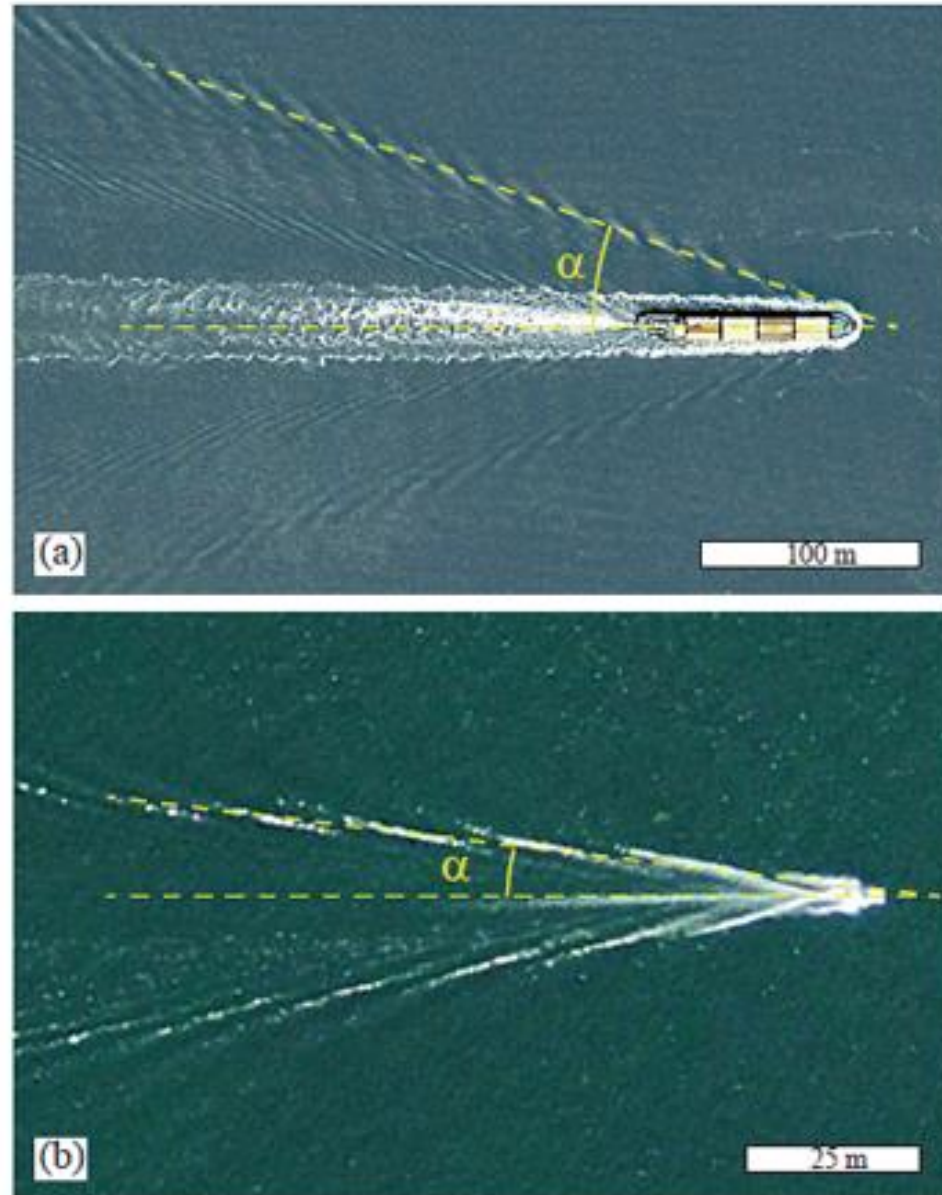
Waves created by a ship



Careful google-earth data analysis by Moisy and Rabaud (2013)



Careful google-earth data analysis by Moisy and Rabaud (2013)



Froude number

$$Fr = \frac{U}{c} \quad \text{c: celerity of wave}$$

$$Fr = \frac{U}{\sqrt{gd}} \quad \text{d: size of ship}$$

$$Fr = \frac{U}{\sqrt{gl}} \quad \text{shallow water}$$

Froude decomposition

$$C_T(F, Re) = C_D(Re) + C_W(F, Re)$$

Total drag Drag in Wave drag
 absence of
 free surface

Froude hypothesis

$$C_T(F, Re) = C_D(Re) + C_W(F, Re)$$

Total drag Drag in Wave drag
 absence of
 free surface

Alternative expression

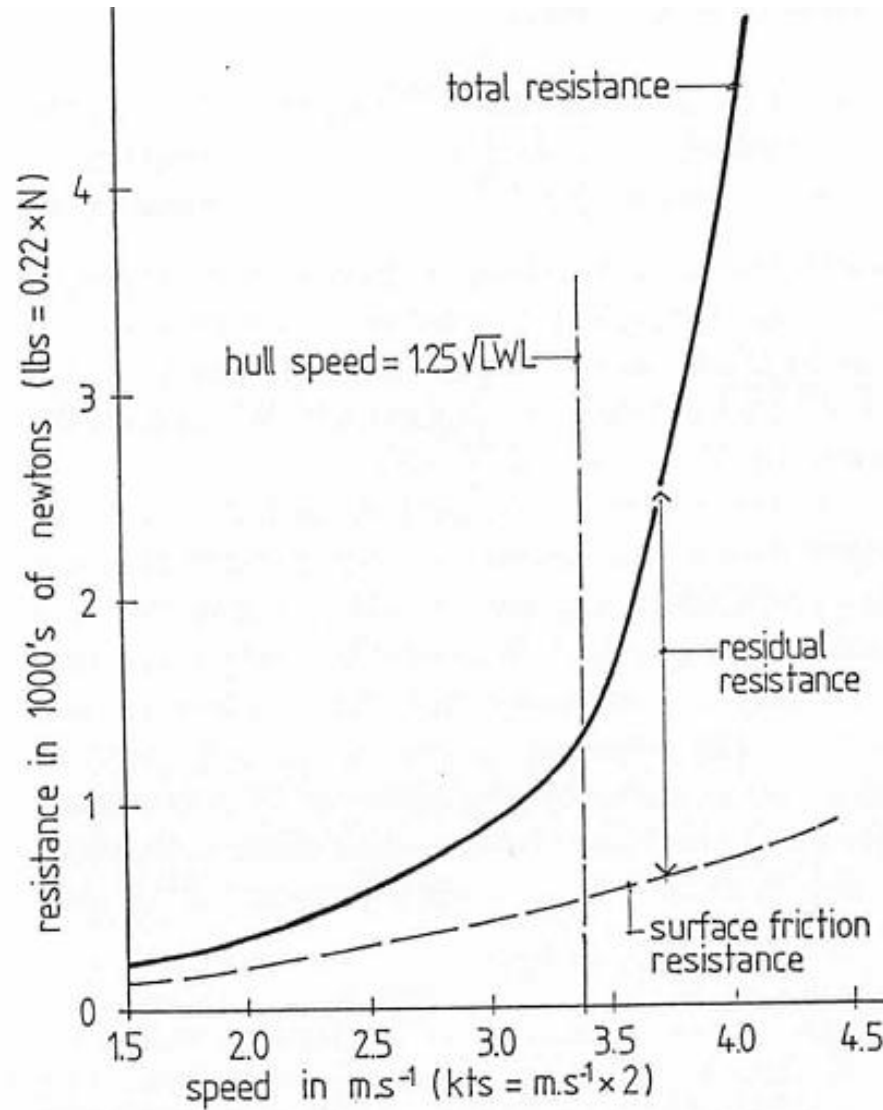
$$C_T(F, Re) = C_F(Re) + C_P(F, Re)$$

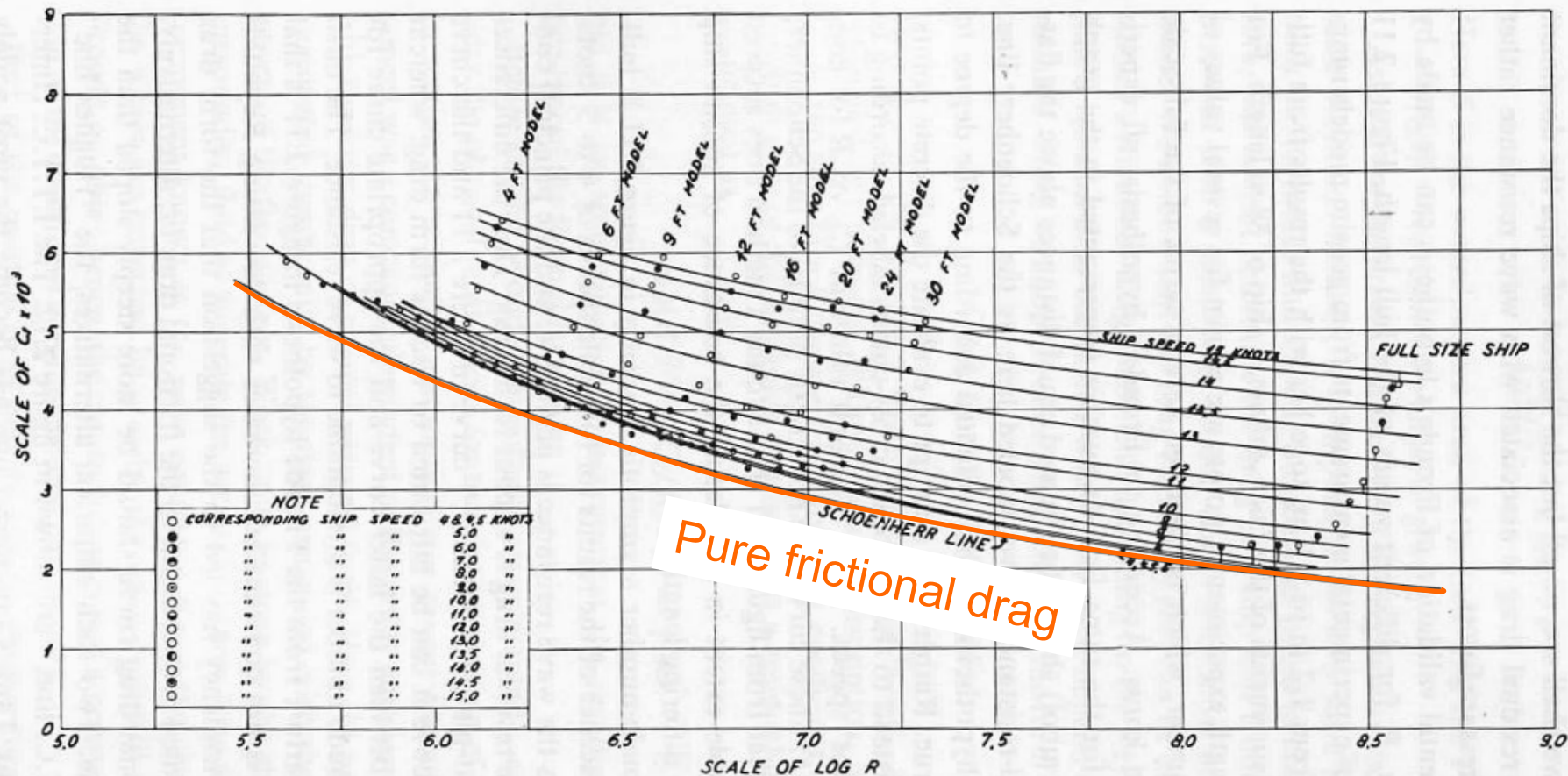
Total drag Friction drag Pressure drag

$$C_P(F, Re) \approx C_W(F)$$

Pressure drag Wave drag
 (Residual drag)

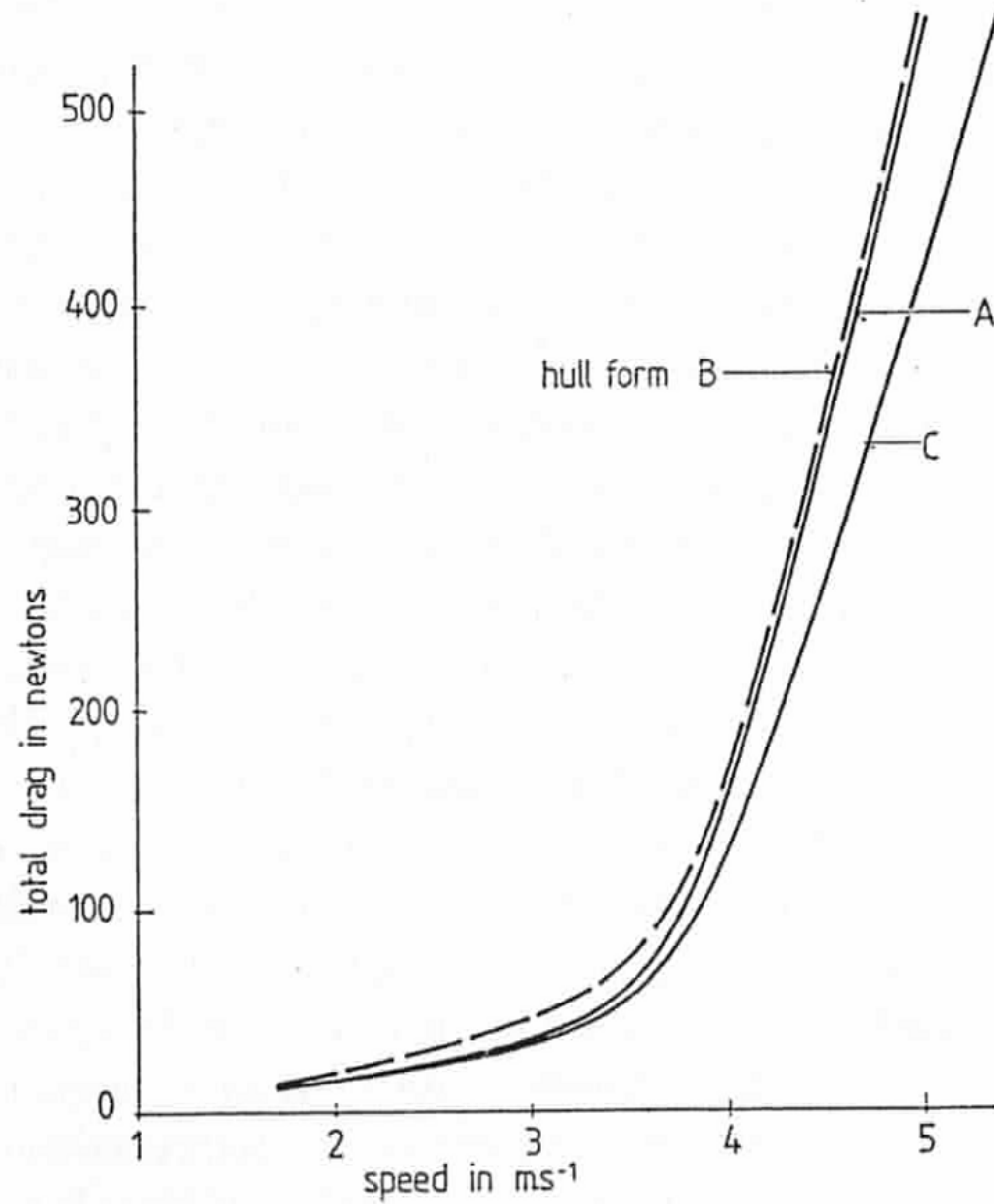
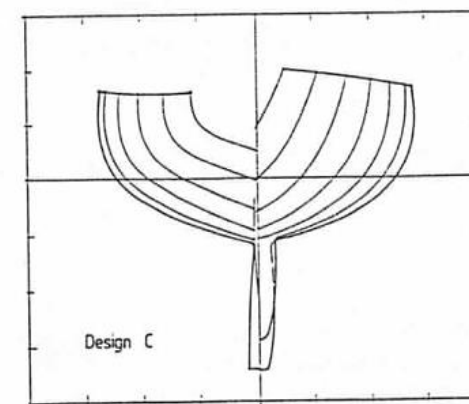
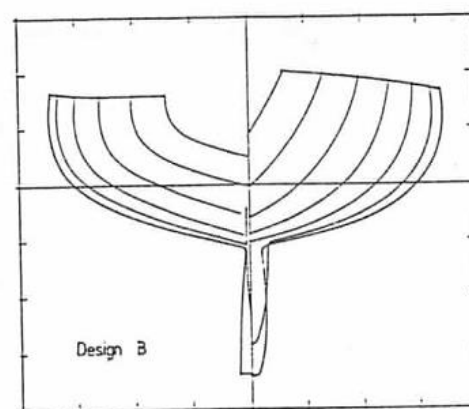
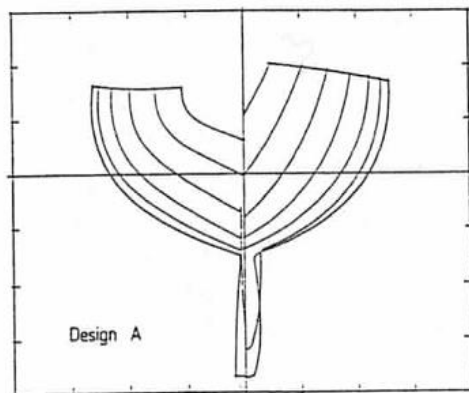
Friction resistance and residual resistance





2.11

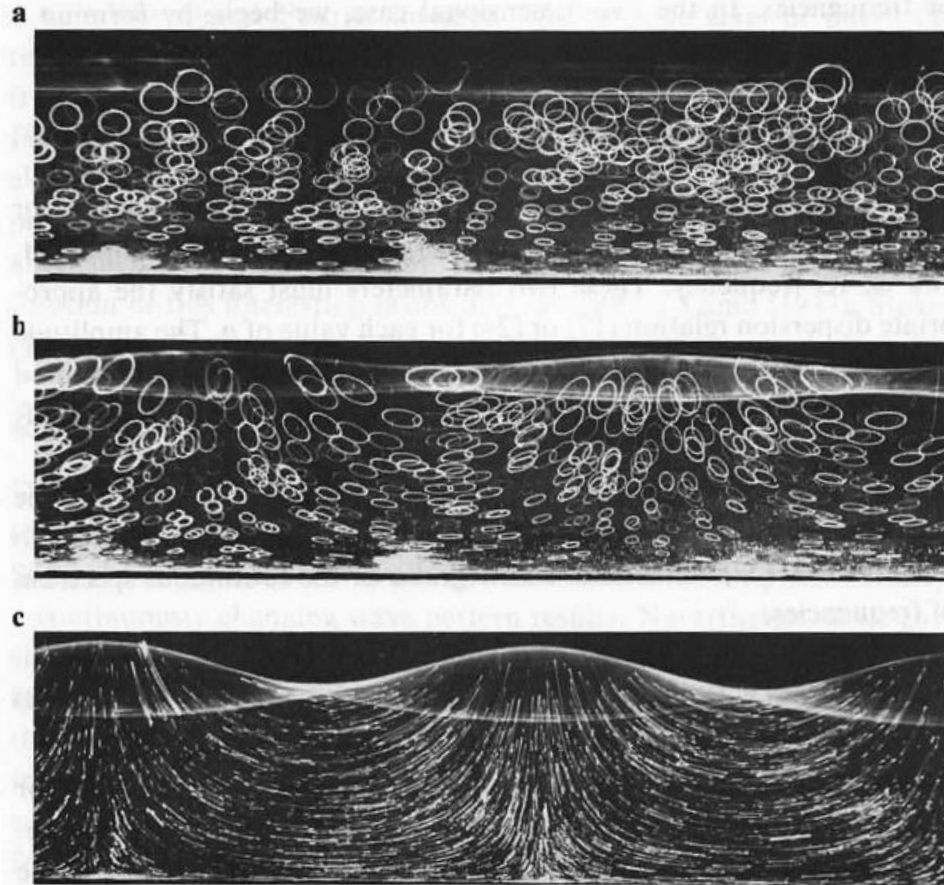
Total drag coefficients of the *Lucy Ashton* and several geosim models of the same vessel (from Troost and Zakay 1951). The faired curves represent constant values of the Froude number and, if Froude's hypothesis were strictly valid, these would be parallel with spacing independent of the Reynolds number. Note that, even for this small full-scale vessel (58 m long), there is a large gap between the largest model results and the full-scale results.



Standing waves

$$\begin{aligned}\eta &= A \cos(kx - \omega t) + A \cos(kx + \omega t) \\ &= 2A \cos kx \cos \omega t,\end{aligned}$$

Reflected waves



6.7

Particle trajectories in a plane progressive wave (a), a partial reflected wave (b), and a standing wave (c). These correspond respectively to a reflection coefficient of 0, 0.38, and 1.0 in equation (56). Note that the reflection coefficient can be measured from the maximum and minimum of the envelope, using (56). These photographs are based on time exposures, and are reproduced from a more extensive series of observations made by Ruellan and Wallet (1950).

Reflected waves

$$\eta = A \operatorname{Re}[e^{-ikx+i\omega t} + R e^{ikx+i\omega t}]$$

R: complex reflection coefficient

$$\eta = A \operatorname{Re}[e^{-ikx+i\omega t}(1 + R e^{2ikx})]$$

Wave energy

$$KE + PE = \rho \iiint_{\mathcal{V}} \left(\frac{1}{2} V^2 + gy \right) d\mathcal{V}$$

$$E = \rho \int_{-h}^{\eta} \left(\frac{1}{2} V^2 + gy \right) dy = \frac{1}{2} \rho \int_{-h}^{\eta} V^2 dy + \frac{1}{2} \rho g \left(\eta^2 - \cancel{h^2} \right).$$

Neglect potential energy of fluid at rest

Wave energy

$$KE + PE = \rho \iiint_{\mathcal{V}} \left(\frac{1}{2} V^2 + gy \right) d\mathcal{V}$$

$$E = \rho \int_{-h}^{\eta} \left(\frac{1}{2} V^2 + gy \right) dy = \frac{1}{2} \rho \int_{-h}^{\eta} V^2 dy + \frac{1}{2} \rho g \left(\eta^2 - \cancel{h^2} \right).$$

Neglect potential energy of fluid at rest

$$E = \frac{\rho \omega^2 A^2}{4k} e^{2k\eta} + \frac{1}{2} \rho g \eta^2.$$

Wave energy

$$KE + PE = \rho \iiint_{\mathcal{V}} \left(\frac{1}{2} V^2 + gy \right) d\mathcal{V}$$

$$E = \rho \int_{-h}^{\eta} \left(\frac{1}{2} V^2 + gy \right) dy = \frac{1}{2} \rho \int_{-h}^{\eta} V^2 dy + \frac{1}{2} \rho g \left(\eta^2 - \cancel{h^2} \right).$$

Neglect potential energy of fluid at rest

$$E = \frac{\rho \omega^2 A^2}{4k} e^{2k\eta} + \frac{1}{2} \rho g \eta^2.$$

$$E = \frac{1}{4} \rho g A^2 + \frac{1}{2} \rho g A^2 \cos^2(kx - \omega t)$$

Wave energy

$$KE + PE = \rho \iiint_{\mathcal{V}} \left(\frac{1}{2} V^2 + gy \right) d\mathcal{V}$$

$$E = \rho \int_{-h}^{\eta} \left(\frac{1}{2} V^2 + gy \right) dy = \frac{1}{2} \rho \int_{-h}^{\eta} V^2 dy + \frac{1}{2} \rho g \left(\eta^2 - \cancel{h^2} \right).$$

Neglect potential energy of fluid at rest

$$E = \frac{1}{4} \rho g A^2 + \frac{1}{2} \rho g A^2 \cos^2(kx - \omega t)$$

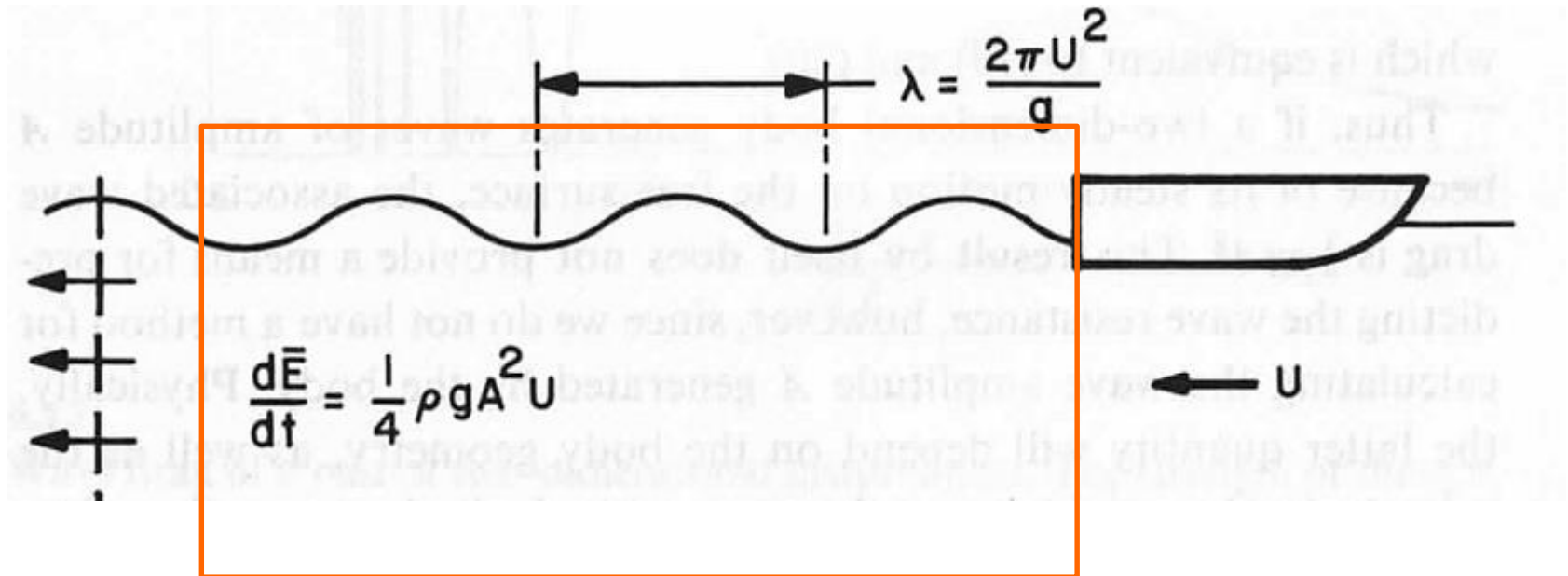
On average:

$$\bar{E} = \frac{1}{2} \rho g A^2$$

Wave energy flux

One can show that the energy travels at the group velocity...

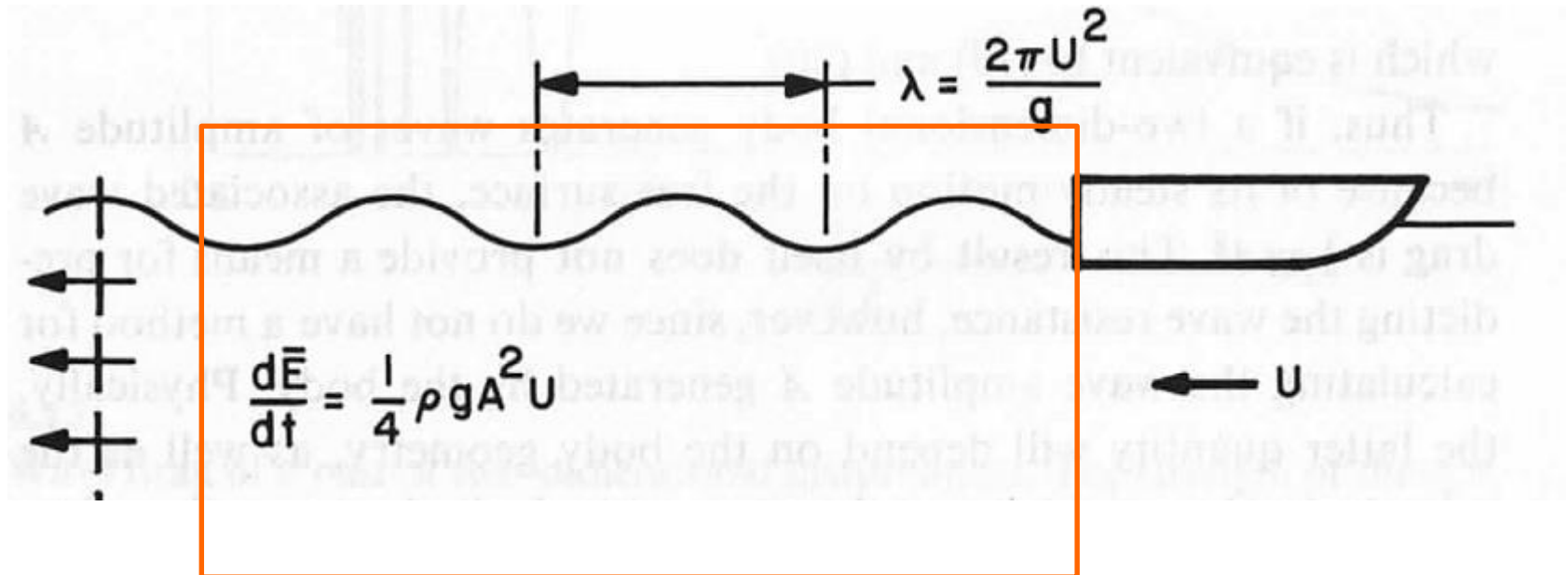
Energy balance (in ship's frame)



Energy input : -work of drag= -D.U

Energy output: (Vg-U)E

Energy balance (in ship's frame)



Energy input : -work of drag= -D.U

Energy output: $(V_g - U)E$

Deep water: $V_g = U/2$

$D = E/2$

$$D = \frac{1}{4} \rho g A^2 U$$

Wave production from bow

displacement

$$\eta = a \cos(kx + \varepsilon)$$

dispersion
relation

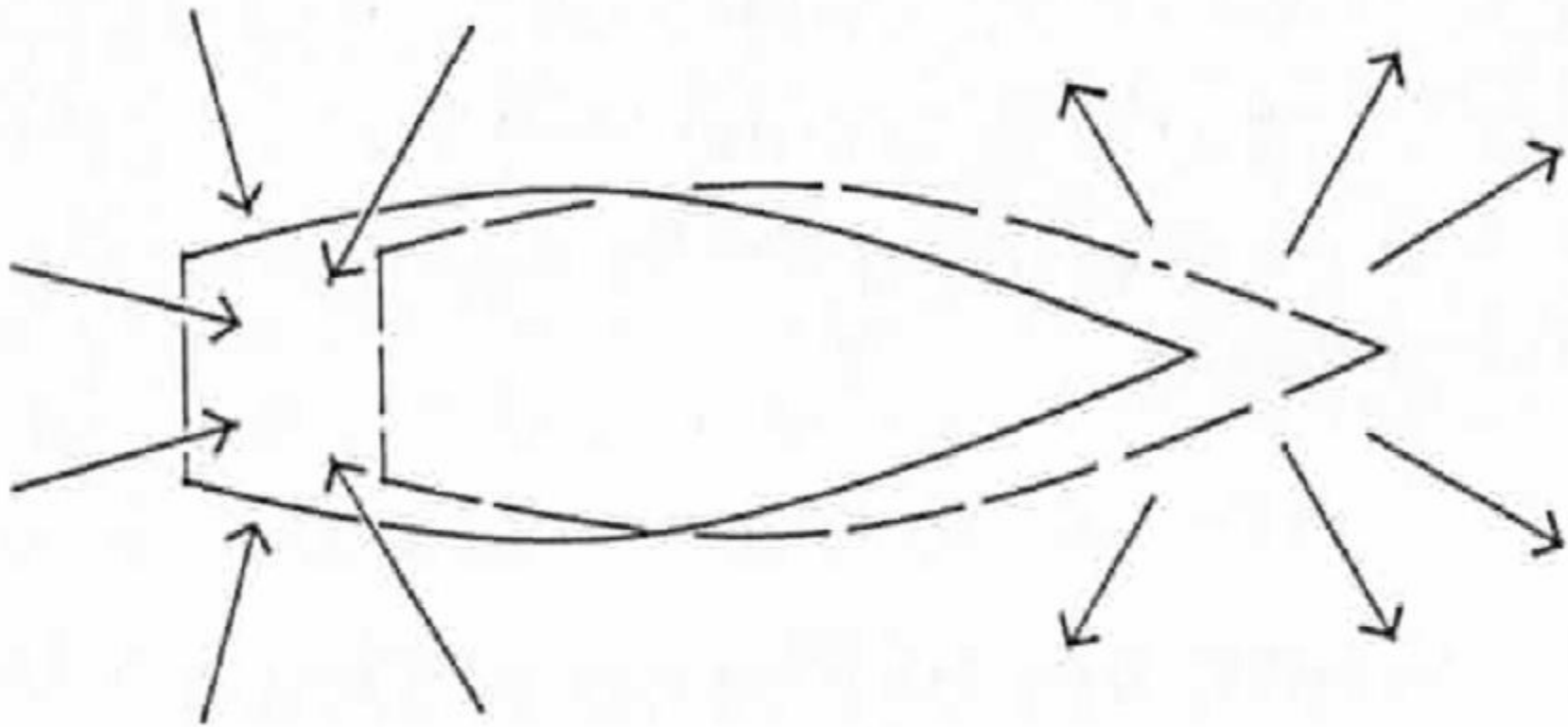
$$k = g/U^2$$

wave amplitude

$$a$$

wave energy

$$\frac{1}{4} \rho g a^2$$

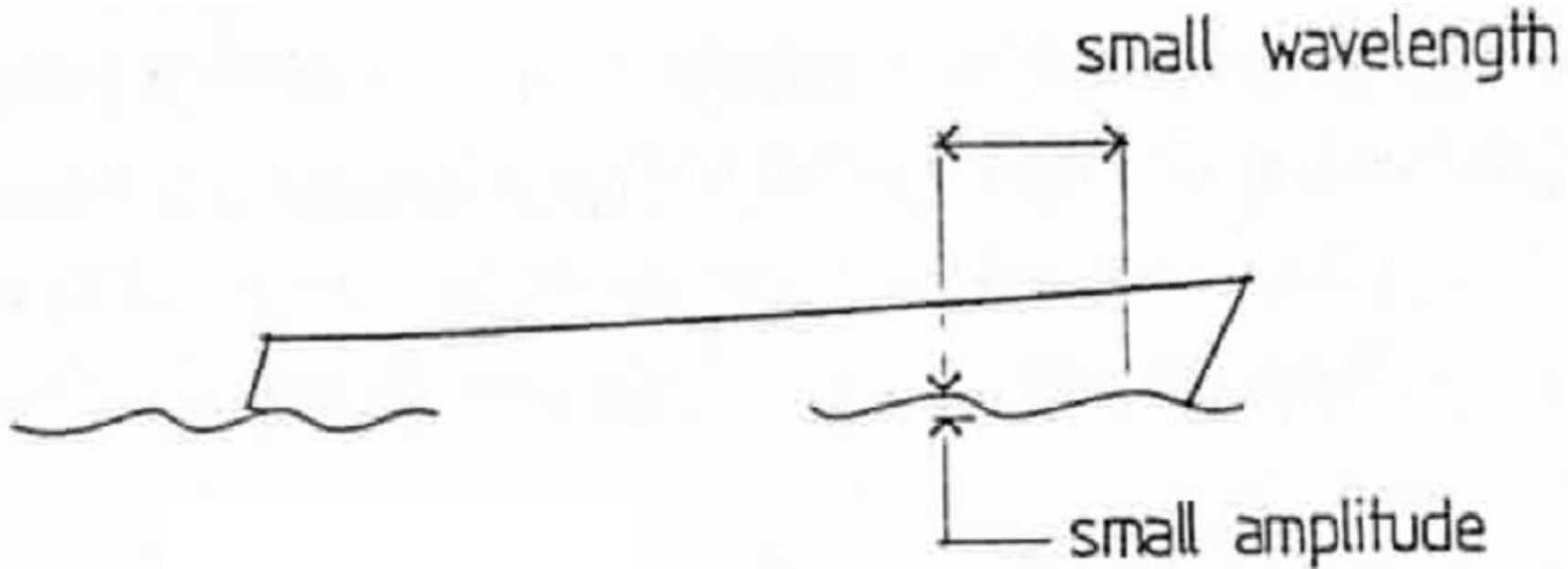


Stern
“Hole”

Bow
“Bump”

Wavelength selection as a function of velocity

Which of these two ships has the highest velocity?



Superimpose wave of opposite amplitude
from stern located at a distance l

displacement

$$\begin{aligned}\eta &= a \cos(kx + \varepsilon) - a \cos(kx + \varepsilon + kl) \\ &= \operatorname{Re}\{ae^{i(kx+\varepsilon)}(1 - e^{ikl})\}.\end{aligned}$$

dispersion
relation

$$k = g/U^2$$

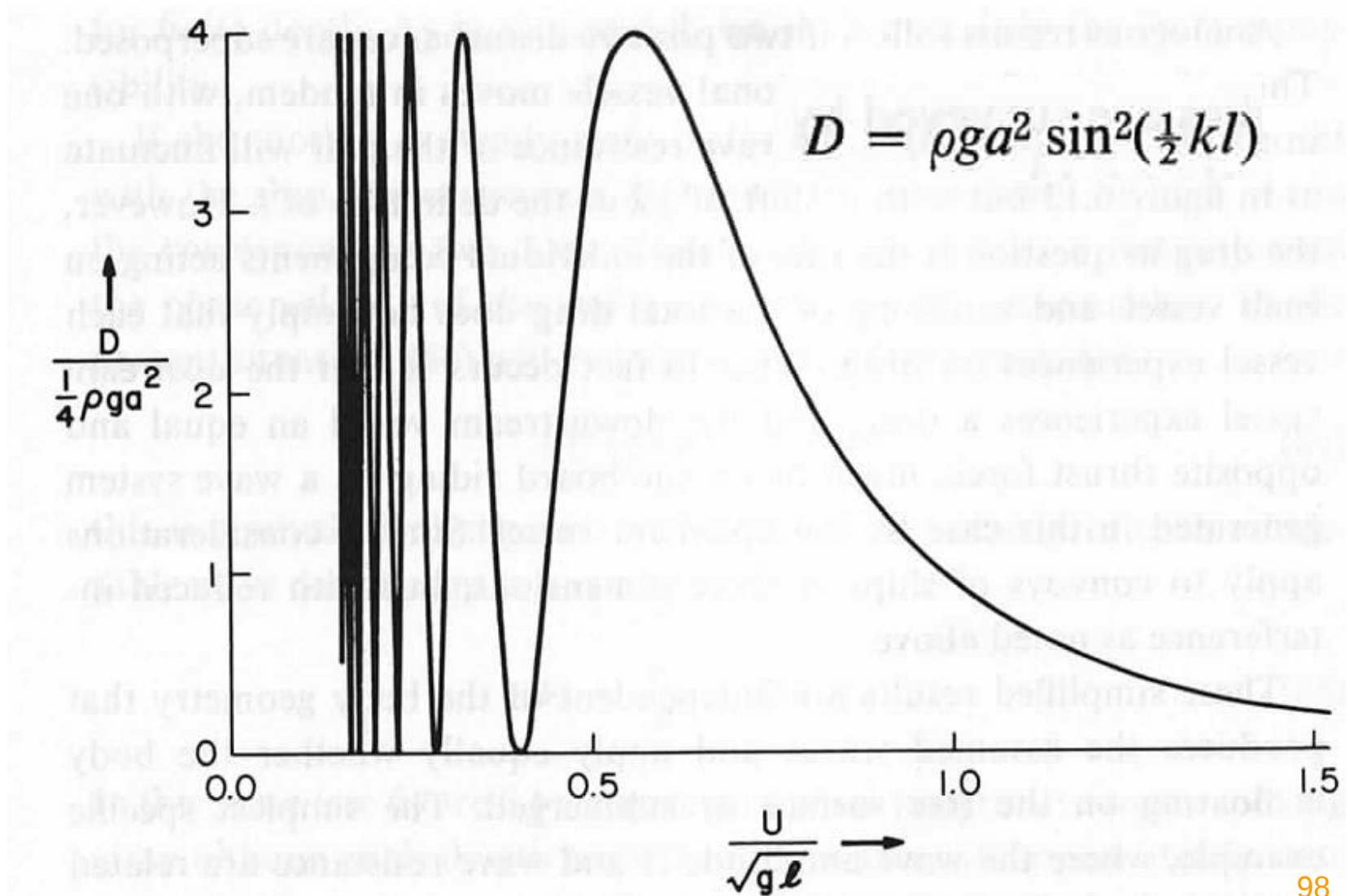
wave amplitude

$$A = a|1 - e^{ikl}| = 2a|\sin(\tfrac{1}{2}kl)|$$

wave energy

$$D = \rho g a^2 \sin^2(\tfrac{1}{2}kl)$$

Superimpose wave of opposite amplitude
from stern located at a distance l



The missing ingredient is now to determine
the wave amplitude!

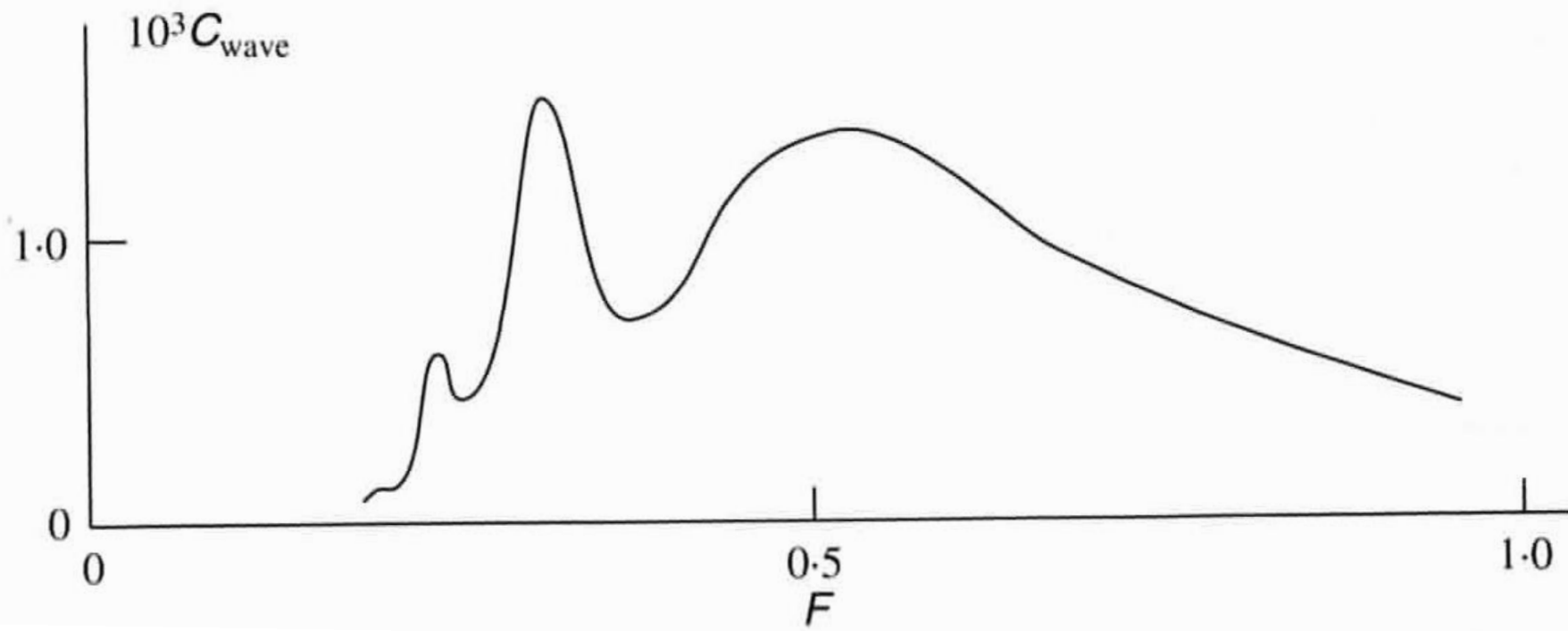
Combination of theoretical and empirical
results.

Generalisation to 3D

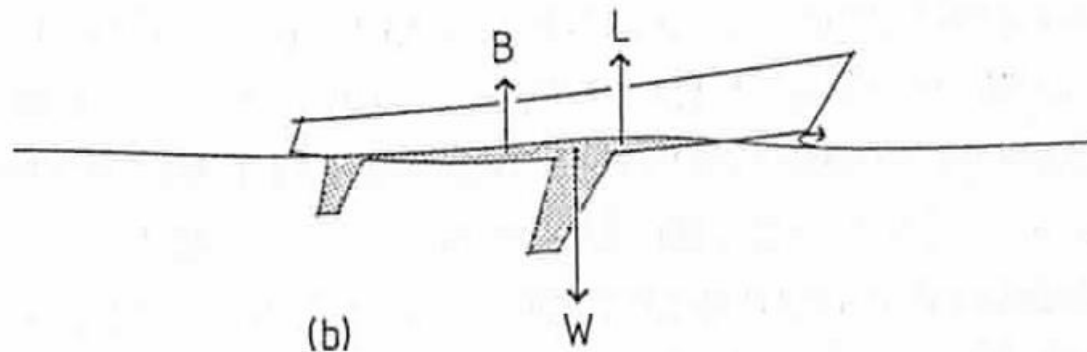
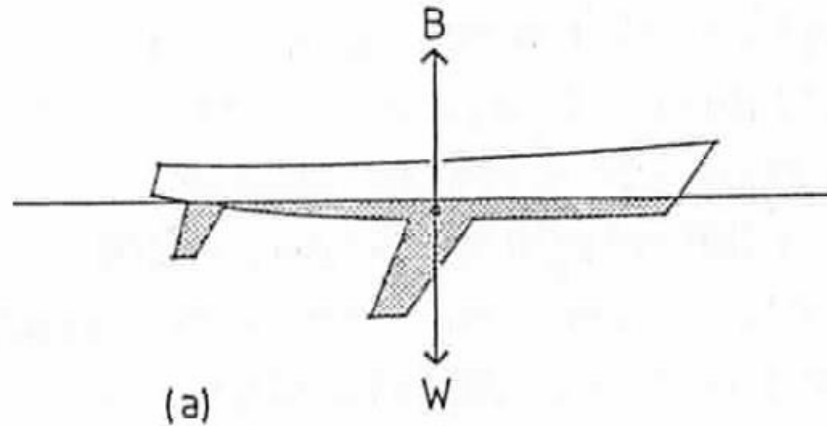
$$\frac{dE}{dt} = \frac{1}{2} \rho g \int_{-\infty}^{\infty} A^2 (V_g \cos \theta - U) dz.$$

$$D = \frac{1}{2} \pi \rho U^2 \int_{-\pi/2}^{\pi/2} |A(\theta)|^2 \cos^3 \theta d\theta.$$

Less interference



This picture is in reality complicated by the floating position of the ship



Which opens the field of dynamical reaction of ships to waves....

