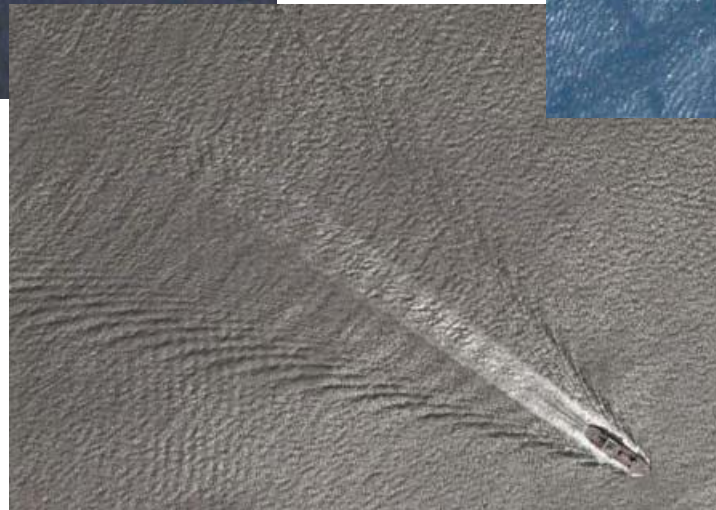
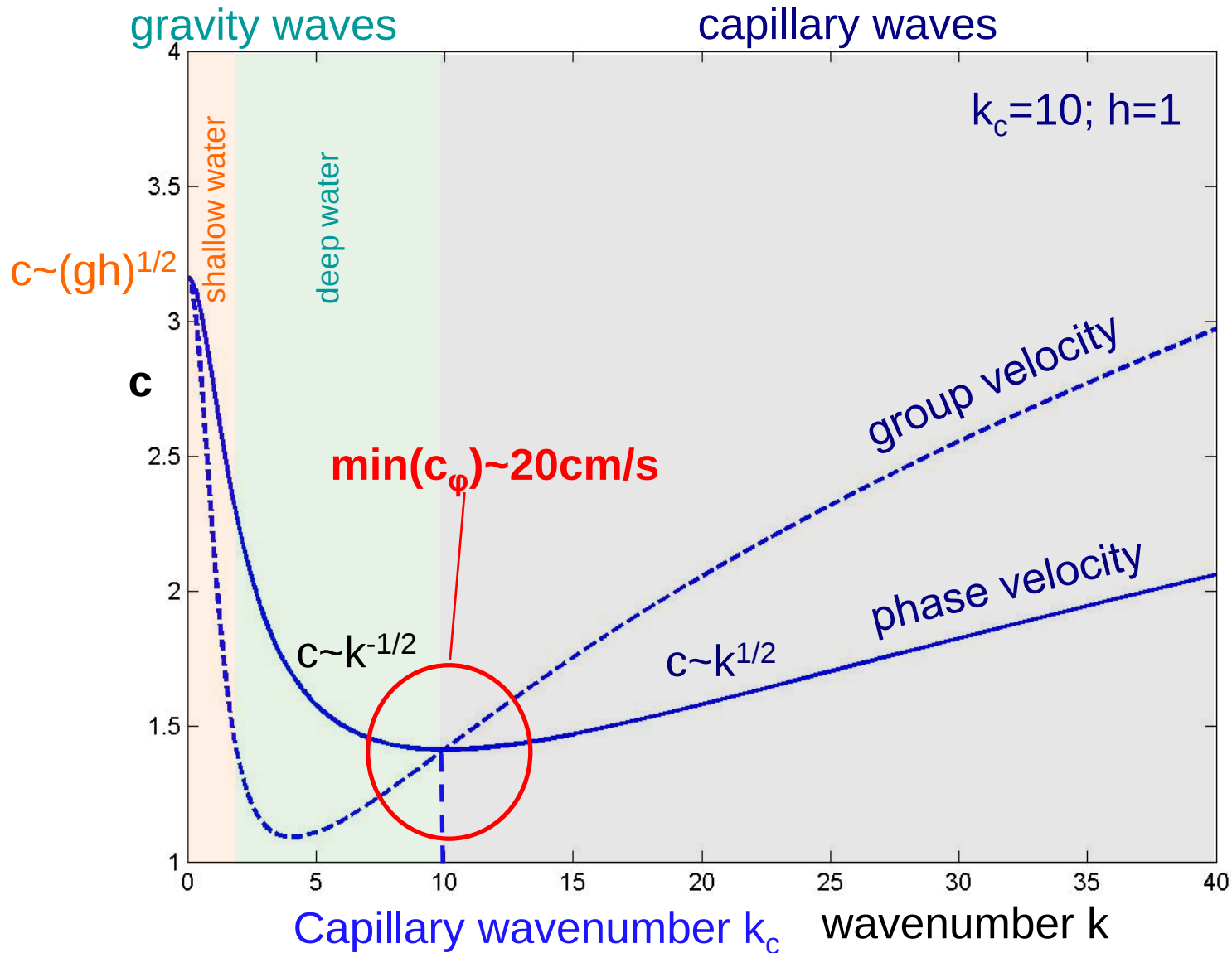


Hydrodynamics 14

Waves



Dispersion relation



Conditions for wave pattern formation?



$$V_{\text{duck}} \leq c_{\text{min}} \quad ?$$

Dispersion relation

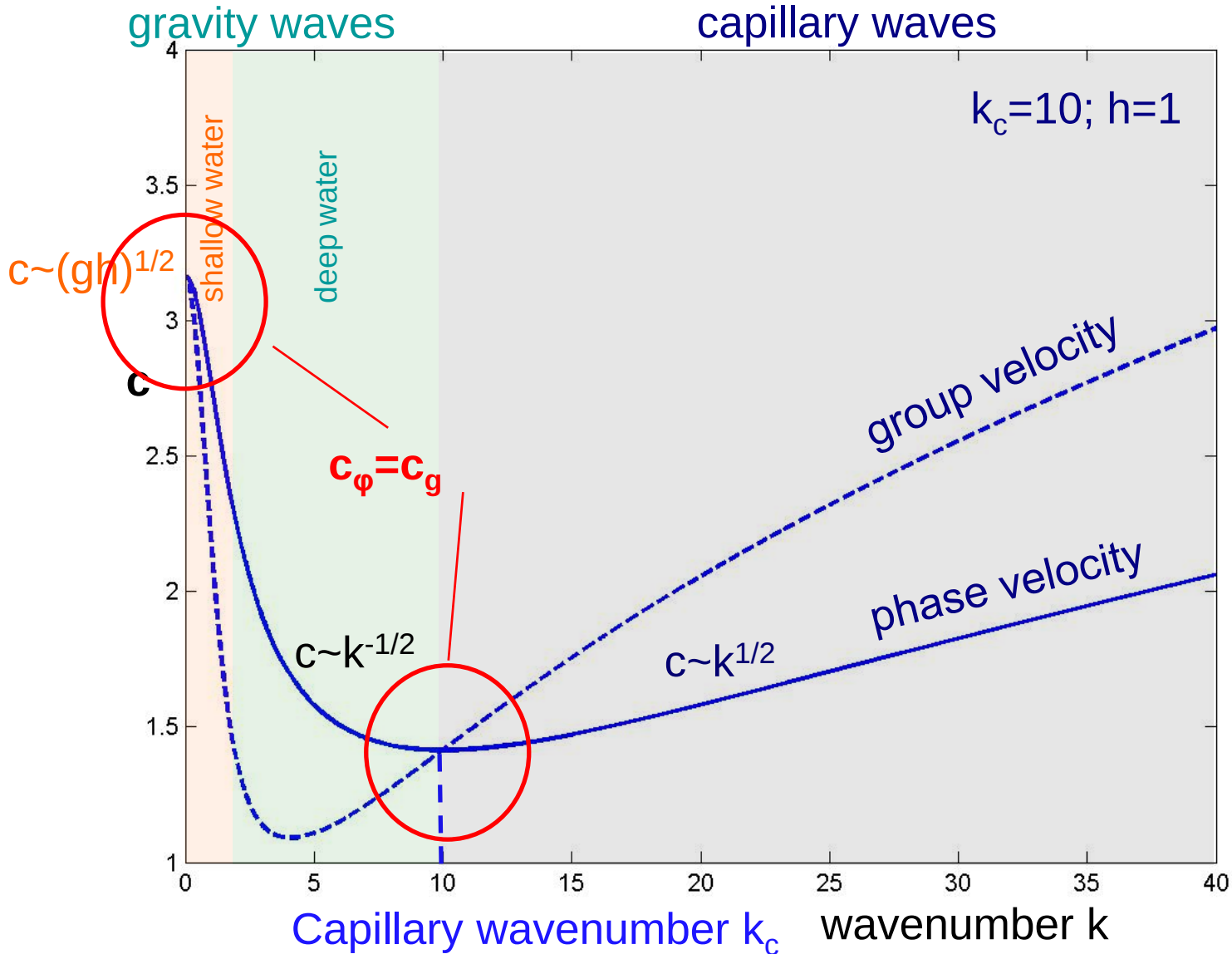


Diagramme spatio-temporel

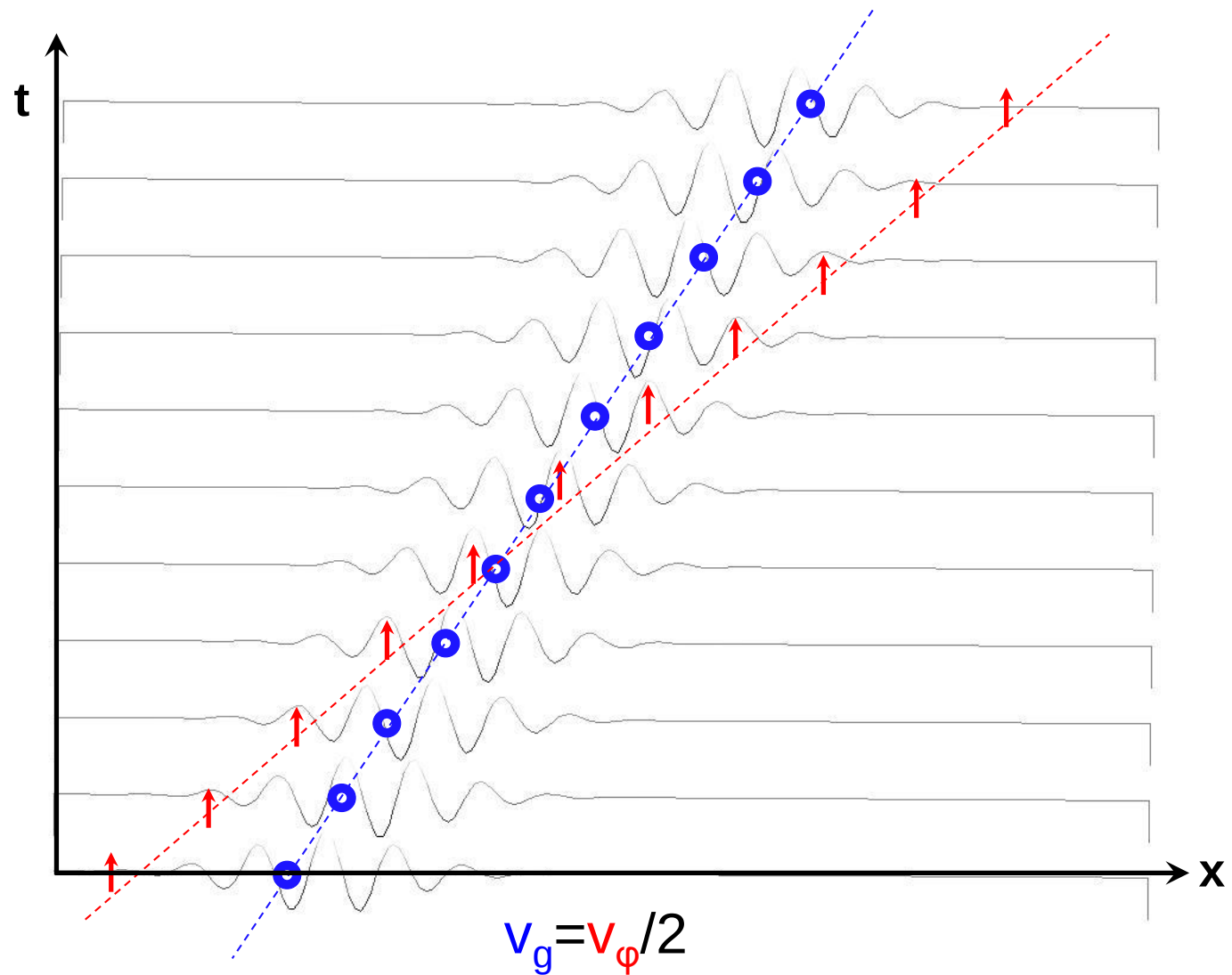


Diagramme spatio-temporel

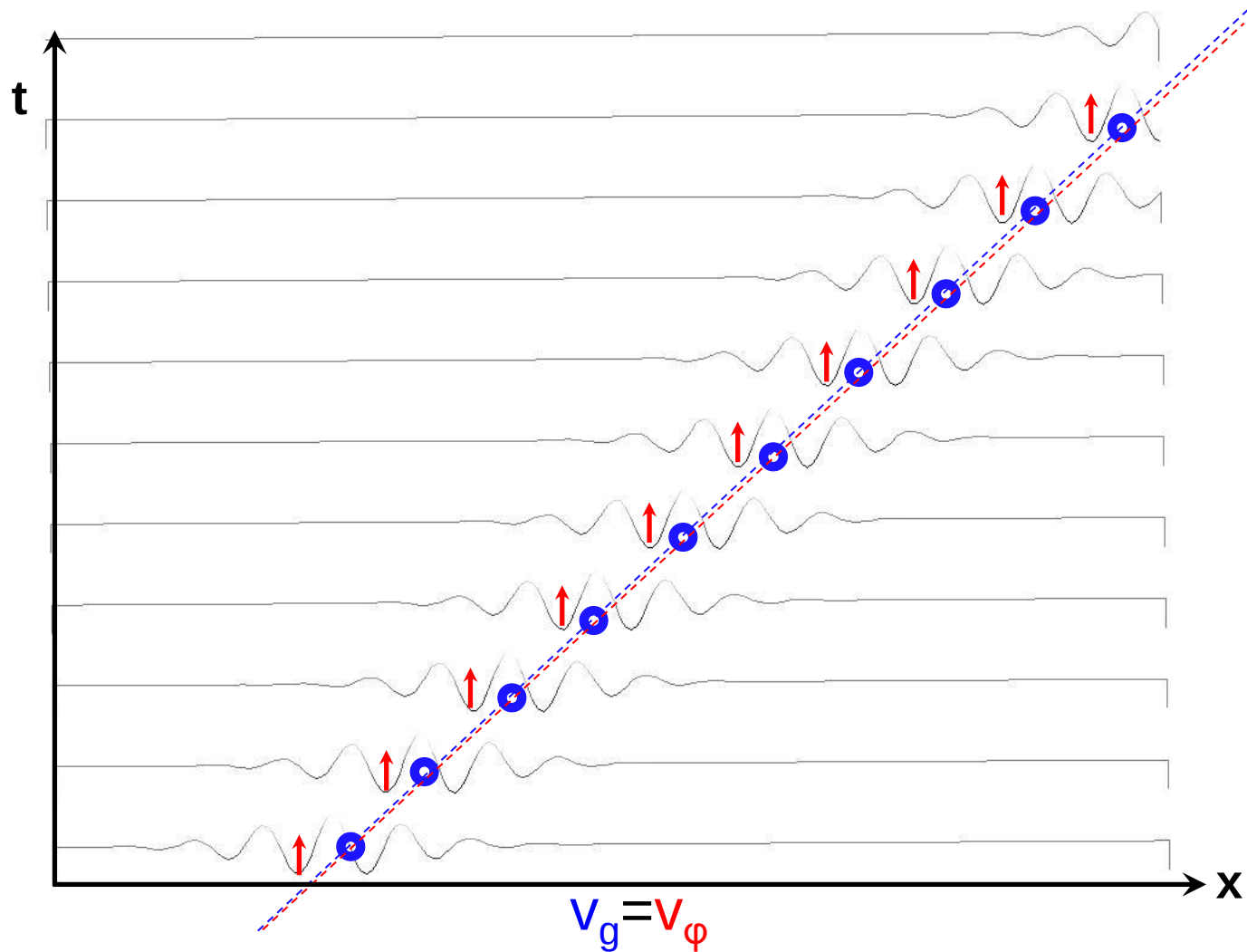
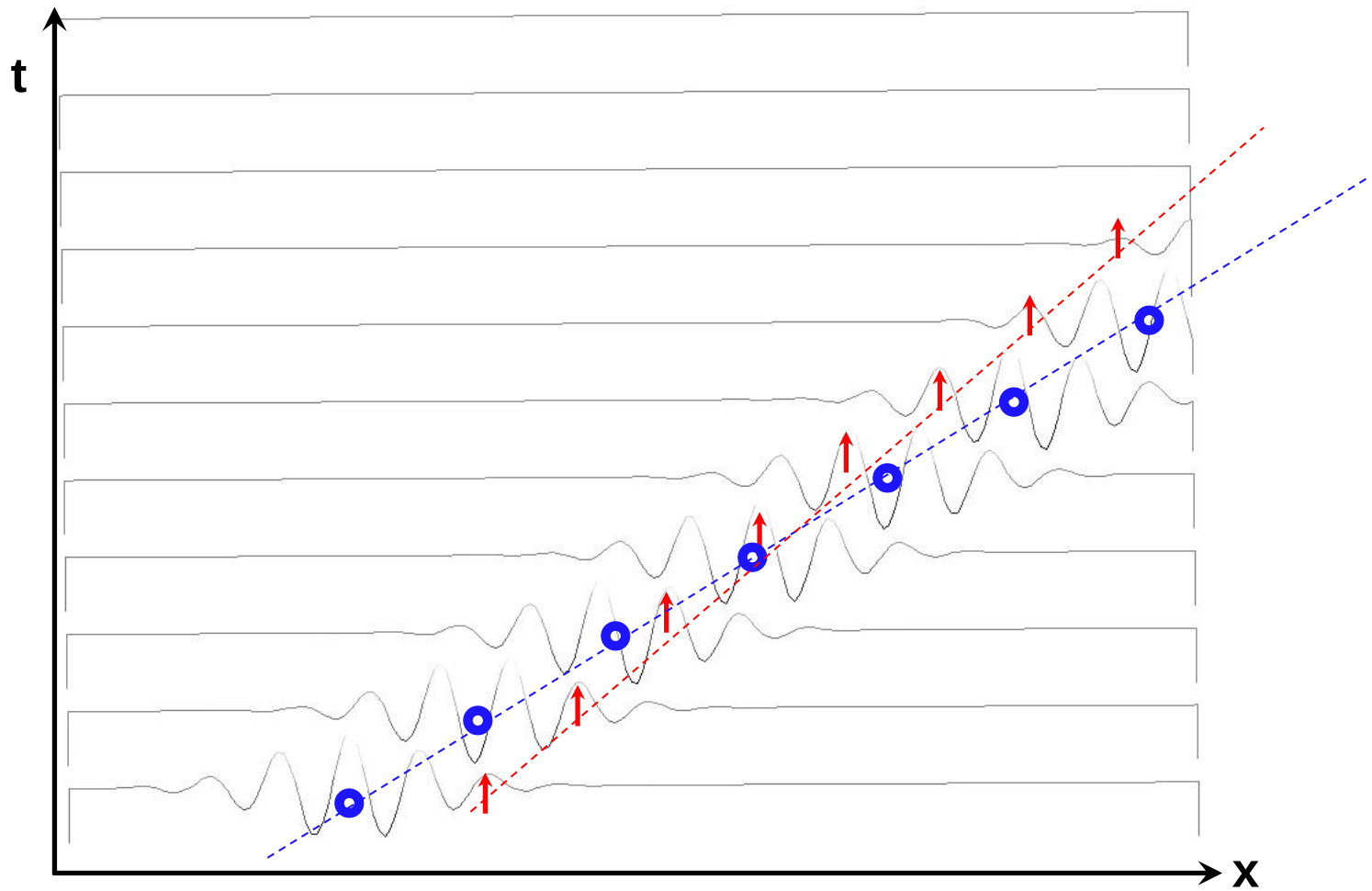


Diagramme spatio-temporel



$$v_g = \frac{3}{2} v_\phi$$

Spectral analysis

Fourier transform:

$$u(x, t) = \frac{1}{2} \int_0^{+\infty} \hat{u}(k) e^{i(kx - \omega(k)t)} dk + c.c.$$

$\hat{u}(k)$ is given by Fourier transform at time $t=0$

Carrier/enveloppe :

$$u(x, t) = \frac{1}{2} A(x, t) e^{i(k_0 x - \omega_0 t)} + c.c.$$

Enveloppe :

$$A(x, t) = \int_0^{\infty} \hat{u}(k) e^{i(k-k_0)x - i(\omega-\omega_0)t} dk.$$

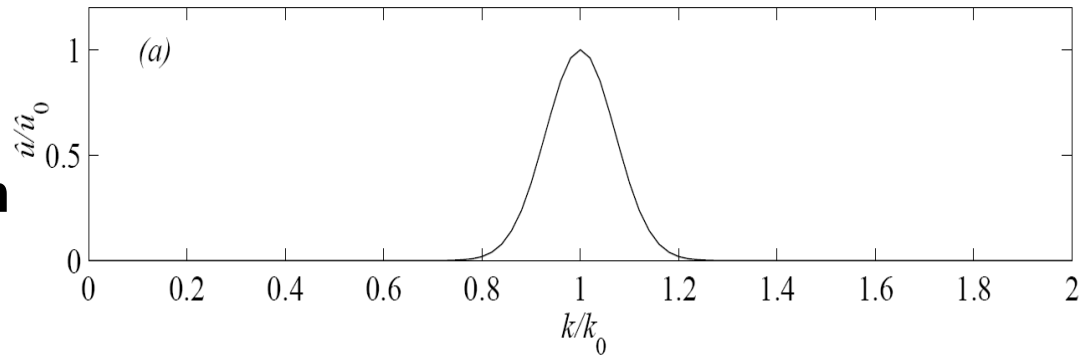
Spectral analysis

Gaussian spectrum: $\hat{u}(k) = u_0 e^{-\sigma^2(k-k_0)^2}$

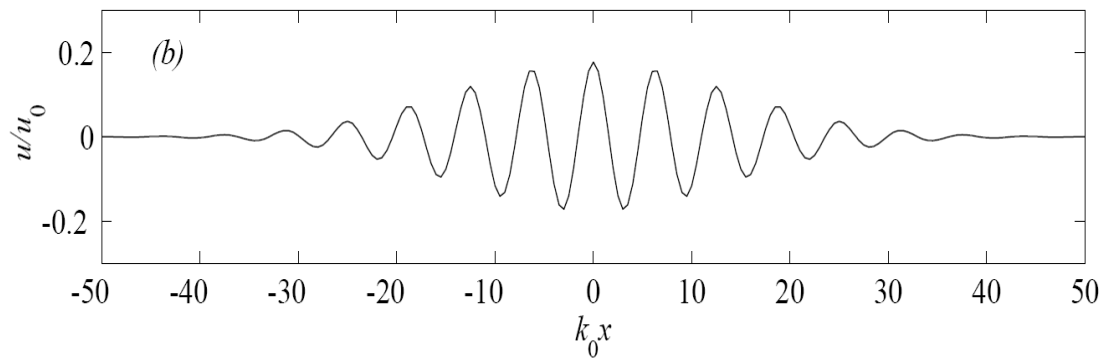
Initial envelope : $A(x, 0) = \frac{u_0 \sqrt{\pi}}{2\sigma} e^{-\frac{x^2}{4\sigma^2}}$

Gaussian spectrum

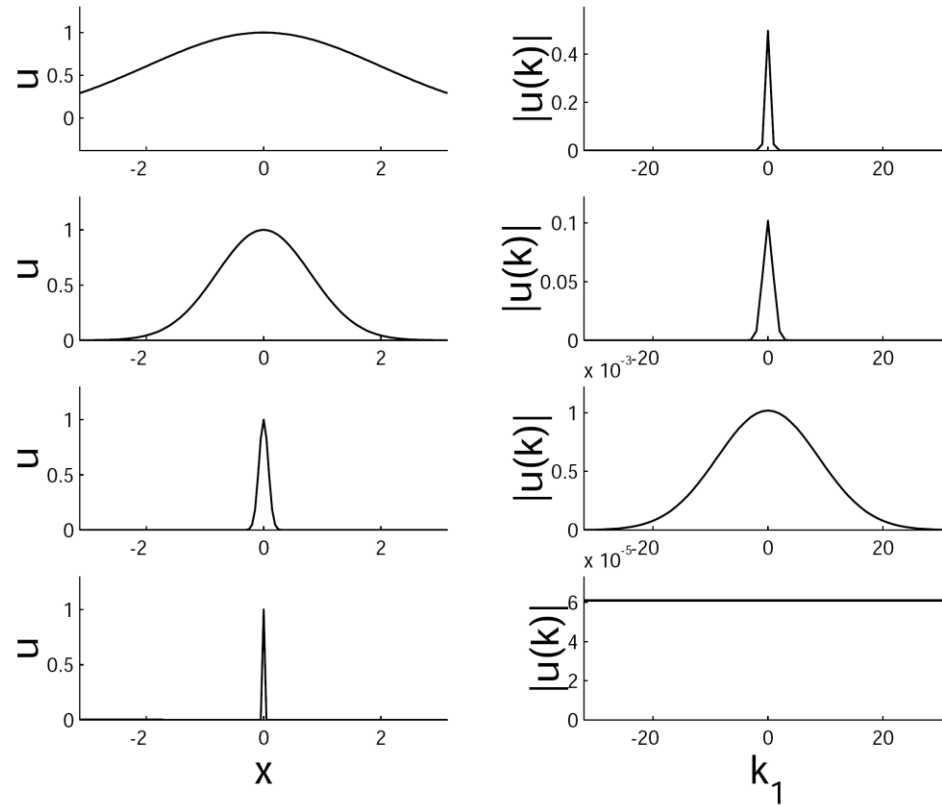
spectrum



wave

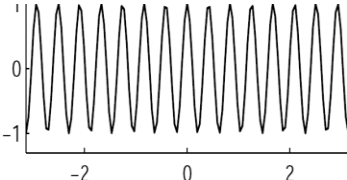
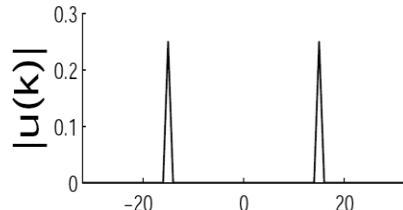
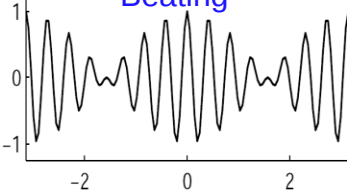
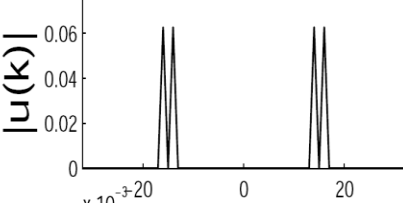
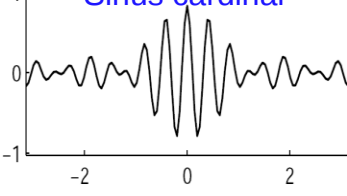
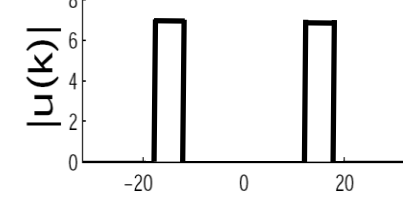
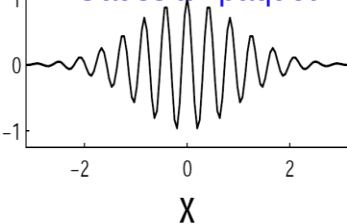
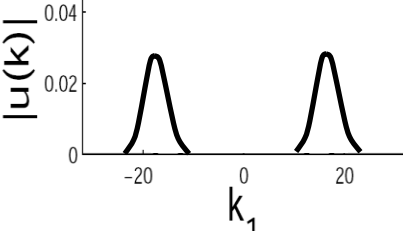


Paquets d'ondes gaussiens



$u_0(x)$	$\hat{u}_0(k_1)$
$\exp\left(-\frac{x^2}{2\sigma^2}\right)$	$\frac{\sigma}{\sqrt{2\pi}} \exp\left(-\frac{\sigma^2 k^2}{2}\right)$
$\delta(x)$	$\frac{1}{2\pi}$

Waves and spectrum

$2 \cos(k_0 x)$	Monochromatic wave 		$\delta(k_1 + k_0) + \delta(k_1 - k_0)$
$\cos[(k_0 - \kappa)x] + \cos[(k_0 + \kappa)x]$ $= 2 \cos(k_0 x) \cos(\kappa x)$	Beating 		$\delta(k_1 + k_0 + \kappa) + \delta(k_1 - k_0 - \kappa)$ $+ \delta(k_1 + k_0 - \kappa) + \delta(k_1 - k_0 + \kappa)$
$\int_{k_0 - \kappa}^{k_0 + \kappa} \cos(k_1 x) dk_1$ $= 4 \kappa \cos(k_0 x) \frac{\sin(\kappa x)}{\kappa x}$	Sinus cardinal 		$= 1 \text{ pour } k_1 \pm k_0 \leq \kappa$ $= 0 \text{ sinon}$
$2 \kappa \sqrt{2\pi} \cos(k_0 x) \exp\left(-\frac{\kappa^2 x^2}{2}\right)$	Gaussian paquet 		$\exp\left[-\frac{(k_1 + k_0)^2}{2 \kappa^2}\right] + \exp\left[-\frac{(k_1 - k_0)^2}{2 \kappa^2}\right]$

Spectral analysis

$$A(x, t) = \int_0^\infty \hat{u}(k) e^{i(k-k_0)x - i(\omega - \omega_0)t} dk.$$

Spectral analysis

$$A(x, t) = \int_0^\infty \hat{u}(k) e^{i(k-k_0)x - i(\omega - \omega_0)t} dk.$$

Definition of group velocity $\omega - \omega_0 = c_g(k - k_0), \quad c_g = \frac{\partial \omega}{\partial k}(k_0)$

Spectral analysis

$$A(x, t) = \int_0^{\infty} \hat{u}(k) e^{i(k-k_0)x - i(\omega - \omega_0)t} dk.$$

$$\hat{u}(k) = u_0 e^{-\sigma^2 (k-k_0)^2}$$

Spectral analysis

Gaussian spectrum: $\hat{u}(k) = u_0 e^{-\sigma^2(k-k_0)^2}$

Initial envelope : $A(x, 0) = \frac{u_0 \sqrt{\pi}}{2\sigma} e^{-\frac{x^2}{4\sigma^2}}$

Spectral analysis

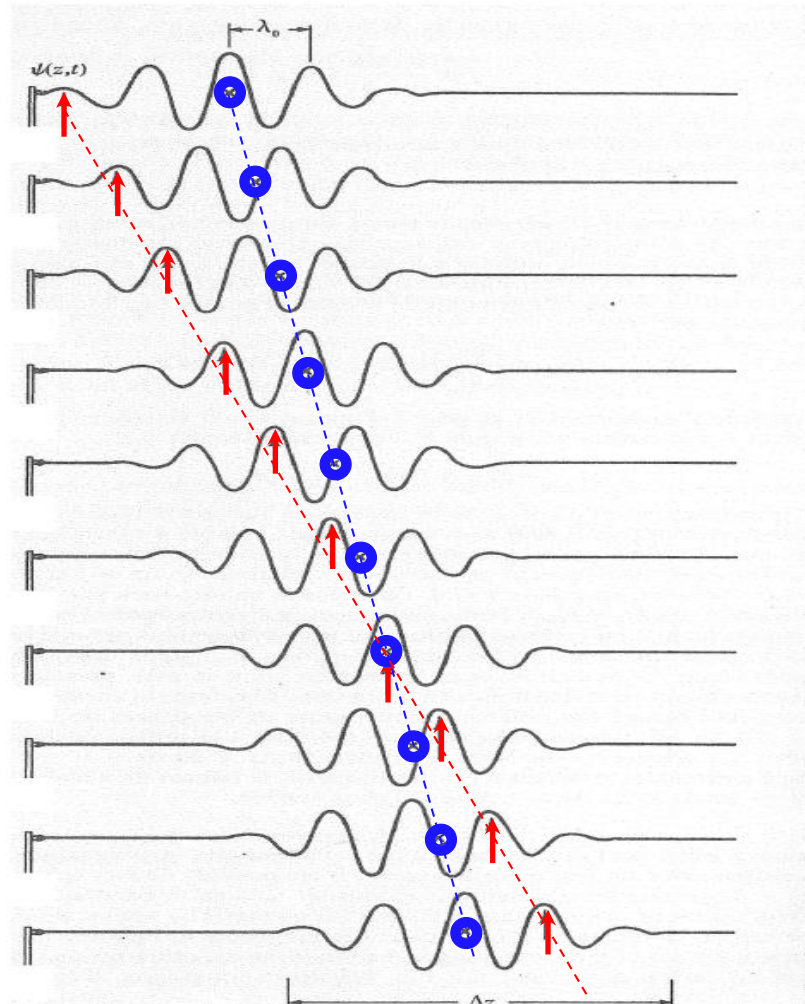
$$A(x, t) = \int_0^\infty \hat{u}(k) e^{i(k-k_0)x - i(\omega - \omega_0)t} dk.$$

Definition of group velocity $\omega - \omega_0 = c_g(k - k_0), \quad c_g = \frac{\partial \omega}{\partial k}(k_0)$

$$\hat{u}(k) = u_0 e^{-\sigma^2(k-k_0)^2}$$

$$A(x, t) = \frac{u_0 \sqrt{\pi}}{2\sigma} e^{-\frac{(x - c_g t)^2}{4\sigma^2}}$$

Group velocity



Wave packet

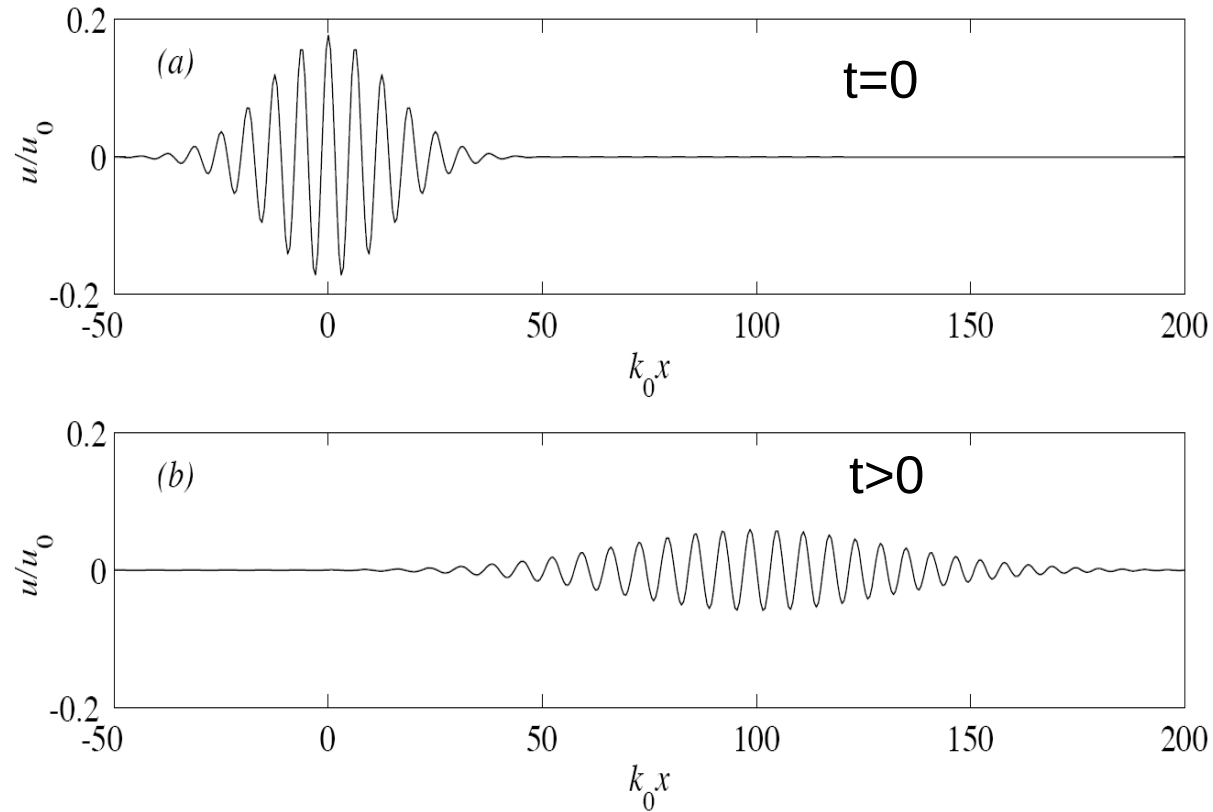
Spectral analysis

Higher order
development

$$\omega - \omega_0 = c_g(k - k_0) + \frac{\omega_0''}{2}(k - k_0)^2$$
$$c_g = \frac{\partial \omega}{\partial k}(k_0), \quad \omega_0'' = \frac{\partial^2 \omega}{\partial k^2}(k_0)$$

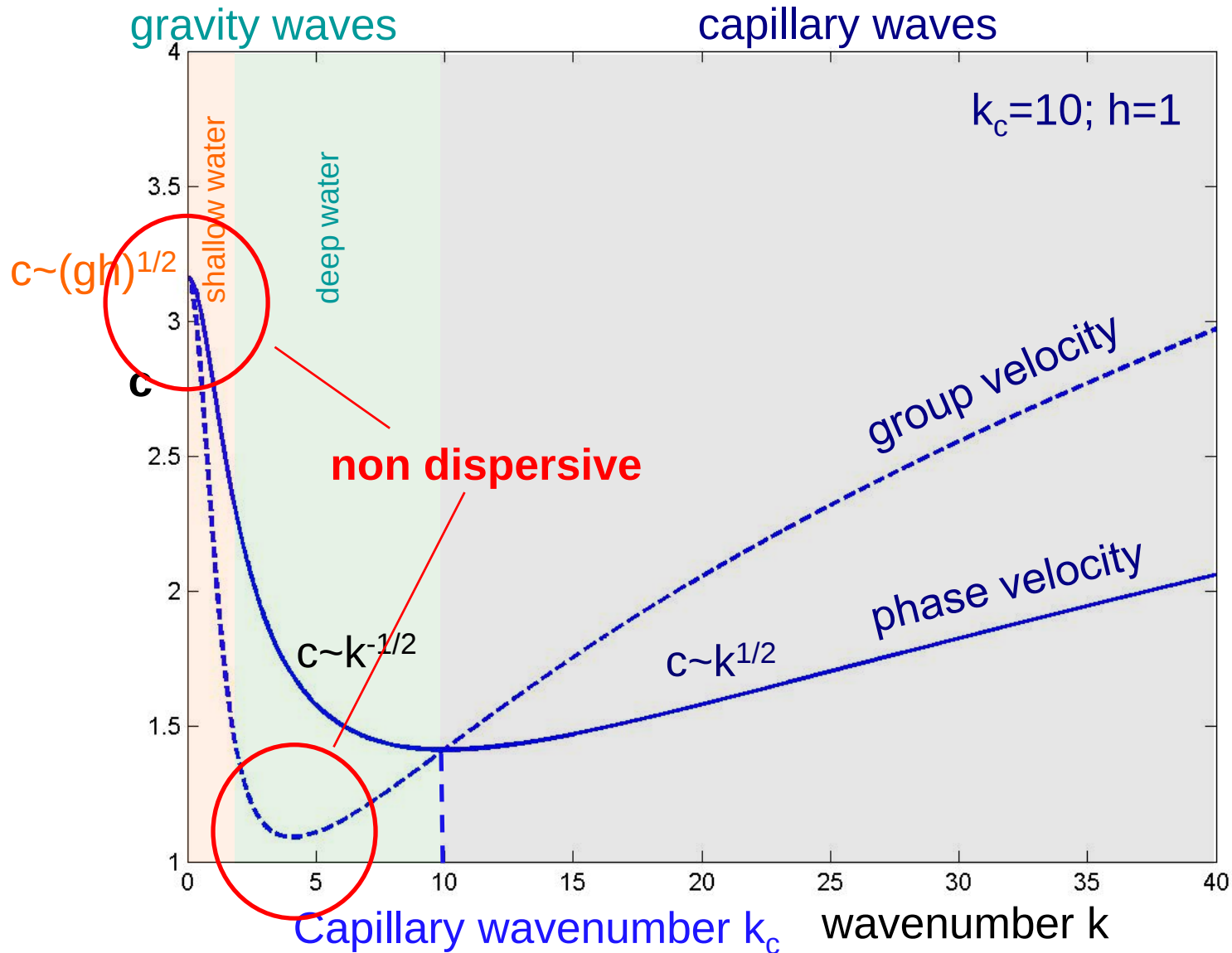
$$A(x, t) = \frac{u_0}{2} \sqrt{\frac{\pi}{\sigma^2 + \frac{1}{2}i\omega_0''t}} \exp\left(-\frac{(x - c_g t)^2}{4(\sigma^2 + \frac{1}{2}i\omega_0''t)}\right)$$

Wave packet dispersion

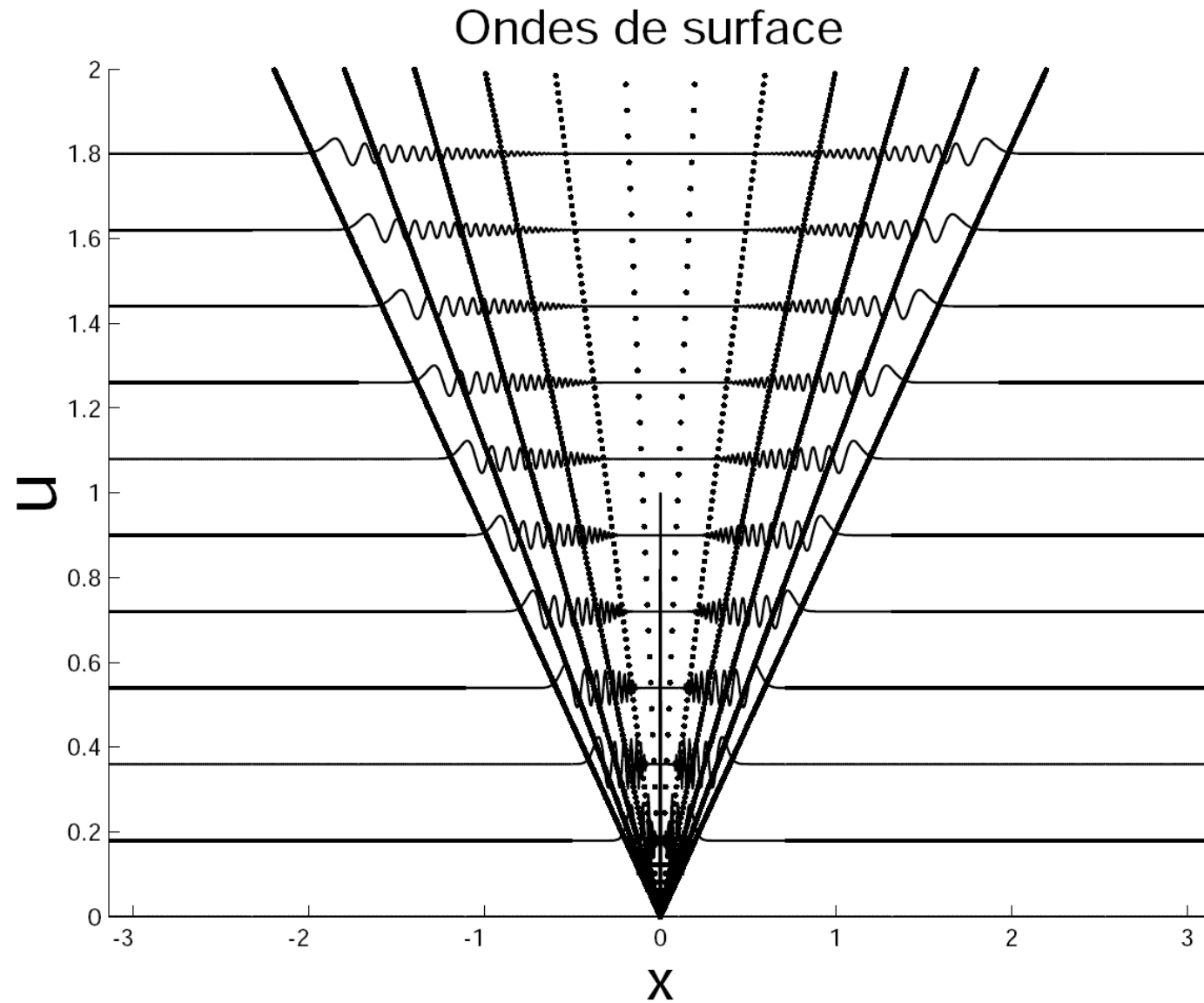


Onde correspondant à l'enveloppe $\left[\frac{1}{2} \exp(-\sigma^2 k_0^2) \right]$ pour $\sigma^{-1} k_0 = 0,1$ et $\omega_0'' = 4c_g/k_0$: (a), instant initial $t = 0$; (b), $c_g t = 100/k_0$.

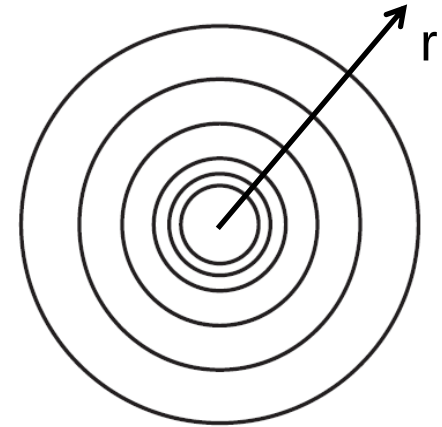
Dispersion relation



Dispersion



Dispersion



Waves with k reach r at time $t=r/v(k)$

For deep gravity waves: $v_{\text{deep/gravity}} \sim \pm \frac{1}{2} \sqrt{\frac{g}{k}}$

$$k = gt^2/4r^2$$

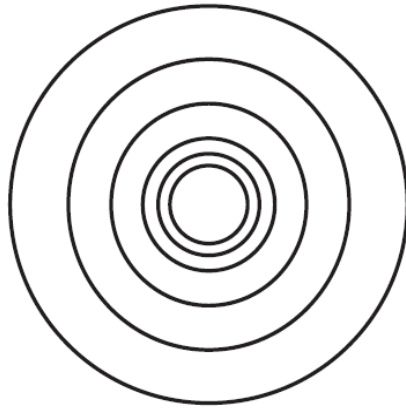
Since $\omega_{\text{deep/gravity}} \sim \pm \sqrt{gk}$

$$\omega = gt/2r$$

⇒ The frequency increases with time

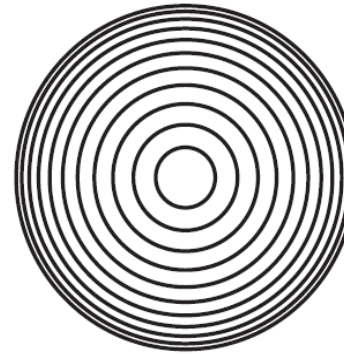
“Ronds dans l’eau”

Gravity
waves

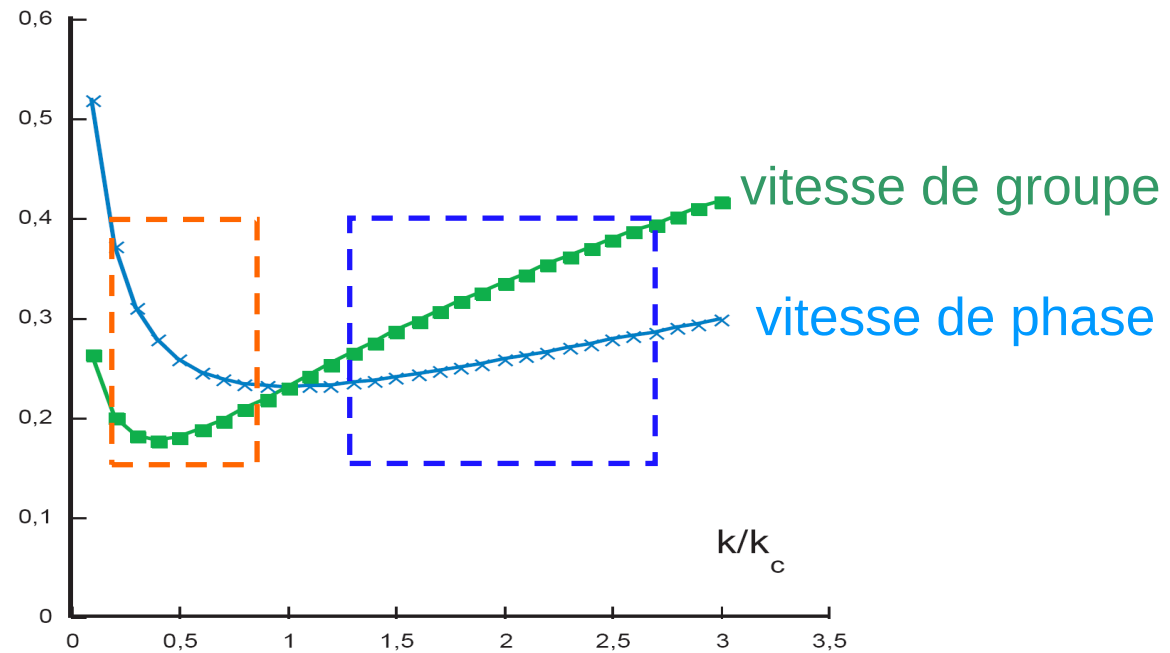


stone $> l_c$

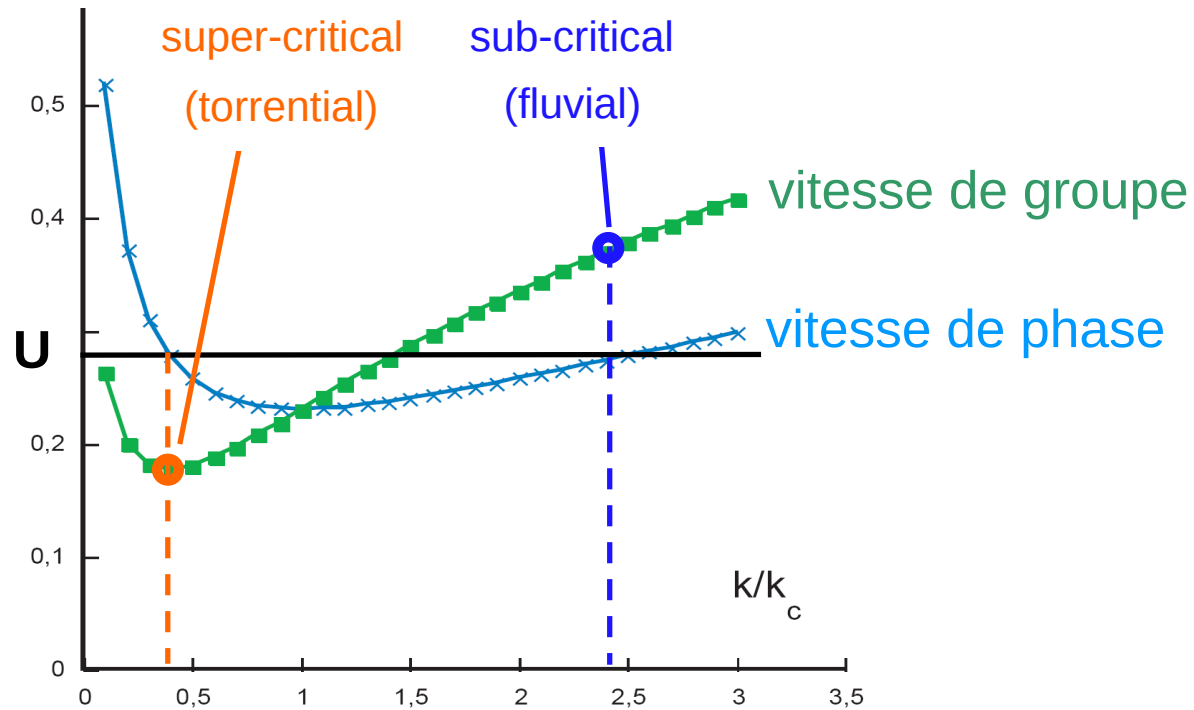
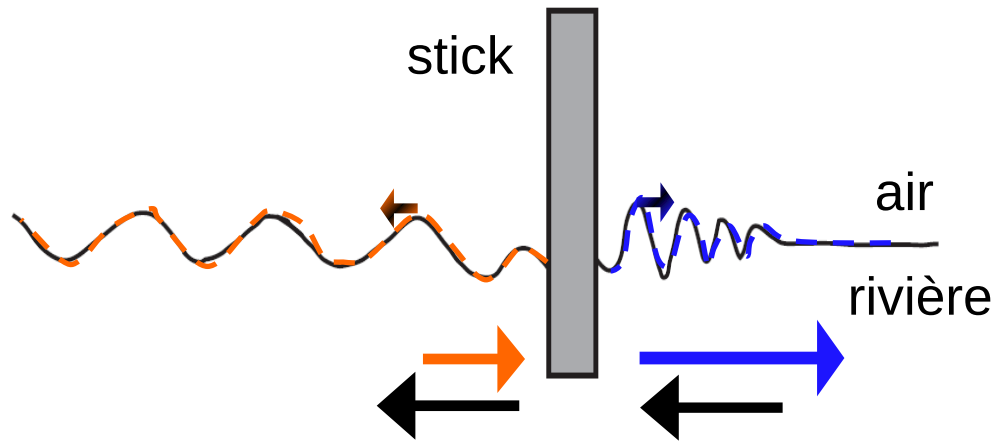
Capillary
waves



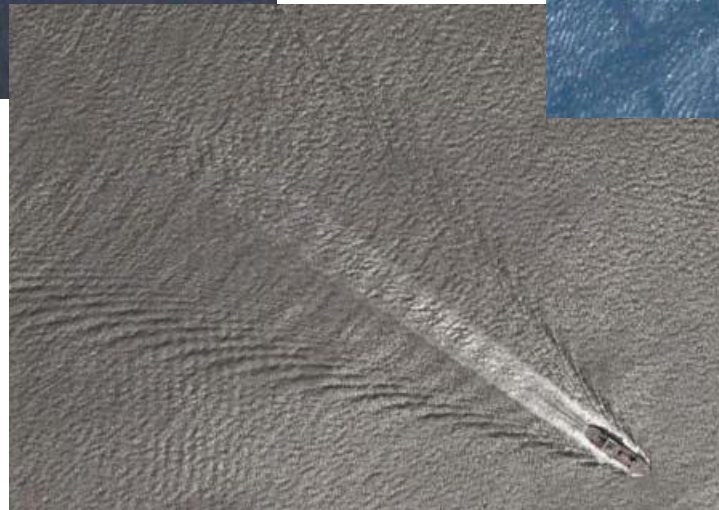
water drop $< l_c$



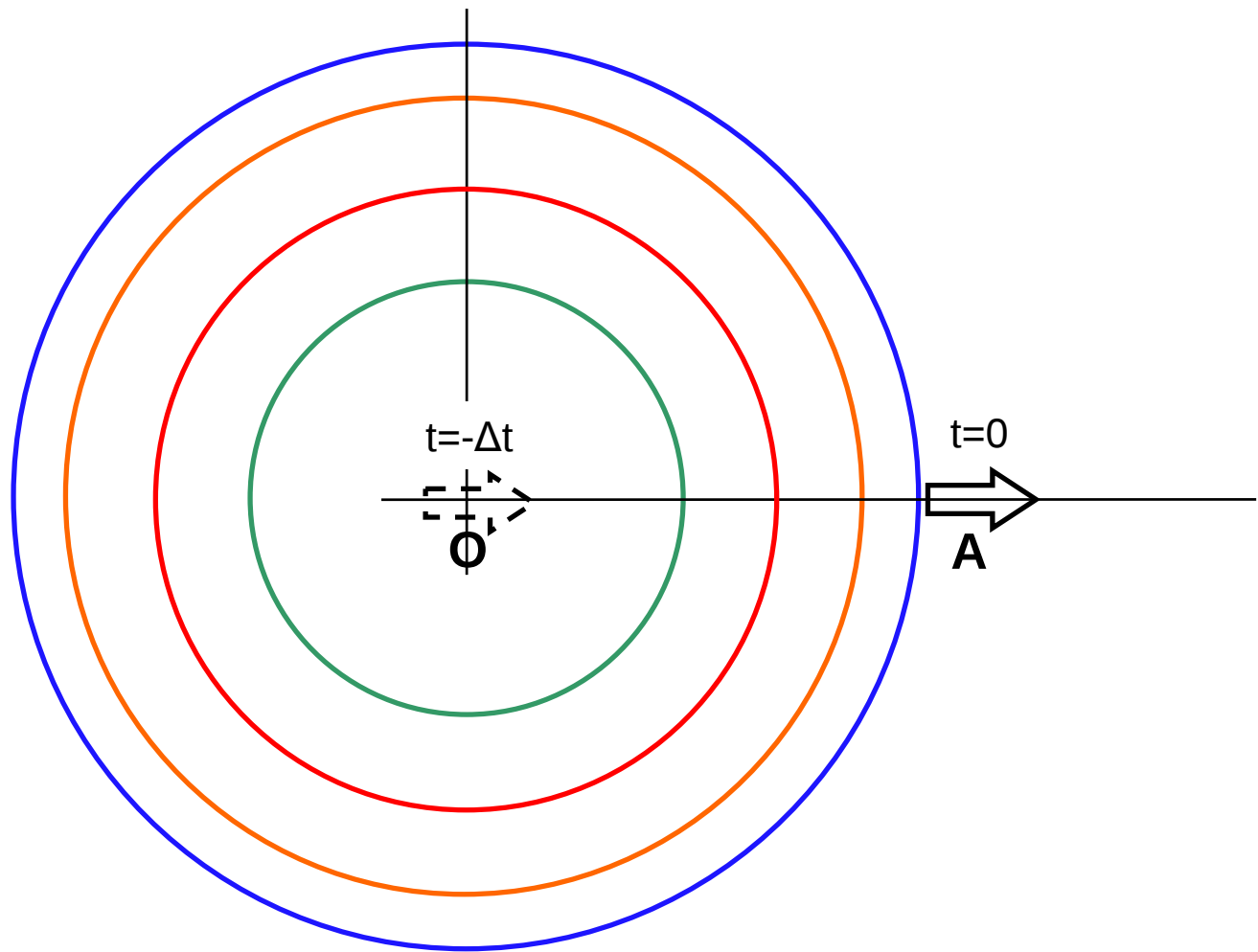
Waves created by an obstacle in a river



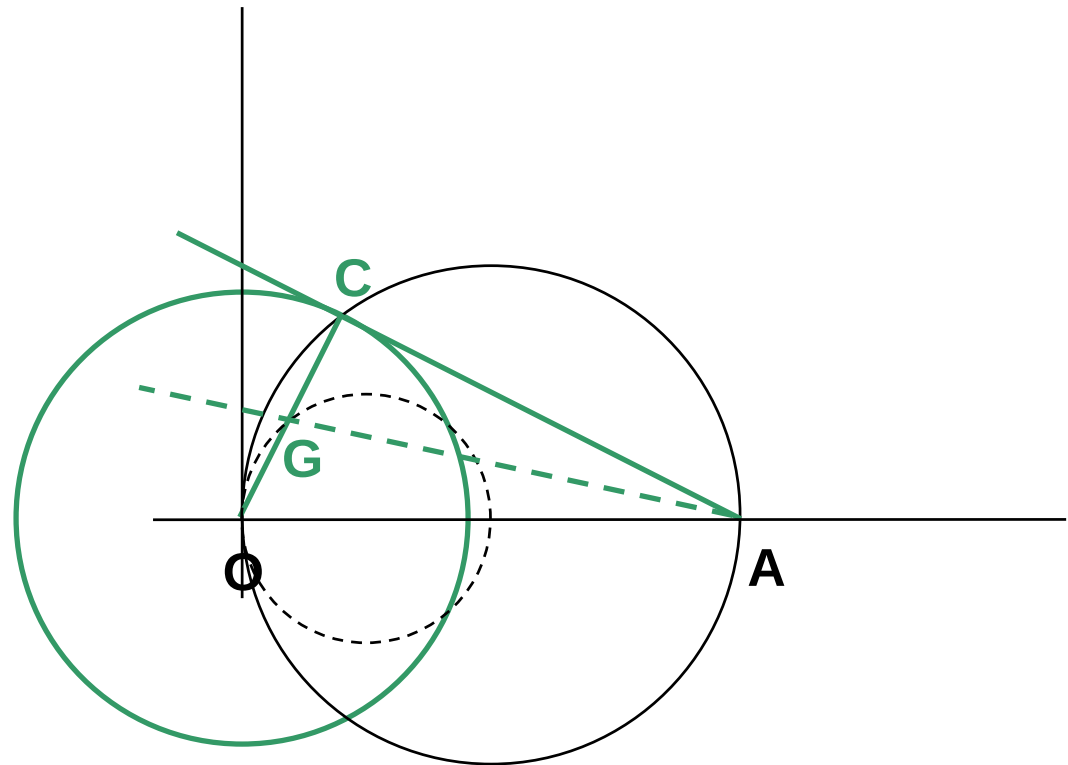
Kelvin wakes



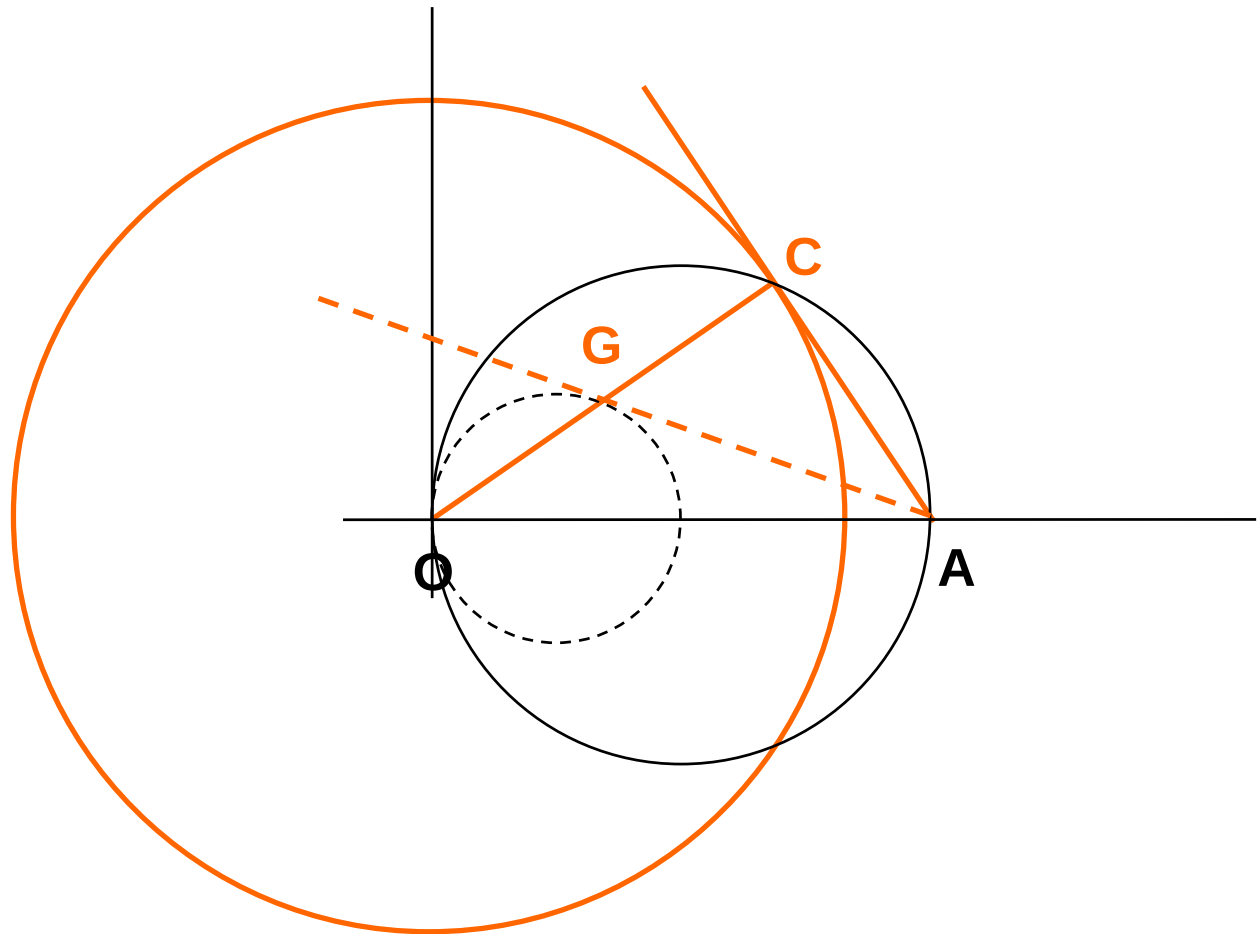
Kelvin wake



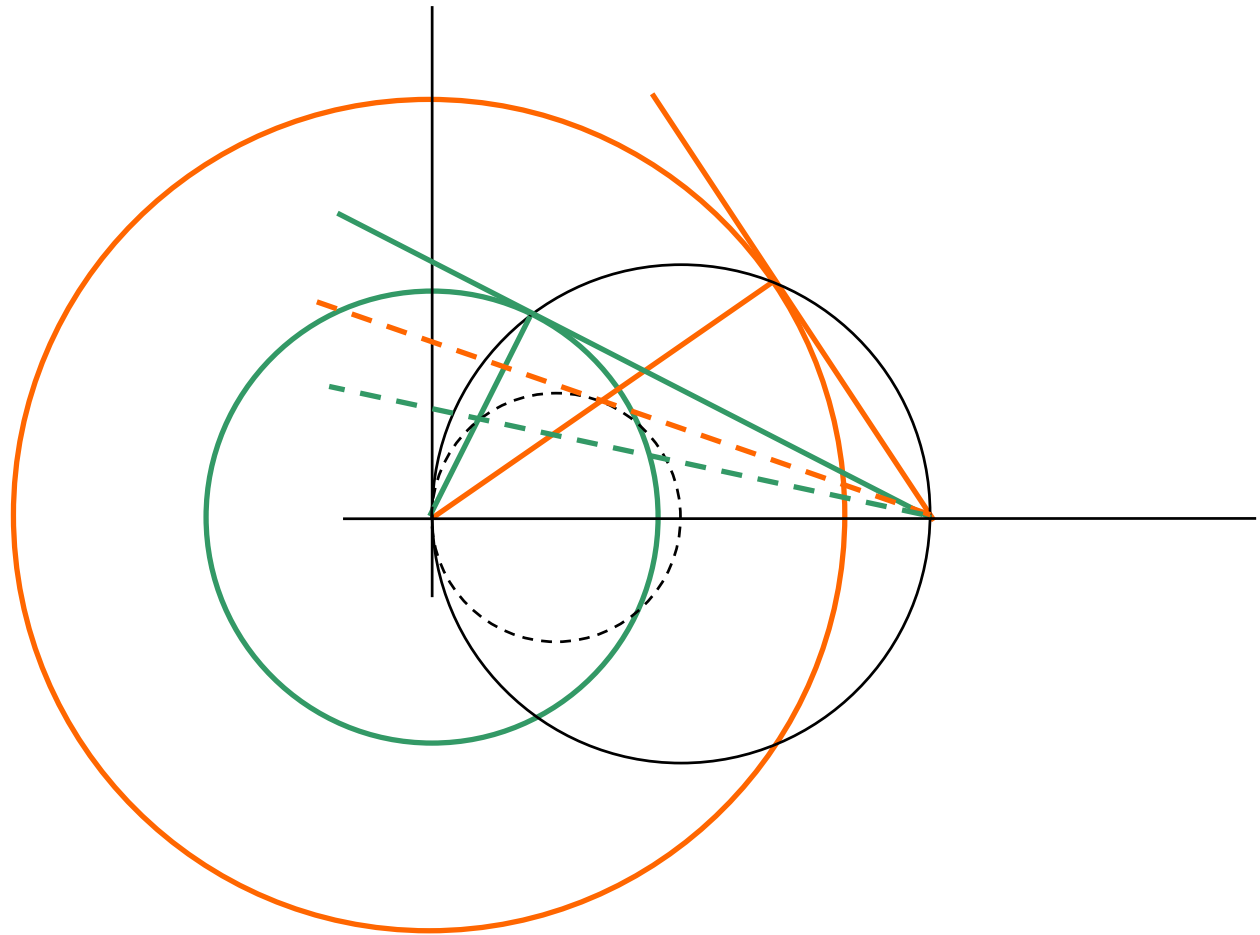
Waves created by a ship



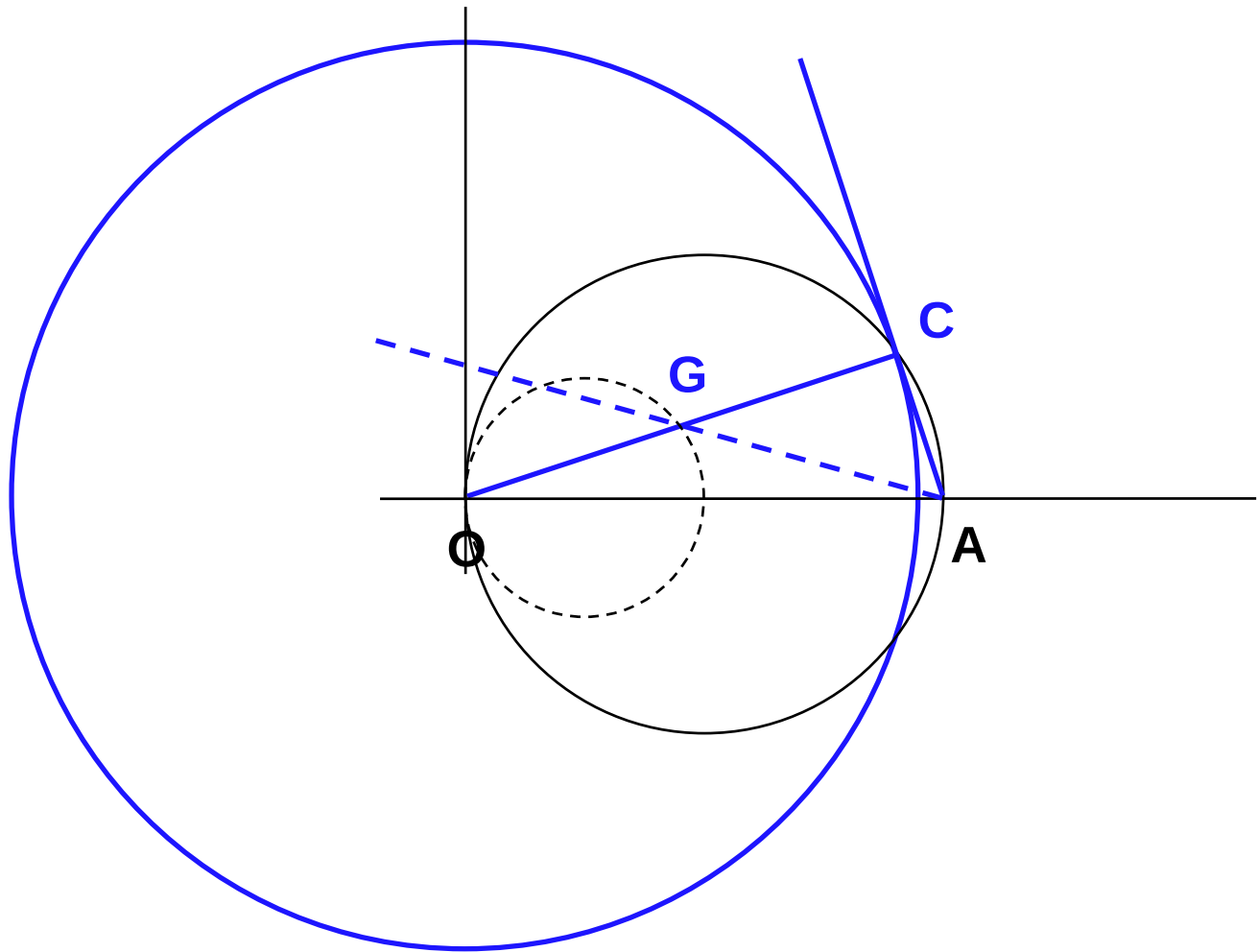
Waves created by a ship



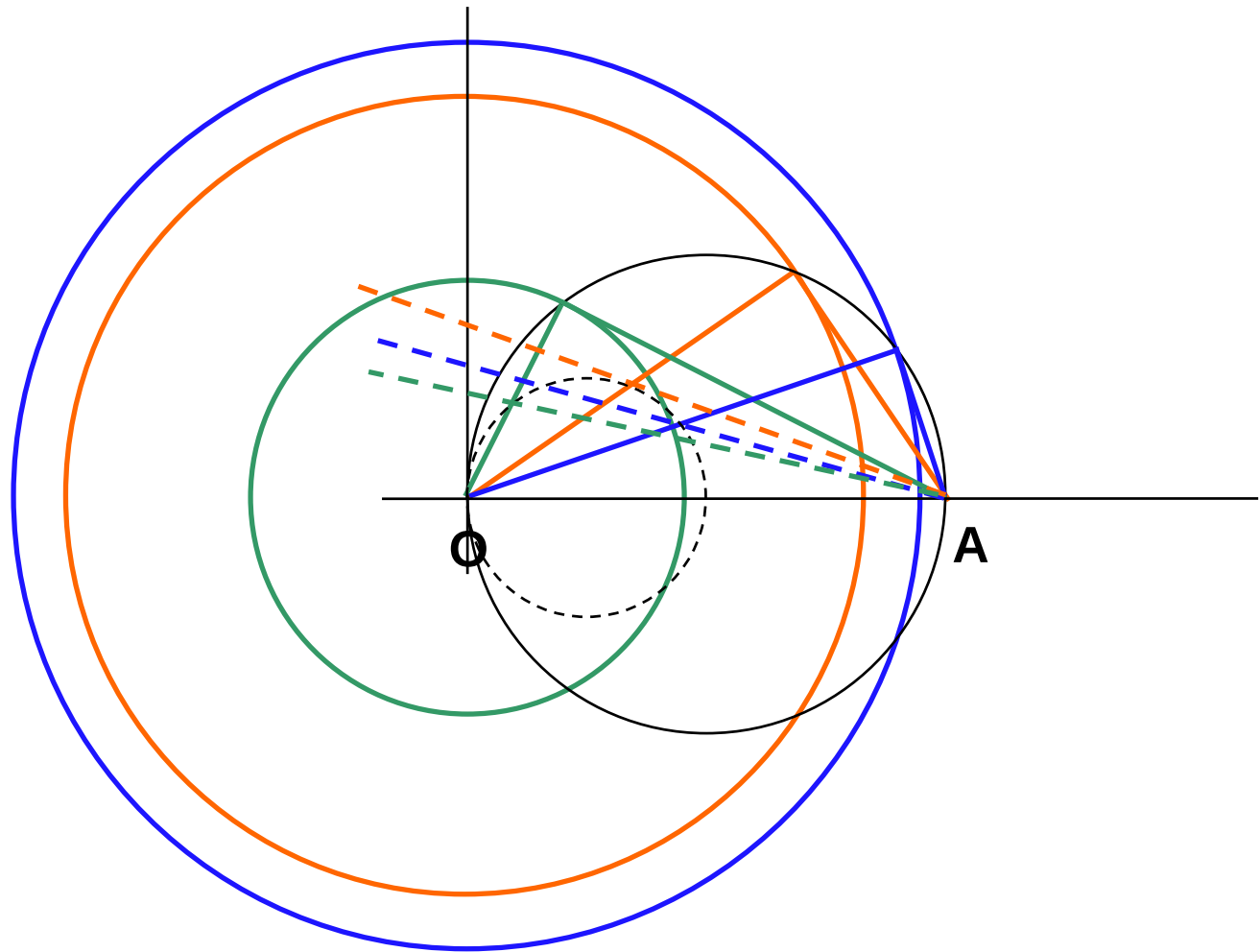
Waves created by a ship



Waves created by a ship



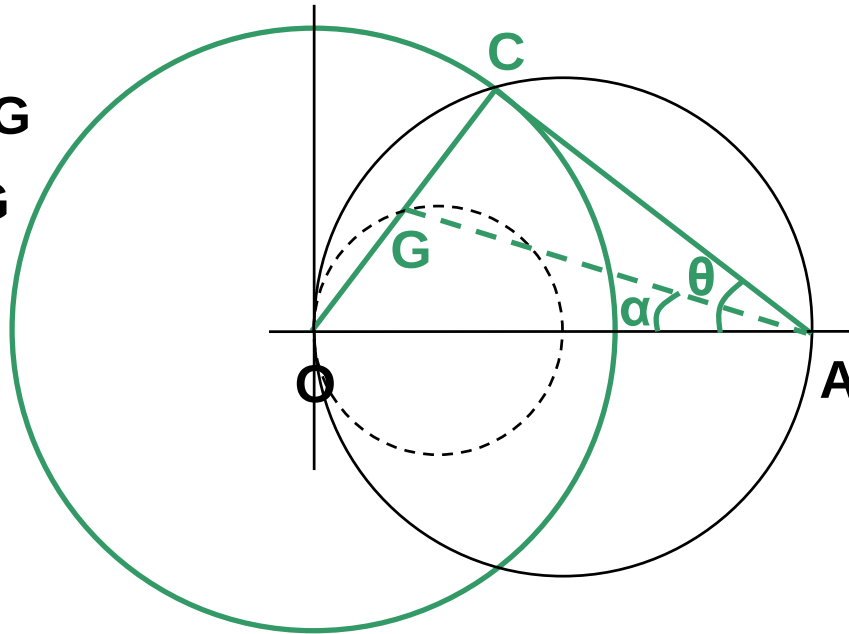
Waves created by a ship



Waves created by a ship

$$\sin(\alpha)/OG = \cos(\theta)/AG$$

$$\sin(\theta - \alpha)AG = GC = OG$$

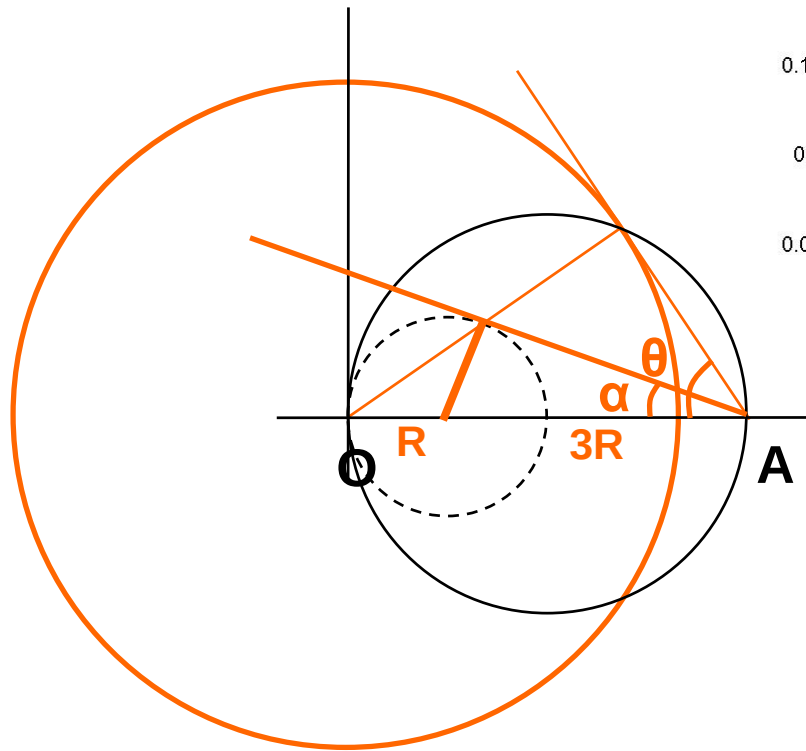


$$\Rightarrow \sin(\alpha) = \cos(\theta) \sin(\theta - \alpha)$$

$$\Rightarrow \sin(\alpha) = \cos(\theta) (\sin(\theta) \cos(\alpha) + \cos(\theta) \sin(\alpha))$$

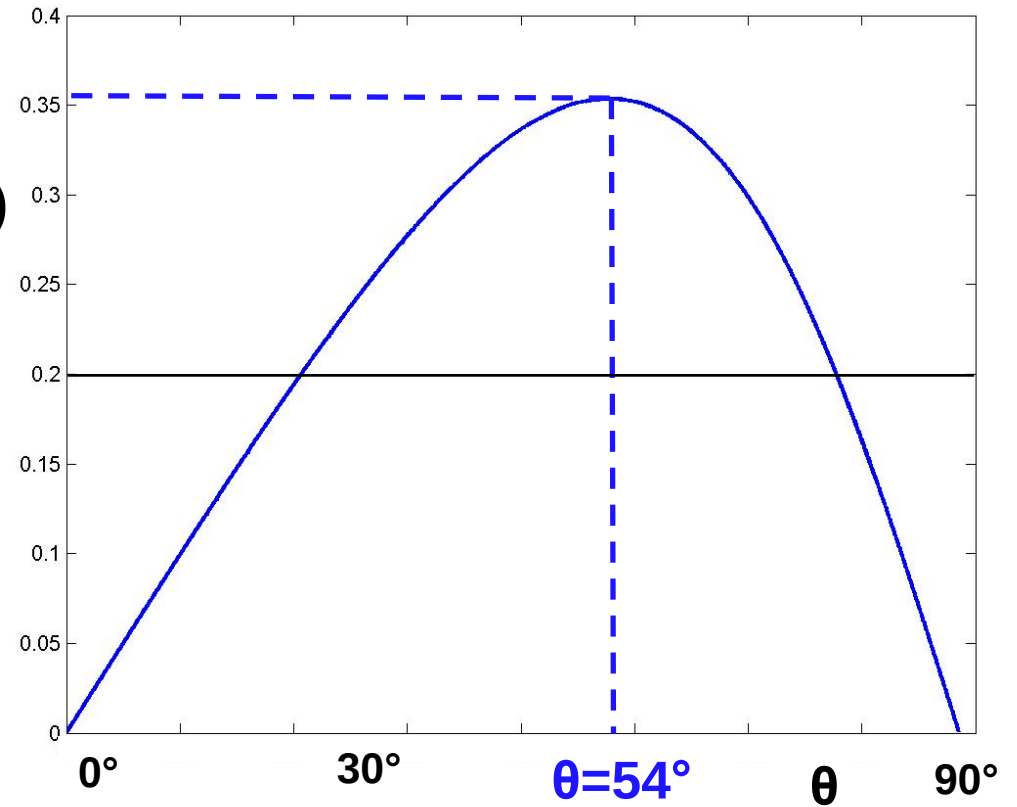
$$\Rightarrow \tan(\alpha) = \cos(\theta) \sin(\theta) / (1 + \cos^2(\theta))$$

Waves created by a ship

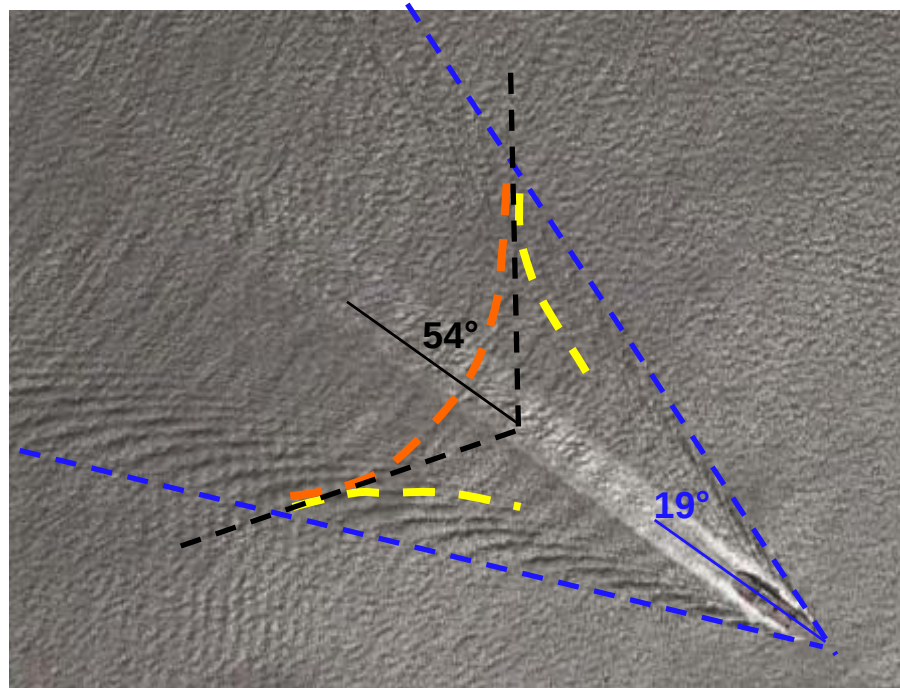


$$\alpha = 19^\circ$$

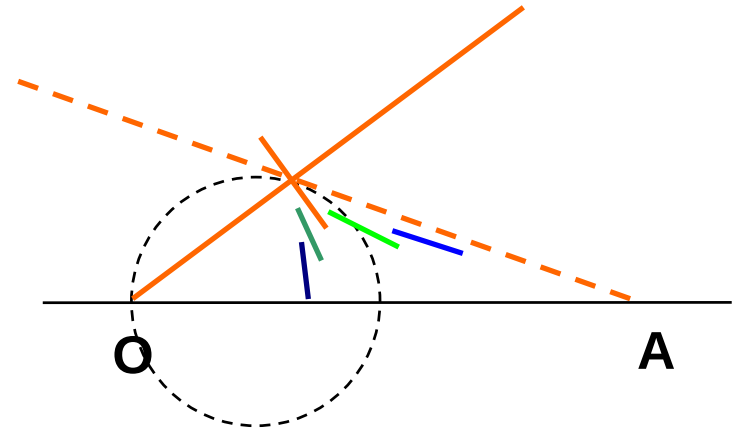
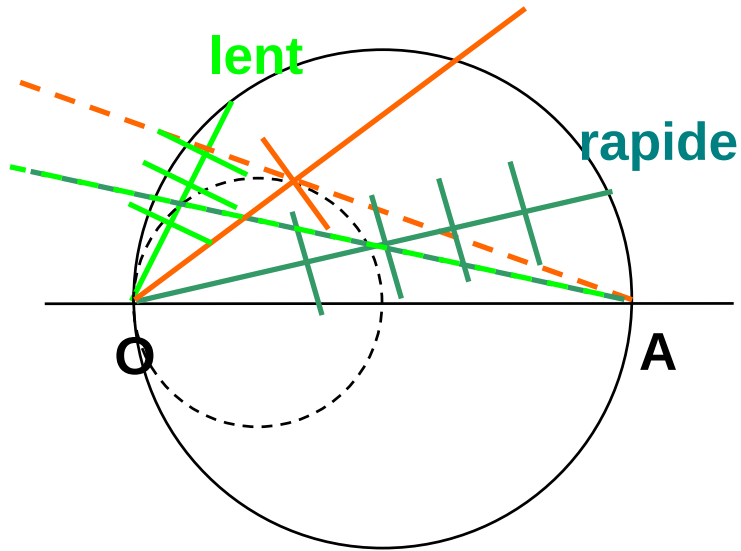
$$\tan(\alpha)$$



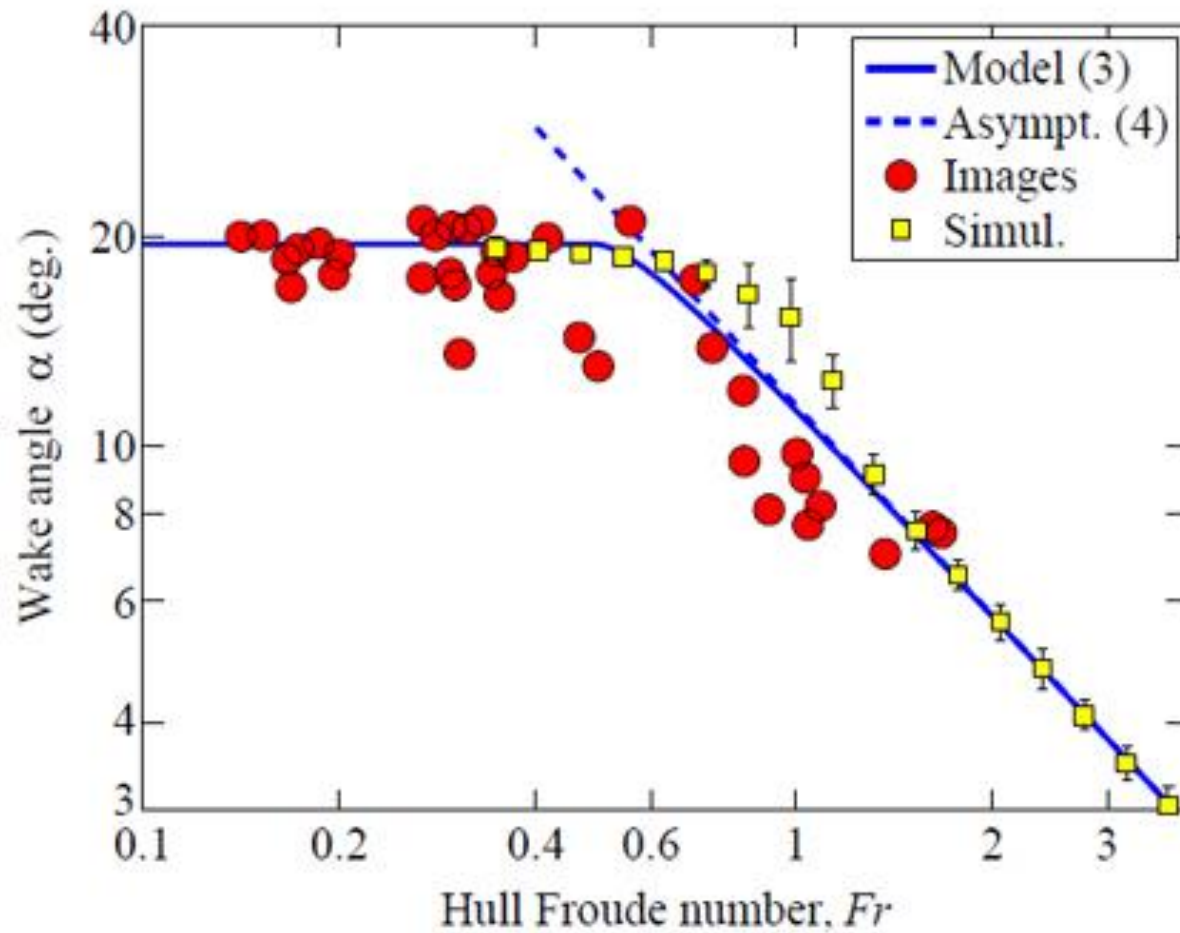
Waves created by a ship



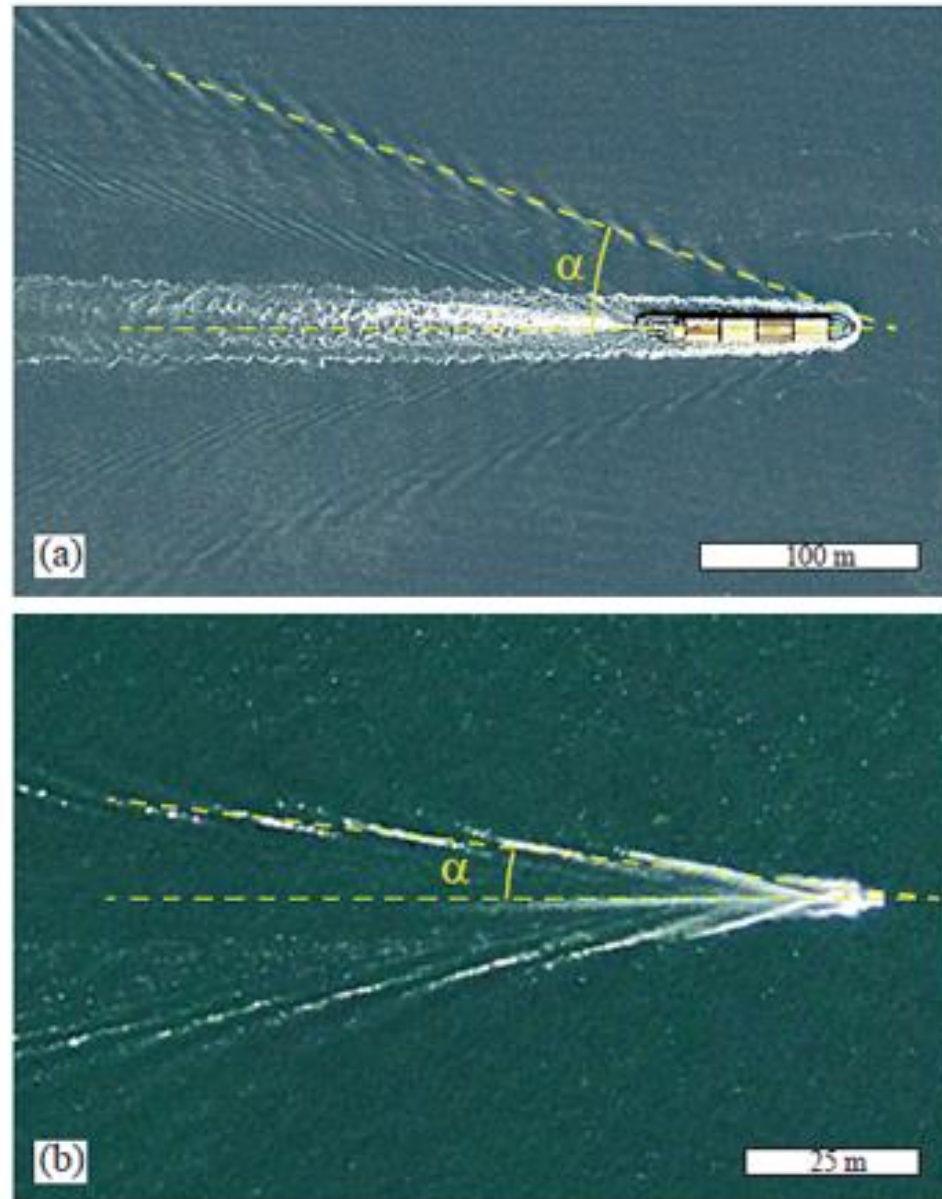
Waves created by a ship



Careful google-earth data analysis by Moisy and Rabaud (2013)



Careful google-earth data analysis by Moisy and Rabaud (2013)



Froude number

$$Fr = \frac{U}{c} \quad \text{c: celerity of wave}$$

$$Fr = \frac{U}{\sqrt{gd}} \quad \text{d: size of ship}$$

$$Fr = \frac{U}{\sqrt{gl}} \quad \text{shallow water}$$

Froude decomposition

$$C_T(F, Re) = C_D(Re) + C_W(F, Re)$$

Total drag Drag in Wave drag
 absence of
 free surface

Froude hypothesis

$$C_T(F, Re) = C_D(Re) + C_W(F, Re)$$

Total drag Drag in Wave drag
 absence of
 free surface

Alternative expression

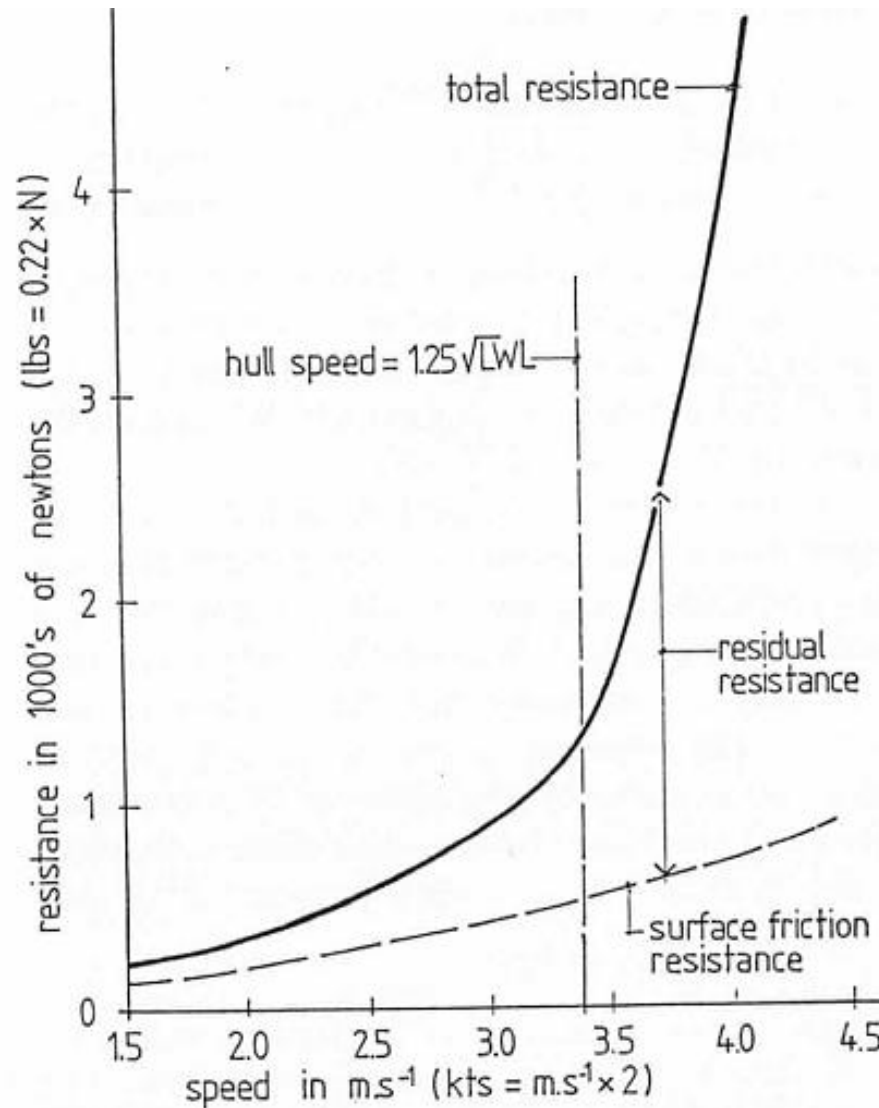
$$C_T(F, Re) = C_F(Re) + C_P(F, Re)$$

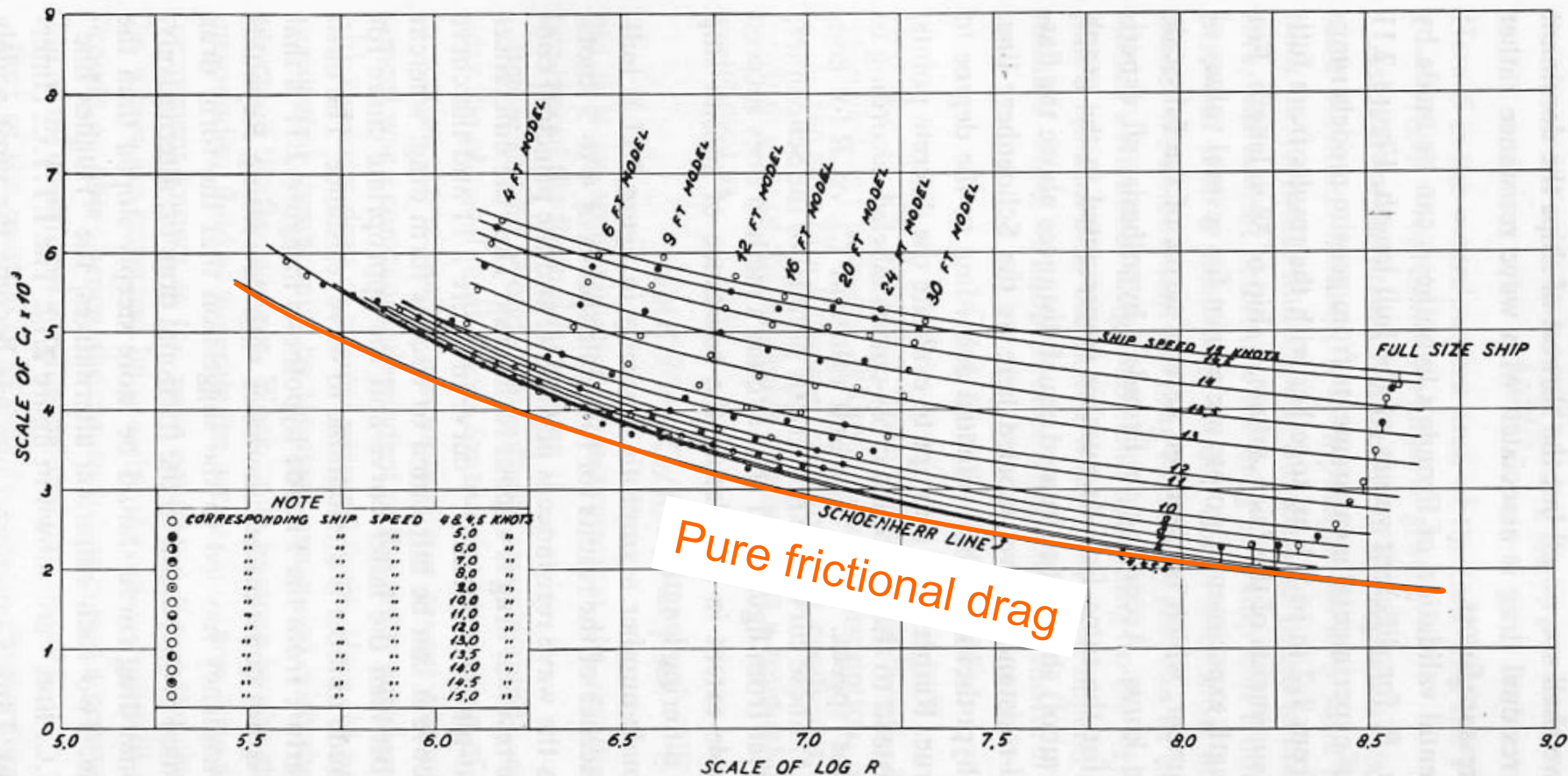
Total drag	Friction drag	Pressure drag
------------	------------------	------------------

$$C_P(F, Re) \approx C_W(F)$$

Pressure drag	Wave drag (Residual drag)
------------------	---------------------------------

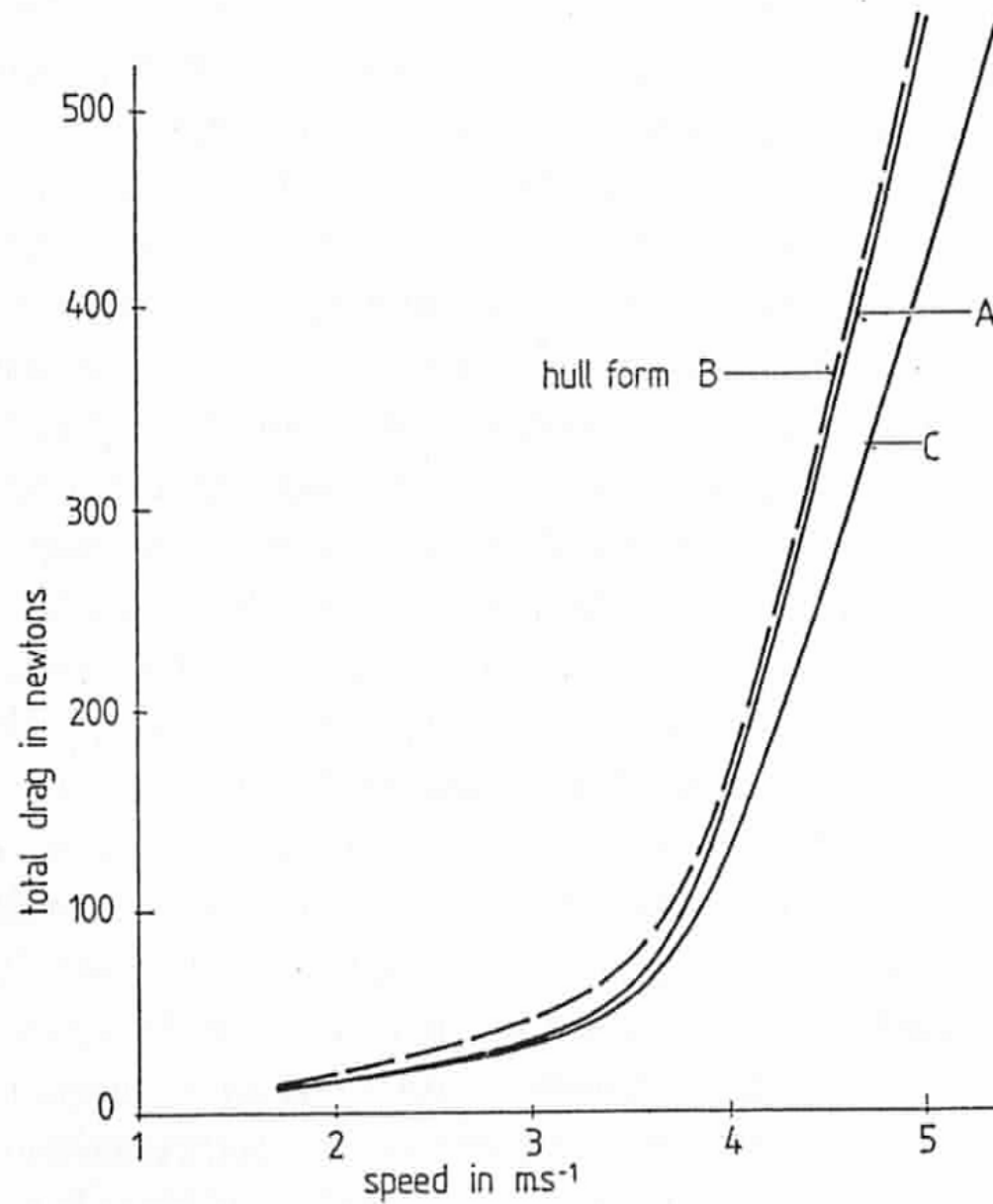
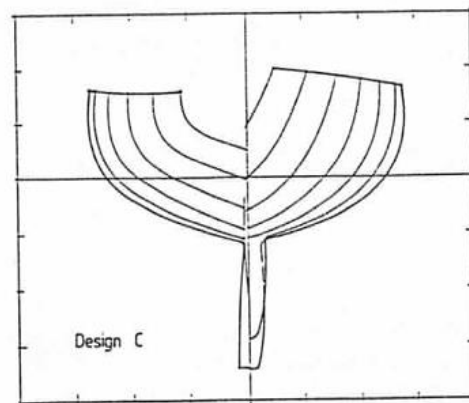
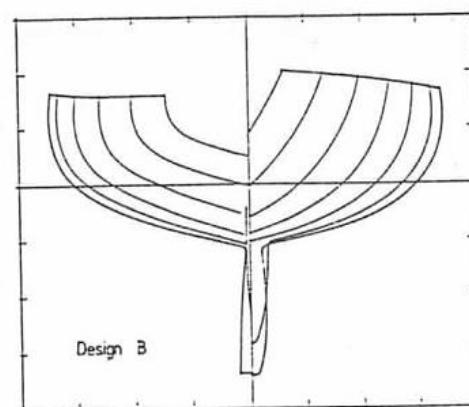
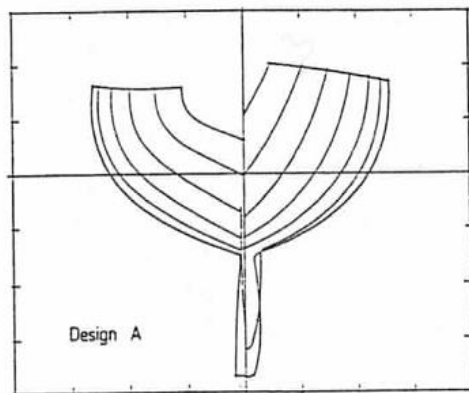
Friction resistance and residual resistance





2.11

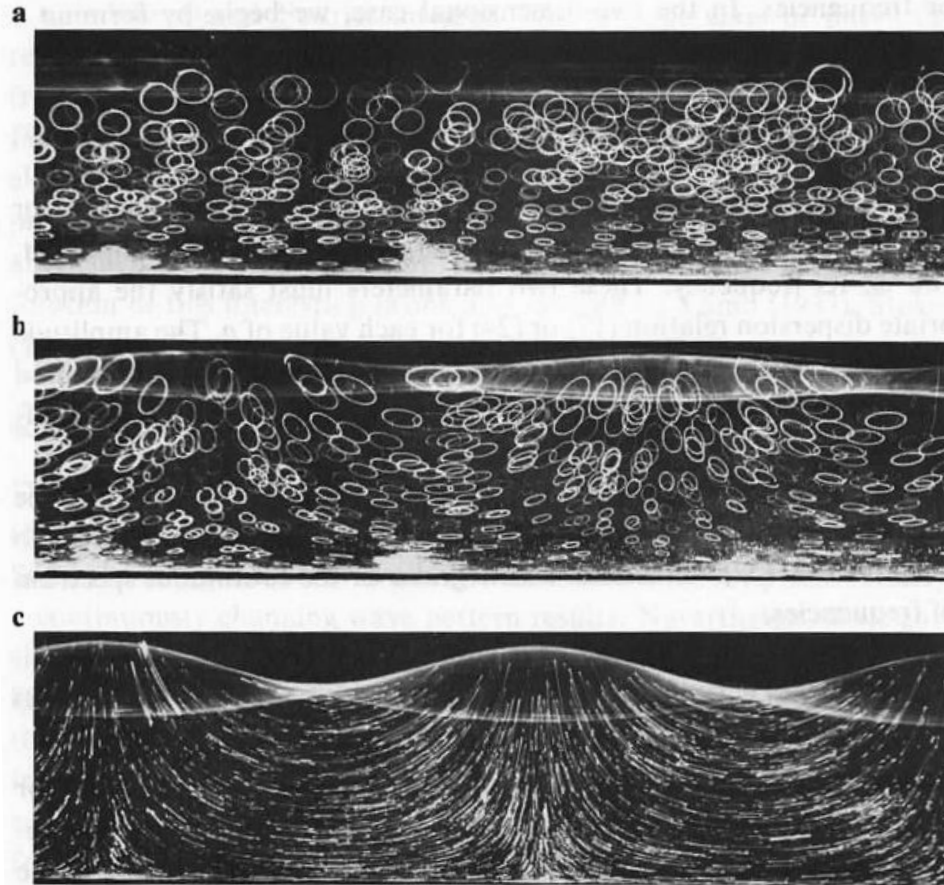
Total drag coefficients of the *Lucy Ashton* and several geosim models of the same vessel (from Troost and Zakay 1951). The faired curves represent constant values of the Froude number and, if Froude's hypothesis were strictly valid, these would be parallel with spacing independent of the Reynolds number. Note that, even for this small full-scale vessel (58 m long), there is a large gap between the largest model results and the full-scale results.



Standing waves

$$\begin{aligned}\eta &= A \cos(kx - \omega t) + A \cos(kx + \omega t) \\ &= 2A \cos kx \cos \omega t,\end{aligned}$$

Reflected waves



6.7

Particle trajectories in a plane progressive wave (a), a partial reflected wave (b), and a standing wave (c). These correspond respectively to a reflection coefficient of 0, 0.38, and 1.0 in equation (56). Note that the reflection coefficient can be measured from the maximum and minimum of the envelope, using (56). These photographs are based on time exposures, and are reproduced from a more extensive series of observations made by Ruellan and Wallet (1950).

Reflected waves

$$\eta = A \operatorname{Re}[e^{-ikx+i\omega t} + R e^{ikx+i\omega t}]$$

R: complex reflection coefficient

$$\eta = A \operatorname{Re}[e^{-ikx+i\omega t}(1 + R e^{2ikx})]$$

Wave energy

$$KE + PE = \rho \iiint_{\mathcal{V}} \left(\frac{1}{2} V^2 + gy \right) d\mathcal{V}$$

$$E = \rho \int_{-h}^{\eta} \left(\frac{1}{2} V^2 + gy \right) dy = \frac{1}{2} \rho \int_{-h}^{\eta} V^2 dy + \frac{1}{2} \rho g \left(\eta^2 - \cancel{h^2} \right).$$

Neglect potential energy of fluid at rest

Wave energy

$$KE + PE = \rho \iiint_{\mathcal{V}} \left(\frac{1}{2} V^2 + gy \right) d\mathcal{V}$$

$$E = \rho \int_{-h}^{\eta} \left(\frac{1}{2} V^2 + gy \right) dy = \frac{1}{2} \rho \int_{-h}^{\eta} V^2 dy + \frac{1}{2} \rho g \left(\eta^2 - \cancel{h^2} \right).$$

Neglect potential energy of fluid at rest

$$E = \frac{\rho \omega^2 A^2}{4k} e^{2k\eta} + \frac{1}{2} \rho g \eta^2.$$

Wave energy

$$KE + PE = \rho \iiint_{\mathcal{V}} \left(\frac{1}{2} V^2 + gy \right) d\mathcal{V}$$

$$E = \rho \int_{-h}^{\eta} \left(\frac{1}{2} V^2 + gy \right) dy = \frac{1}{2} \rho \int_{-h}^{\eta} V^2 dy + \frac{1}{2} \rho g \left(\eta^2 - \cancel{h^2} \right).$$

Neglect potential energy of fluid at rest

$$E = \frac{\rho \omega^2 A^2}{4k} e^{2k\eta} + \frac{1}{2} \rho g \eta^2.$$

$$E = \frac{1}{4} \rho g A^2 + \frac{1}{2} \rho g A^2 \cos^2(kx - \omega t)$$

Wave energy

$$KE + PE = \rho \iiint_{\mathcal{V}} \left(\frac{1}{2} V^2 + gy \right) d\mathcal{V}$$

$$E = \rho \int_{-h}^{\eta} \left(\frac{1}{2} V^2 + gy \right) dy = \frac{1}{2} \rho \int_{-h}^{\eta} V^2 dy + \frac{1}{2} \rho g \left(\eta^2 - \cancel{h^2} \right).$$

Neglect potential energy of fluid at rest

$$E = \frac{1}{4} \rho g A^2 + \frac{1}{2} \rho g A^2 \cos^2(kx - \omega t)$$

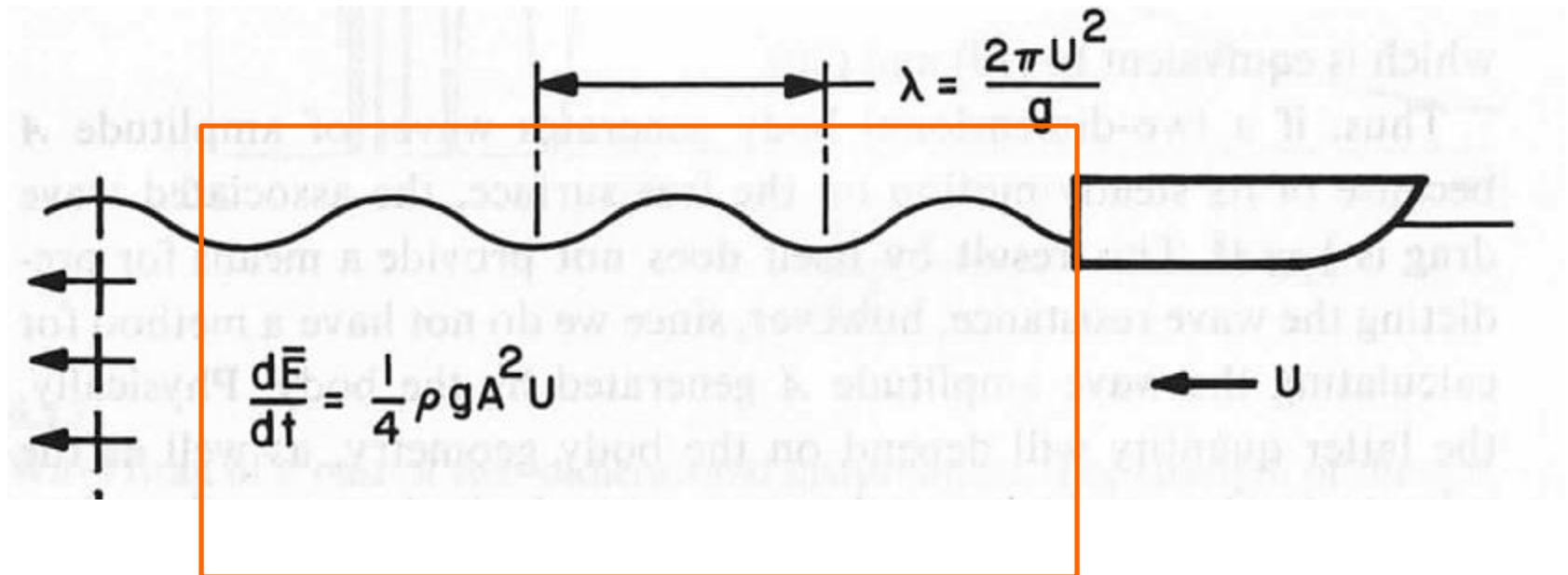
On average:

$$\bar{E} = \frac{1}{2} \rho g A^2$$

Wave energy flux

One can show that the energy travels at the group velocity...

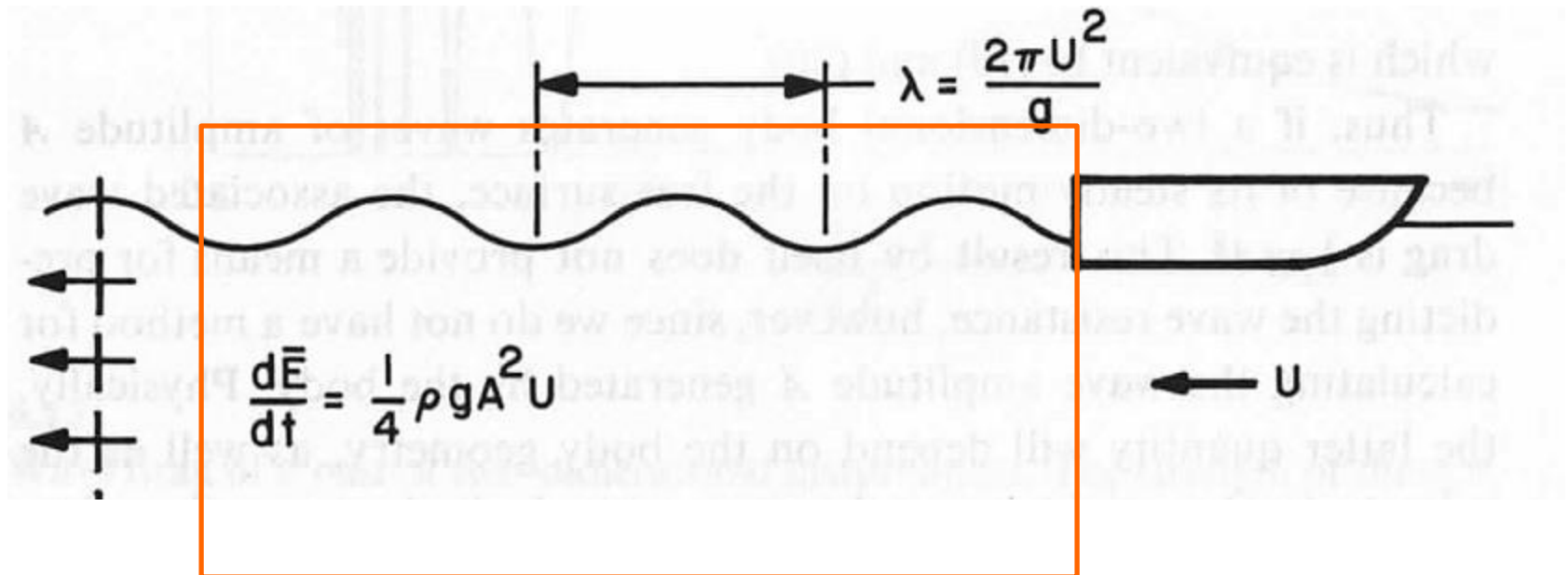
Energy balance (in ship's frame)



Energy input : -work of drag= $-D.U$

Energy output: $(Vg-U)E$

Energy balance (in ship's frame)



Energy input : -work of drag= $-D \cdot U$

Energy output: $(V_g - U)E$

Deep water: $V_g = U/2$

$$D = E/2$$

$$D = \frac{1}{4} \rho g A^2$$

Wave production from bow

displacement

$$\eta = a \cos(kx + \varepsilon)$$

dispersion
relation

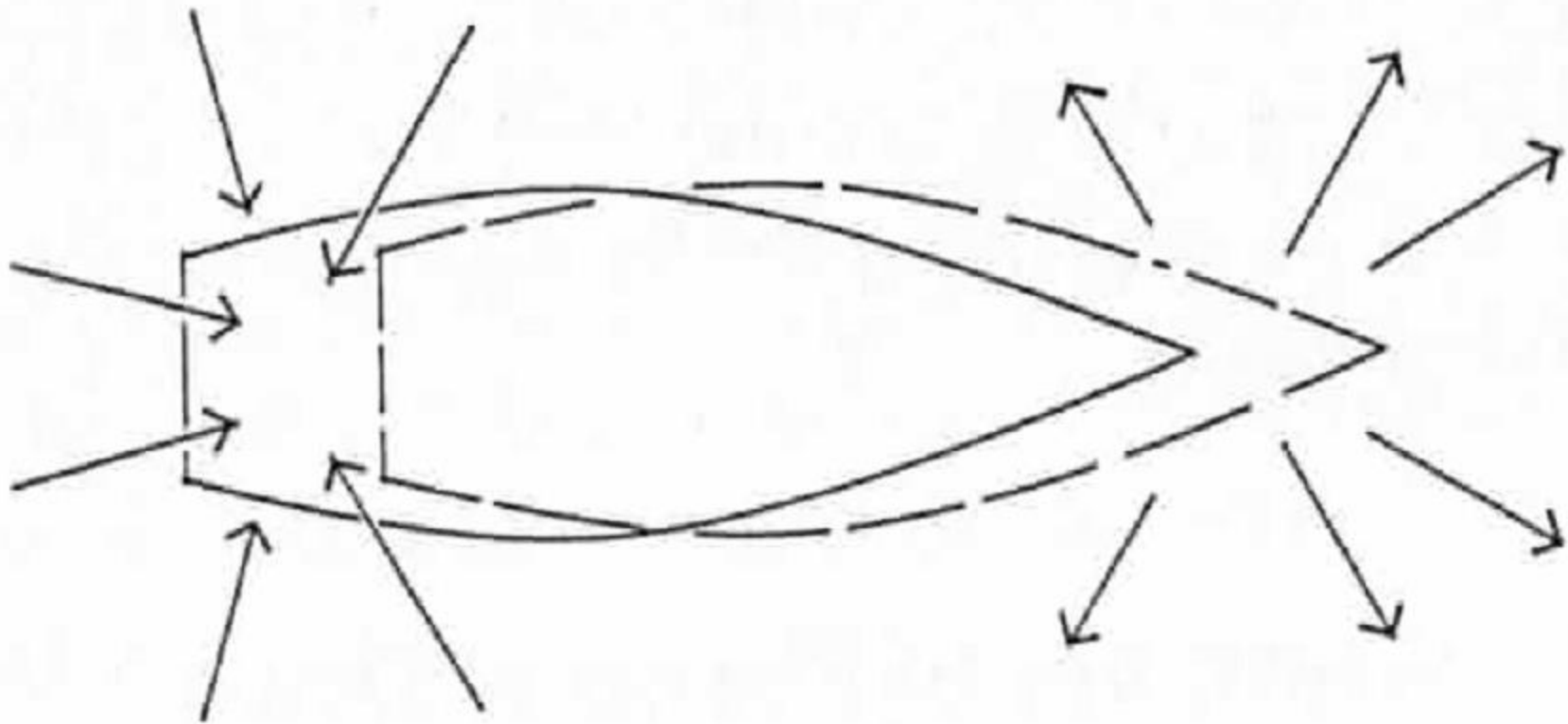
$$k = g/U^2$$

wave amplitude

$$a$$

wave energy

$$\frac{1}{4} \rho g a^2$$

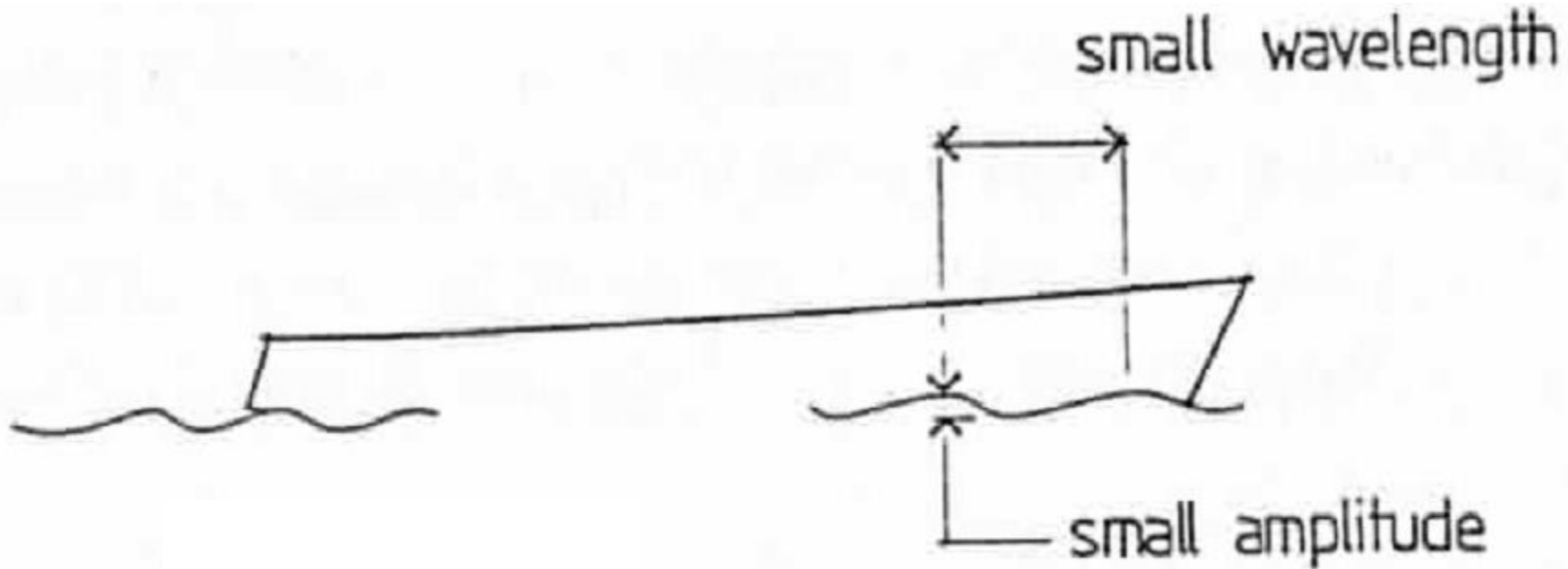


Stern
“Hole”

Bow
“Bump”

Wavelength selection as a function of velocity

Which of these two ships has the highest velocity?



Superimpose wave of opposite amplitude
from stern located at a distance l

displacement

$$\begin{aligned}\eta &= a \cos(kx + \varepsilon) - a \cos(kx + \varepsilon + kl) \\ &= \operatorname{Re}\{ae^{i(kx+\varepsilon)}(1 - e^{ikl})\}.\end{aligned}$$

dispersion
relation

$$k = g/U^2$$

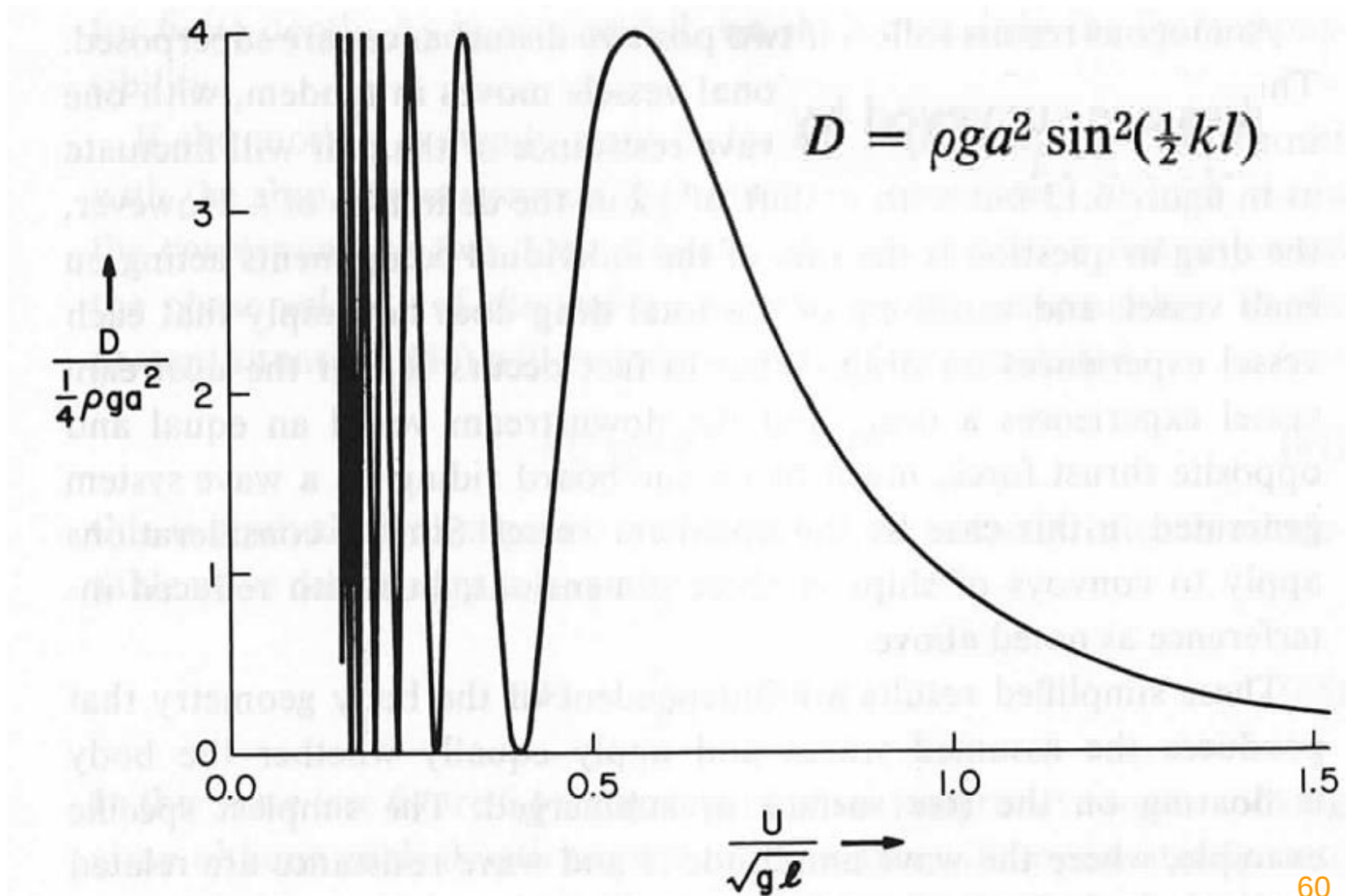
wave amplitude

$$A = a|1 - e^{ikl}| = 2a|\sin(\tfrac{1}{2}kl)|$$

wave energy

$$D = \rho g a^2 \sin^2(\tfrac{1}{2}kl)$$

Superimpose wave of opposite amplitude
from stern located at a distance l



The missing ingredient is now to determine
the wave amplitude!

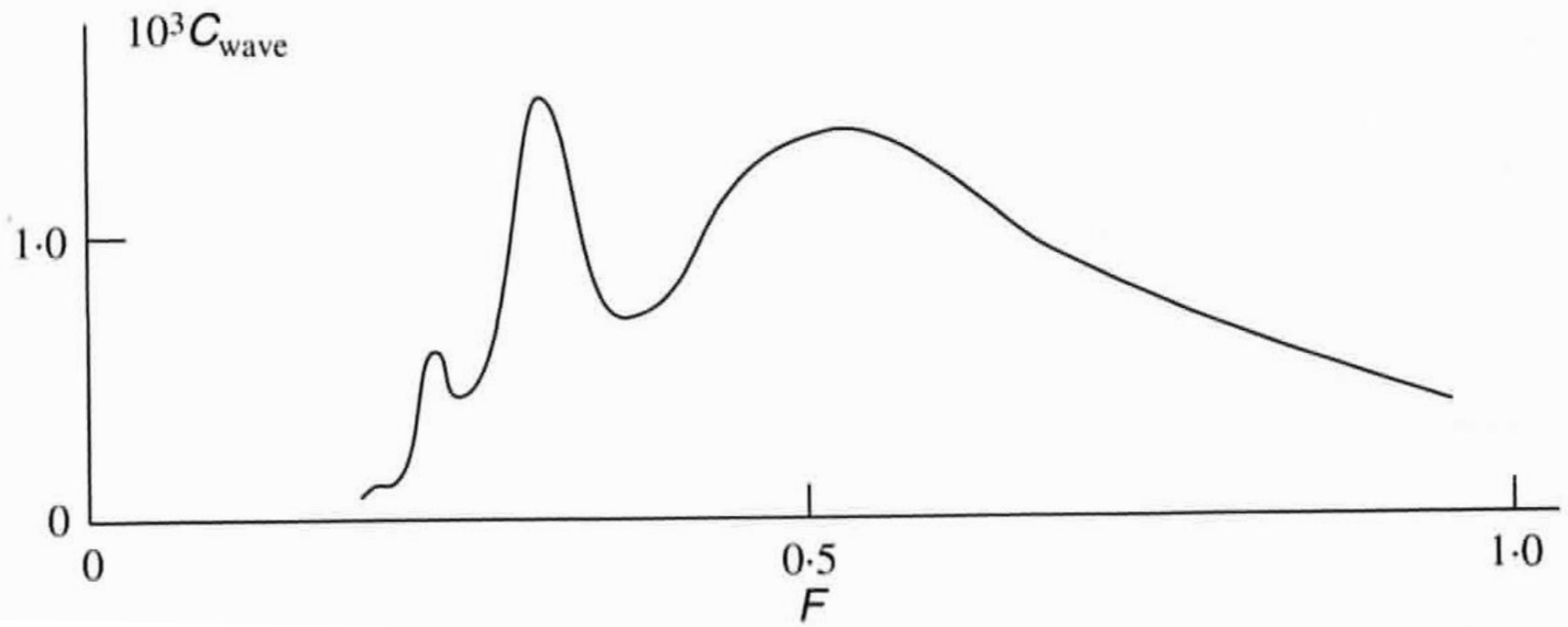
Combination of theoretical and empirical
results.

Generalisation to 3D

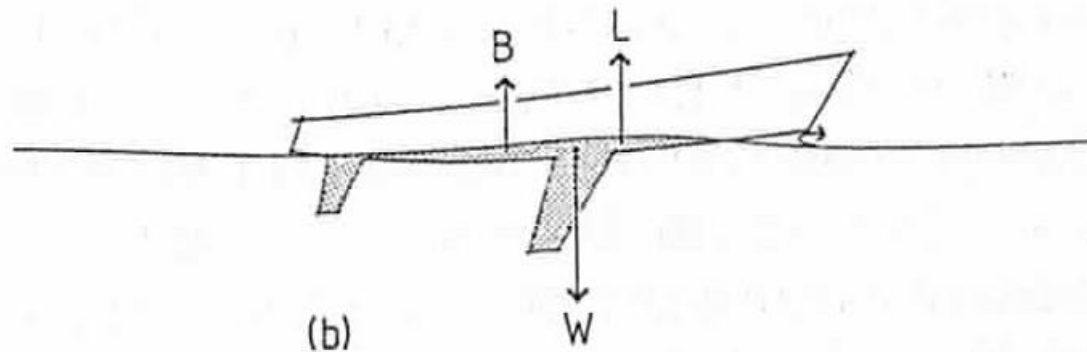
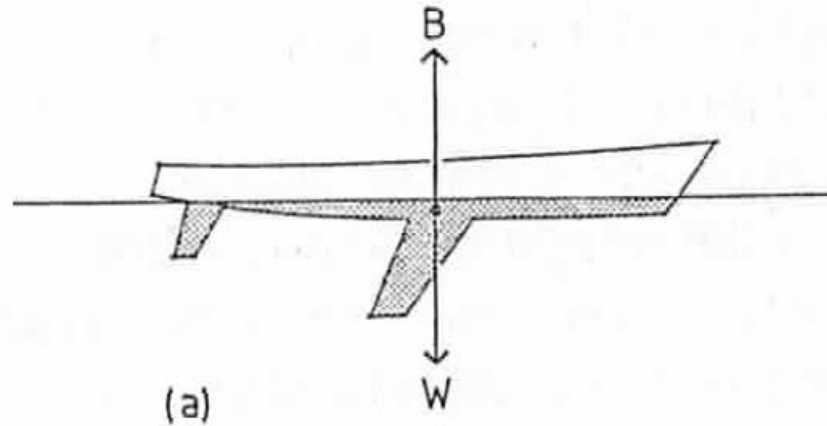
$$\frac{dE}{dt} = \frac{1}{2} \rho g \int_{-\infty}^{\infty} A^2 (V_g \cos \theta - U) dz.$$

$$D = \frac{1}{2} \pi \rho U^2 \int_{-\pi/2}^{\pi/2} |A(\theta)|^2 \cos^3 \theta d\theta.$$

Less interference



This picture is in reality complicated by the floating position of the ship



Which opens the field of dynamical reaction of ships to waves....

