Hydrodynamics

Marmottant and Villermaux (2004)

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Chapter 1:
Introduction

Outline

1. Introduction
2. Fluid: Definition and models
3. Fluid Kinematics

Introduction: Detachment on modern cars

Figure 1:
BMW advertising

Detachment on… les modern cars

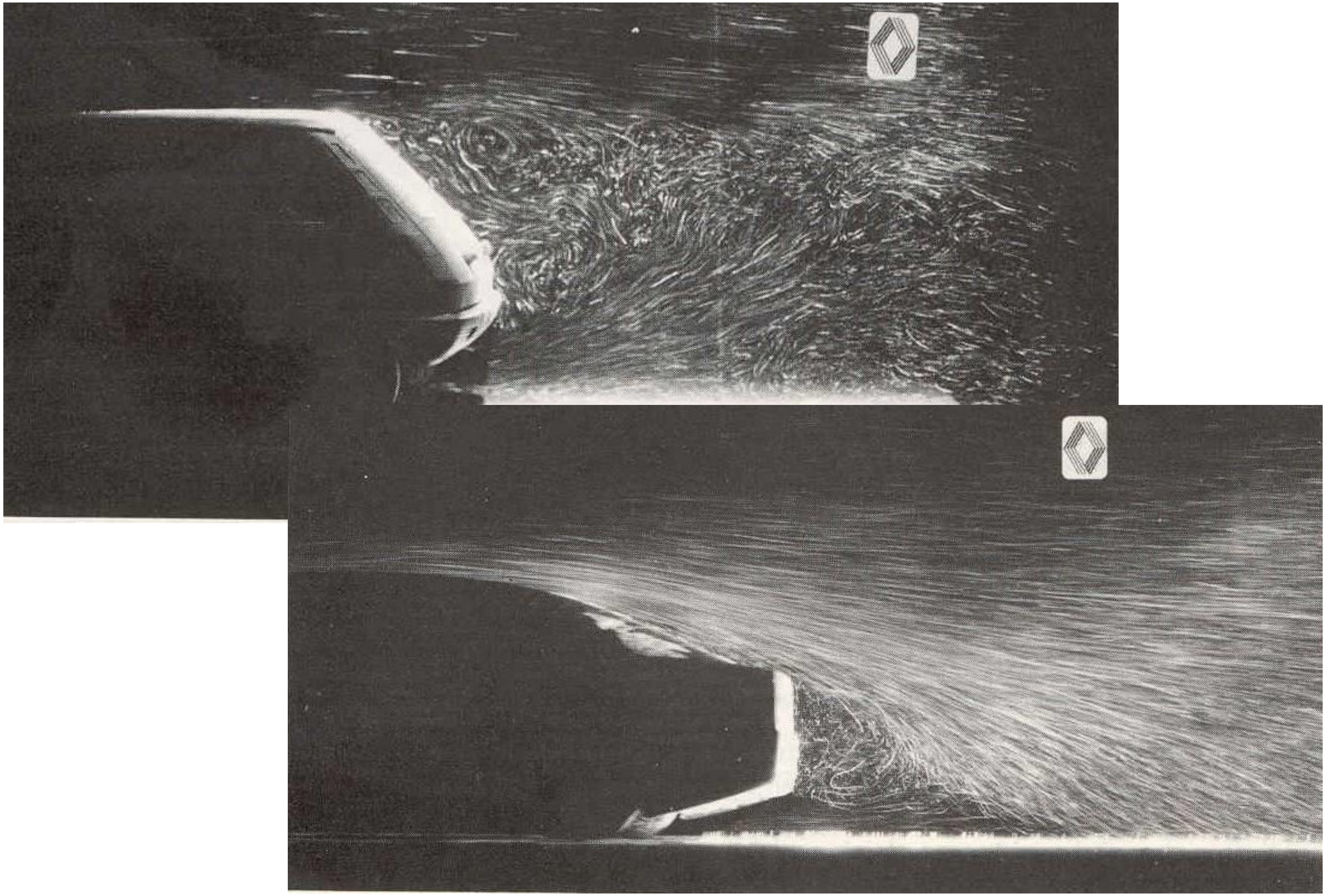


Figure 2: PIV experiment on Renault cars

Aero and/or hydro

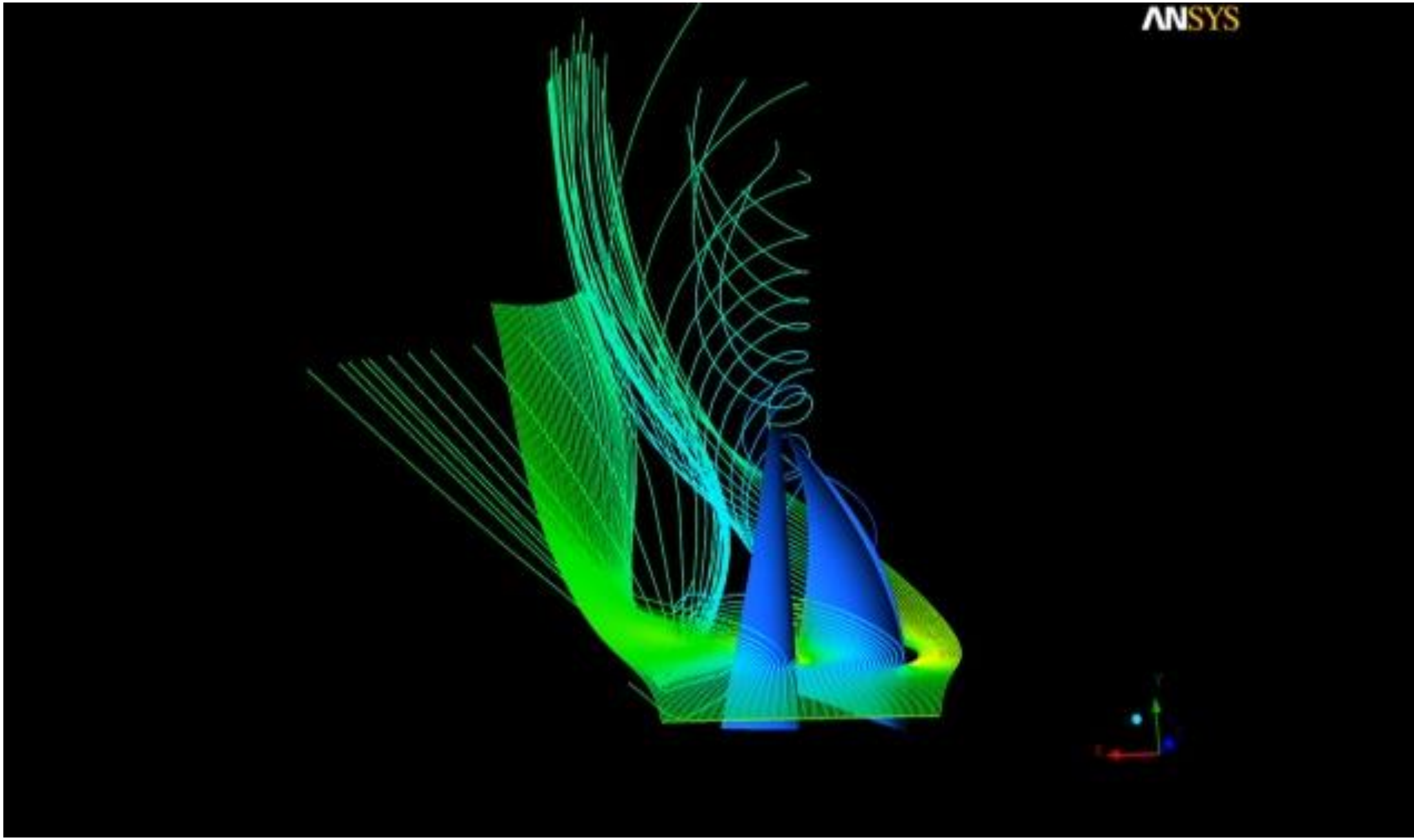


Figure 4:
Alinghi CFD model, EPFL

Drag reduction- waves at the free surface

Figure 5:
Rowing team

Two-phase flows: Turbines, cavitation

Figure 6:
Cavitation erosion on turbine blades

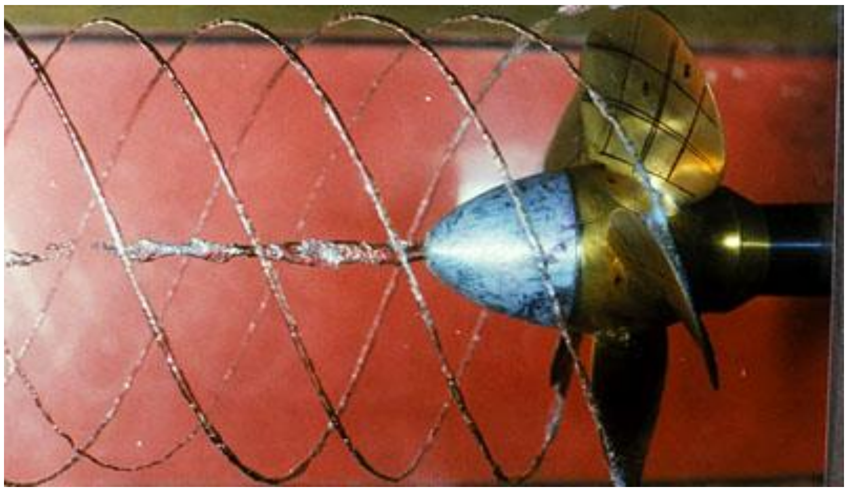


Figure 7:
Tip vortices and cavitation on turbine

Geophysics (atmosphere, ocean, rivers)

Figure 8: Kelvin-Helmholtz instability over mountain

Figure 9:
Rio Negro (slow and clean) meets amazon
(quick and dirty)

Geophysics: tornado

Figure 11: Waterspout

Oil

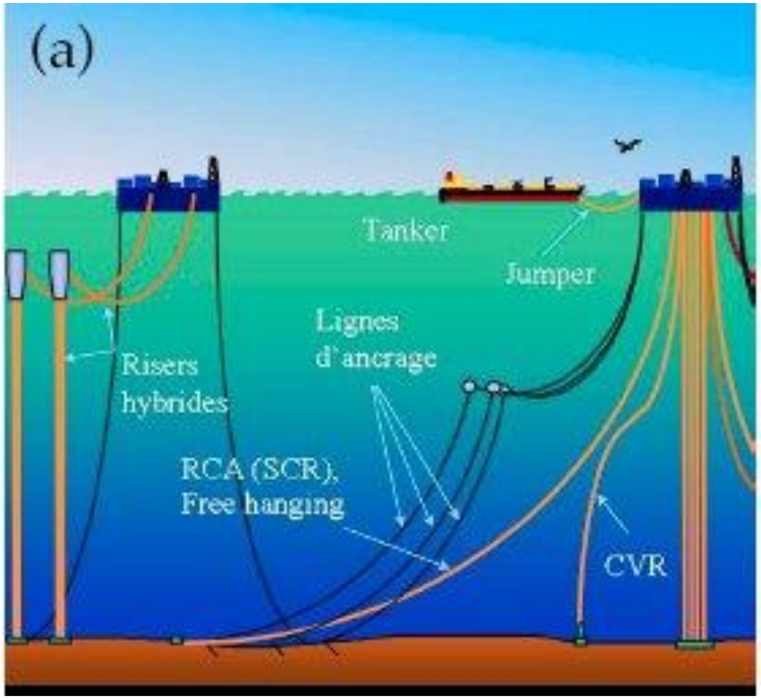


Figure 13: Offshore oil rig

Tidal and ocean waves energy harvesting

Figure 14: Pelamis snake

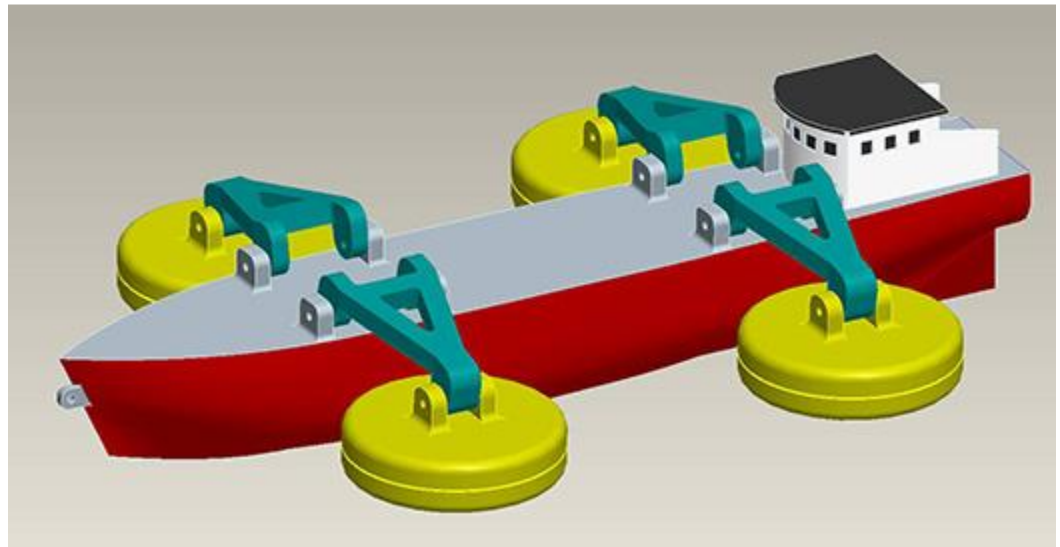


Figure 15: Wave energy harvesting boat concept

Sports

Swimming

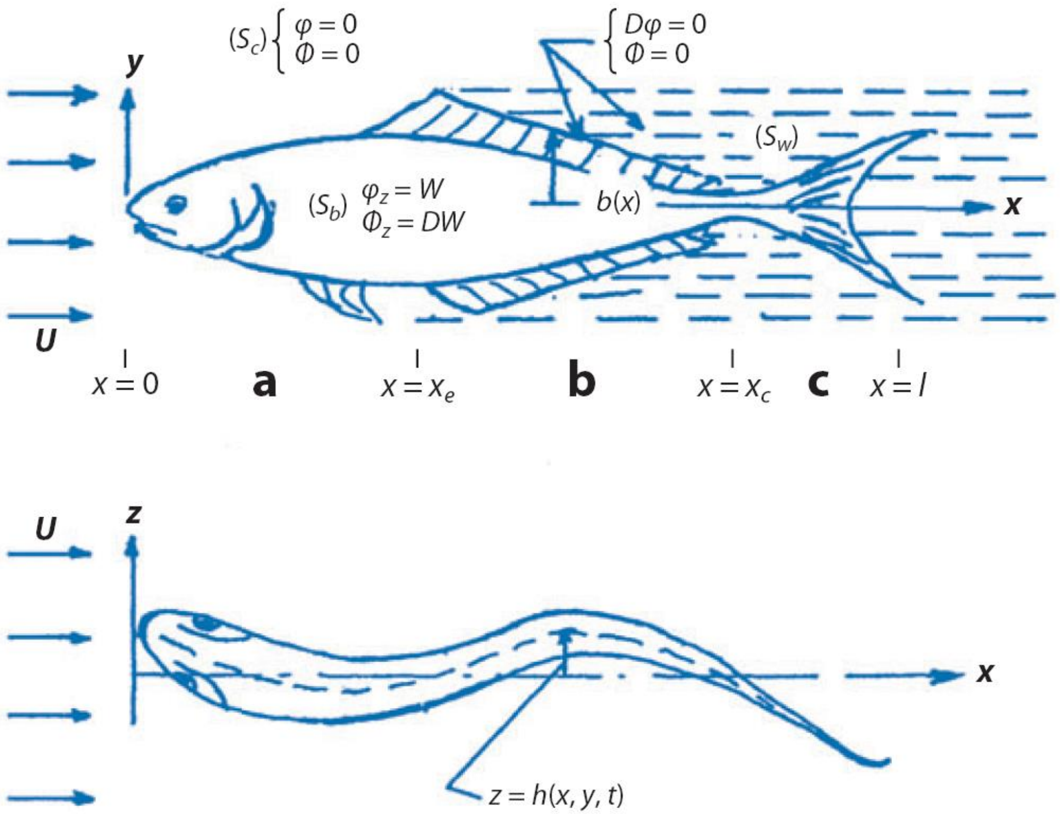


Figure 19:

Flow regions for analyzing fish propulsion: a) Anterior leading-edge section, b) Trailing side-edge section, c) Caudal-fin section

Agriculture

Size of the droplets?

Figure 18:
Irrigation sprinklers, Eggers and Villermaux (2008)

Flow models

- Continuous model
- Newtonian fluid
- Creeping/Stokes flow
- Inviscid fluid
- Unidirectional flow
- Lubrication equation
- Incompressible flow
- Potential flow
- Boundary layer
- Turbulent flow

Classification: Several types of flows

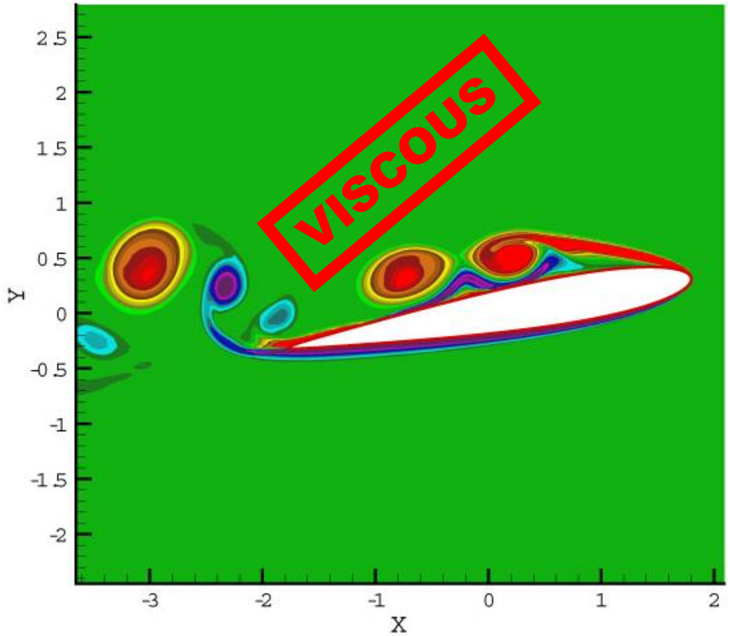
- Compressible/incompressible

Mach > 0.3
« high velocity »
(discontinuities, choc waves…)

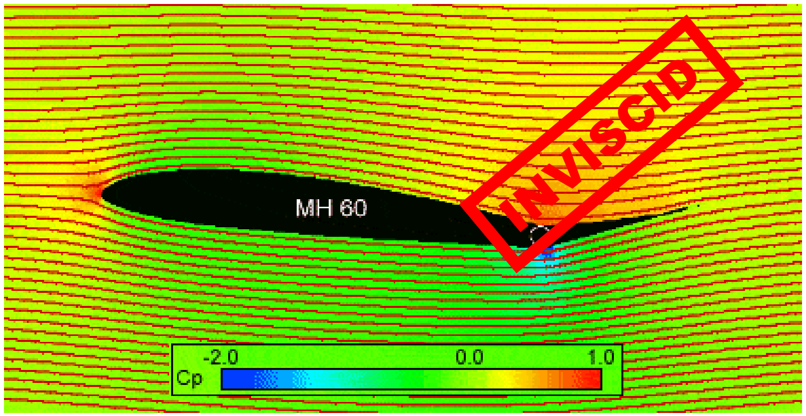
Mach < 0.3
« low velocity »

Classification: Several types of flows

- Viscous/Inviscid



The fluid sticks to the wall, which originates in a boundary layer



The fluid slips at the wall

Instabilities and turbulences

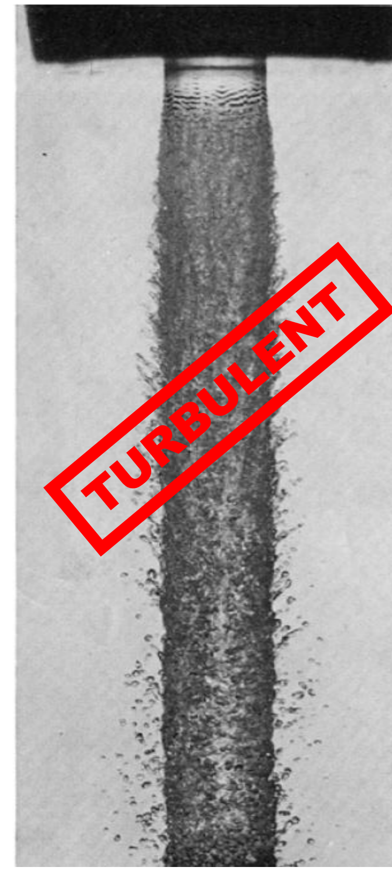
Laminar ⇨ Instability ⇨ Disorder/Pattern/Chaos ⇨Turbulence

Transition

Marmottant and Villermaux (2004)

Rayleigh (1891)

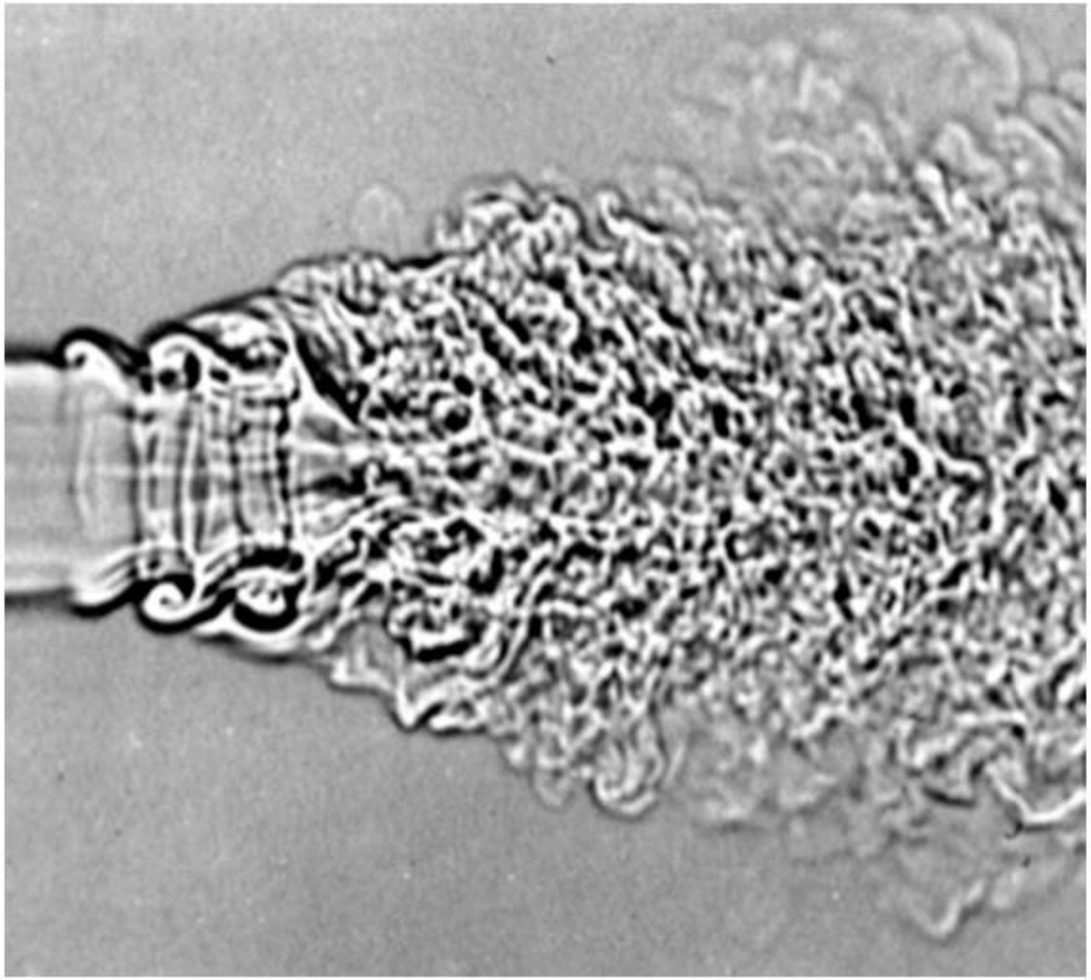
Marmottant and Villermaux (2004)



Hoyt and Taylor (1977)

Transition to turbulence

LAMINAR



TURBULENT

Unsteady, intermittent, no predictability, random

Tools to arrive to or to solve these models

- Integral relations of conservation laws
- Partial differential equations
- Harmonic fields
- Similarity analysis/ nondimensional numbers
- Boundary layers
- Matched Asymptotic expansions
- Self-similar solutions

Beware!

All the flows tackled in this class, although quite far from hydrodynamic applications, will hopefully help you to develop the required intuition to avoid falling into the engineer's most frequent **pitfall**:

Using CFD software without thinking and simplifying

Example

Are you really going to implement a 3D-fluid structure coupling CFD code before:

1. You determine the relevant nondimensional parameters?
2. You estimate the boundary layer thickness and evaluate the feasibility of a correct CFD computation?
3. You model the exact shape by a simplified one where literature might be abundant?

Hydrodynamics

Course: Monday 14h15-16h
Exercises: Monday 16h15-18h
with Alice Marcotte, Timothée Salamon and Simeon Djambov

Grade:

2 intermediate exams, take home (20%)

1 final exam: written, take home (probably) 80%

Books:
- Guyon Hulin & Petit, Physical hydrodynamics [Electronic version on BEAST in french]
- Kundu
- Ryhming PPUR
- Multimedia Fluid Dynamics (DVD or online, I was also upload movies)

Outline

1. Introduction
2. Fluid: Definition and models
3. Fluid Kinematics

What is a fluid? Some definitions

- Dictionary : not solid nor thick, flows easily. Takes the form of its container.

- Physicist : in a fluid, the spatial organization is not that of a solid (crystal) nor the free agitation of molecules of a low pressure gaz.

- Mechanists : a solid is weakly deformable. A fluid is very deformable. Fluids can take any form when they are subjected to forces, regardless of how strong these forces are. Deformation continues until the strain stop (no memory of the reference configuration).

$\boxed{\text{Limits between solid/fluid rather fuzzy}}$

What is a fluid? Some definitions

« **FLUIDE,** *adj. pris subst. (Phys. & Hydrodyn.)* est un corps dont les parties cèdent à la moindre force, & en lui cédant sont aisément mûes entr'elles. Il faut donc pour constituer la fluidité, que les parties se séparent les unes des autres, & cèdent à une impression si petite, qu'elle soit insensible à nos sens ; c'est ce que font l'eau, l'huile, le vin, l'air, le mercure... »

Figure 23:
Definition of a fluid from *l'Encyclopédie Diderot*, d'Alembert.

A Fluid is a body, the constituent parts of which break to the least force, and by breaking are easily moved by one another. In order to constitute fluidity, the parts thus need to separate and break at such a negligible effort that it is unperceivable to our senses; which is what water, oil, air or mercury do…

What is a fluid? Some definitions

- A fluid is a continuum medium that cannot be maintained at rest when stressed.
- In general, this definition is sufficient.
- There exist materials which behave closer to a solid or a fluid, depending on the applied forces, as the so called visco-elastic materials for instance.

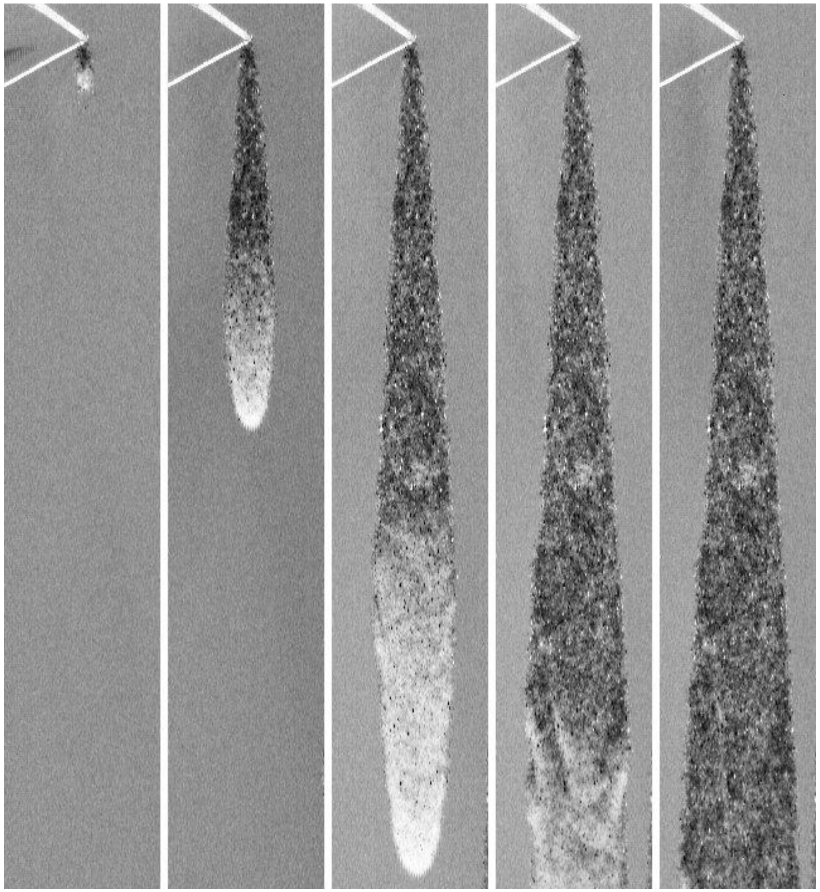
Fluid or solid?

Figure 24: Aletsch Glacier

Fluid or solid?

Figure 25: Granular avalanche

Fluid or not fluid?



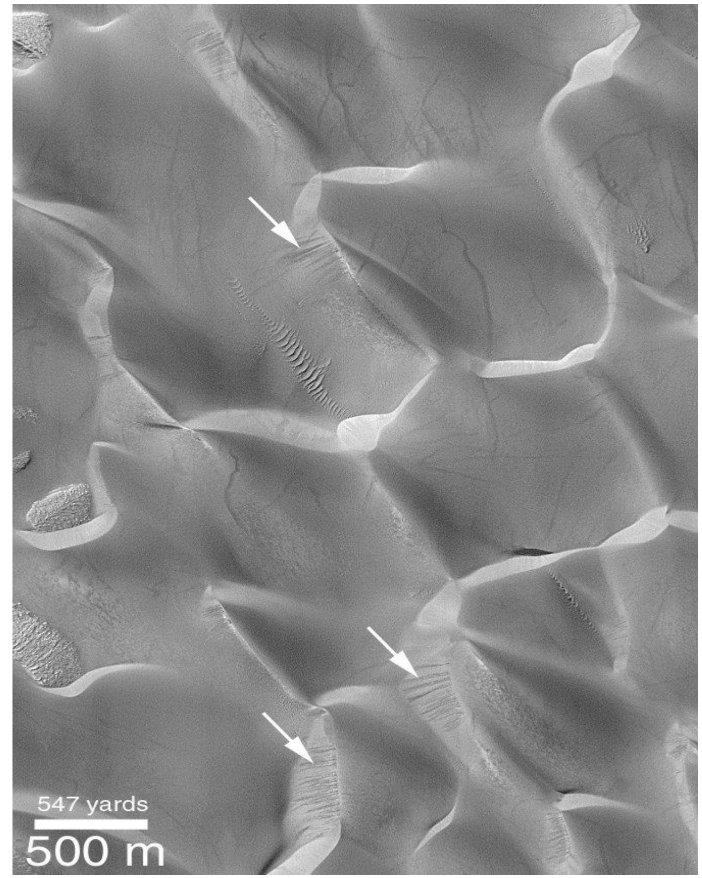


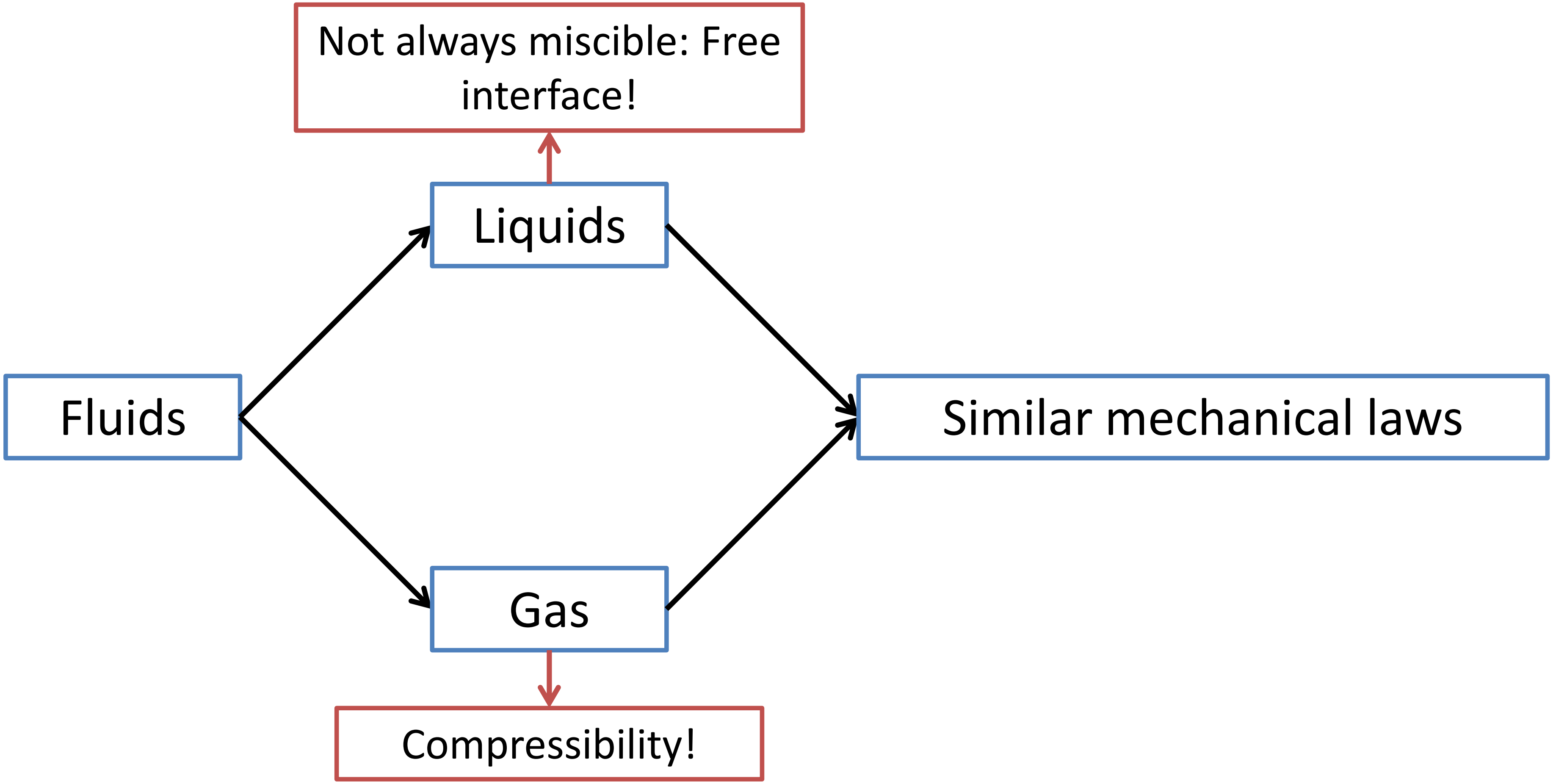
Figure 26: Granular avalanche (PMMH/ESPCI)

Fluid properties

- 3 scalar quantities : p, ρ, T
- 1 vector quantity : **u**
- All these quantities depend on position and time
  → p(x,y,z,t)…

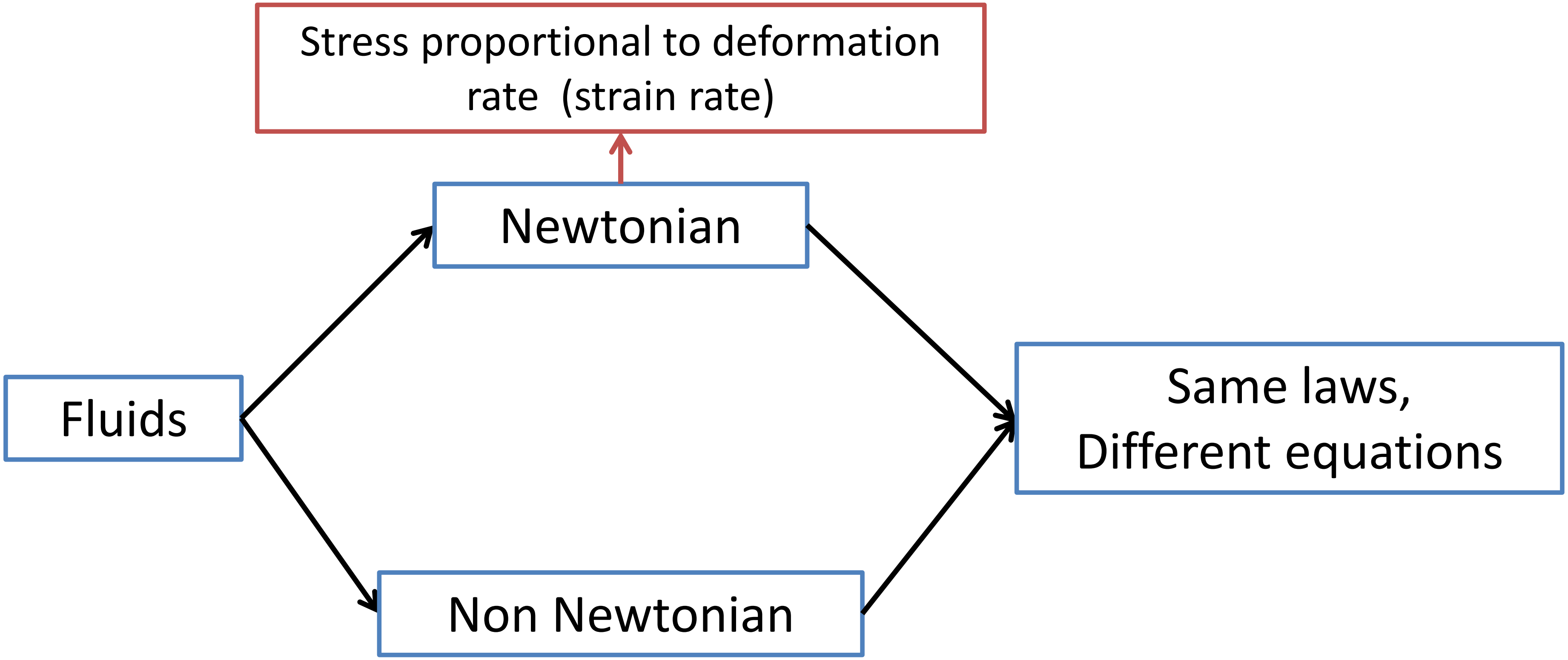
Homogeneous flow : these quantities are independent of the location
p(t)…

Fluid properties

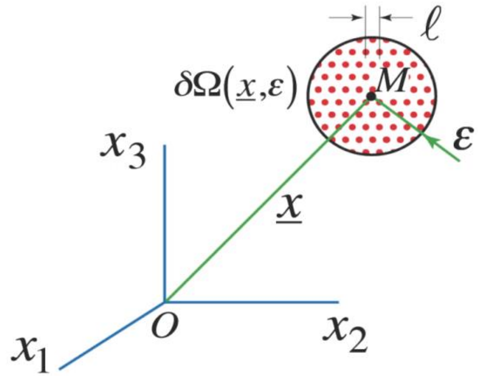


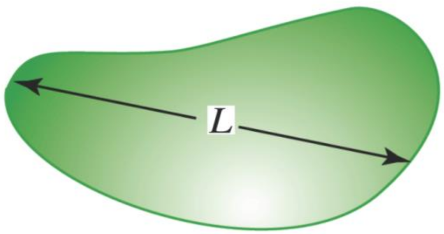
Is there a situation where water is seen to be compressible?

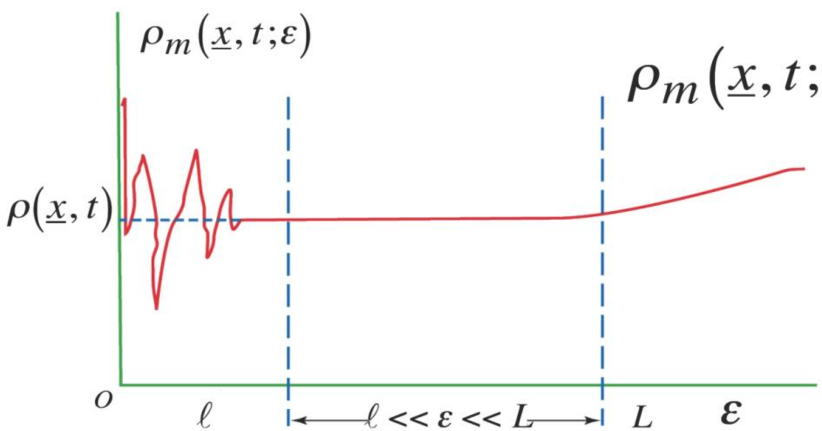
Fluid models: How to relate the deformation of a fluid to the applied stress?



Continuum hypothesis







$$\rho_m(\underline{x},t;\varepsilon) = \frac{\delta m(\underline{x},t;\varepsilon)}{\delta\Omega(\underline{x};\varepsilon)}$$

Knudsen number: $$\boxed{Kn = \frac{l}{L} \ll 1}$$

Continuum hypothesis:

Micro-Electro-Mechanical systems

L~100 nm

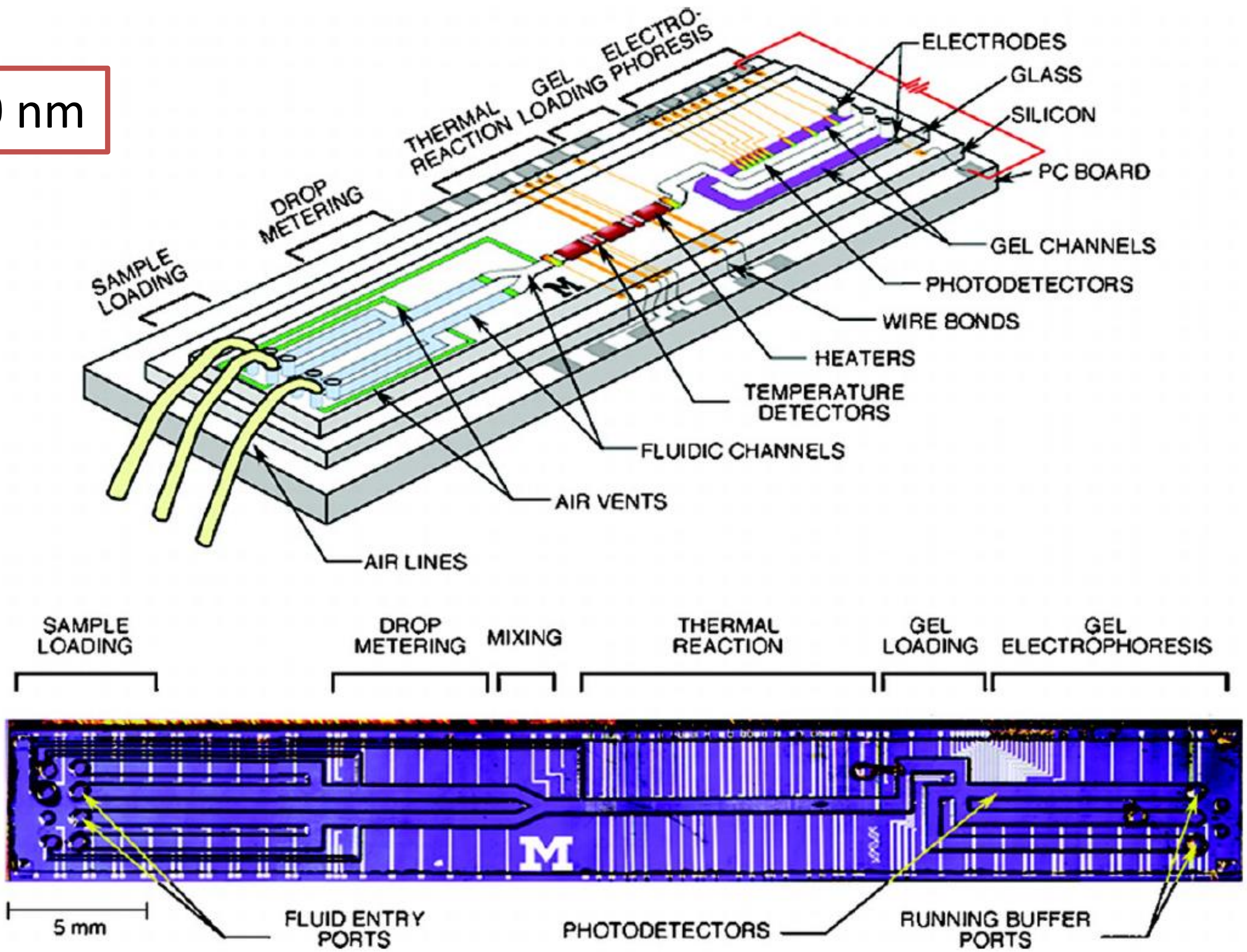


Figure 27: “Lab on a chip” Burns & al (1998)

Q1: By definition the velocity of a fluid can be everywhere 0 if and only if there are no external forces

1. True
2. False

Q2: The constitutive law of a Newtonian fluid links

1. A stress rate tensor to a strain tensor
2. A stress tensor to a strain tensor
3. A stress tensor to a strain-rate tensor

Q3: Thanks to Cauchy’s continuum mechanics tensor formalism, the stress on any infinitesimal surface of arbitrary orientation at a certain point of space requires the knowledge of

1. 9 scalars
2. 6 scalars
3. a 3x3 symmetric tensor

Q4: The flow is always directed from high to low pressure

1. True
2. False

Q5: The drag exerted on a body decreases with the Reynolds number

1. True
2. False

Outline

1. Introduction
2. Fluid: Definition, models and classifications
3. Navier-Stokes

    What do we need?

    1. F=ma and Lavoisier
    2. Fluid Kinematics, Euler-Lagrange, transport theorem
    3. A constitutive model
    4. Differential operators

Fluid kinematics: Euler/Lagrange

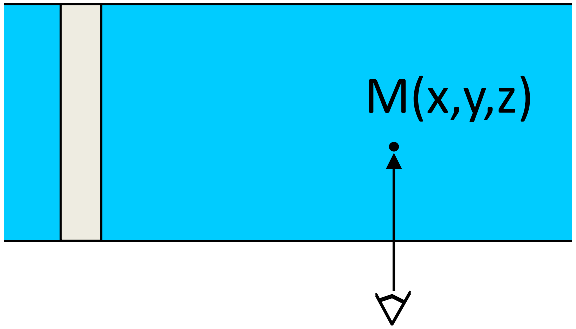
- Fluid kinematics is the study of fluid motion without taking into account of the forces at their origin.

- Two possible approaches:

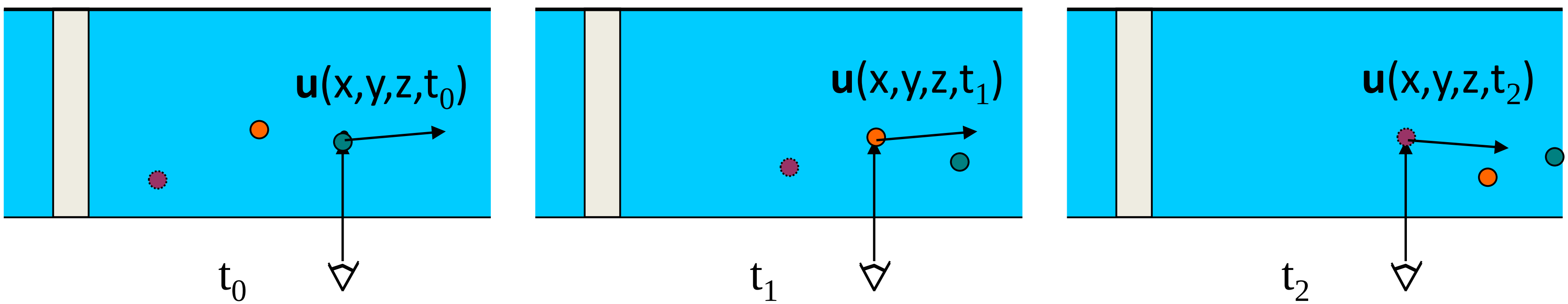
  - Eulerian description
  - Lagrangian description

Eulerian description

- One considers the velocity **u**(x,y,z) at a given fixed location M(x,y,z)

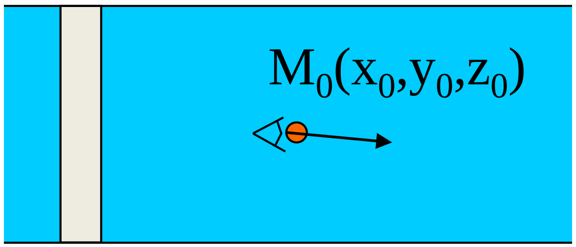


At each time-instant, we consider the velocity of a different fluid parcel



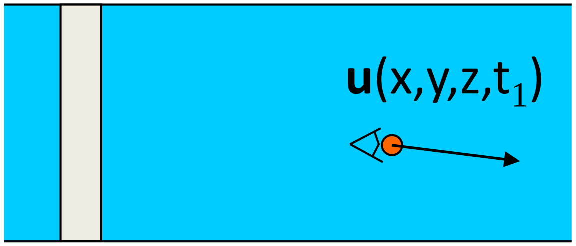
Lagrangian description

- One considers the velocity **u**(x,y,z,t) of a fluid parcel in its motion, by specifying its position $M_0(x_0,y_0,z_0)$ at time t_0.

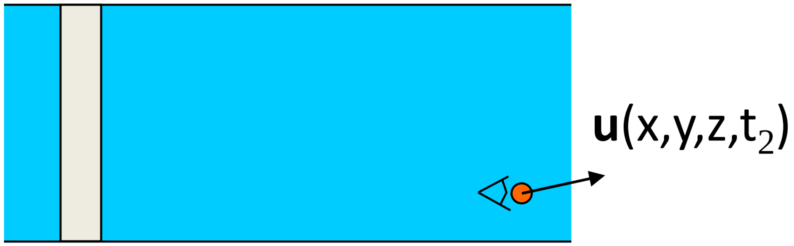


t_0

At each time instant, one considers the same fluid parcel

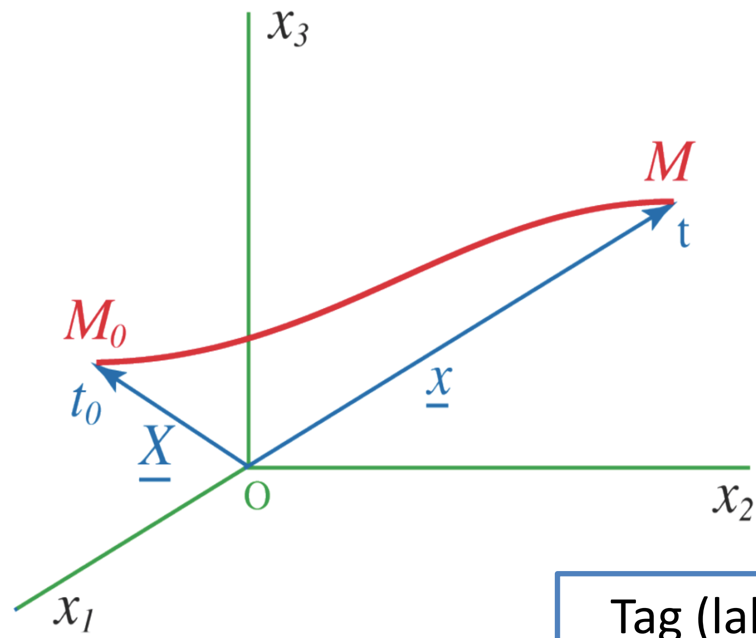


t_1



t_2

Lagrangian description



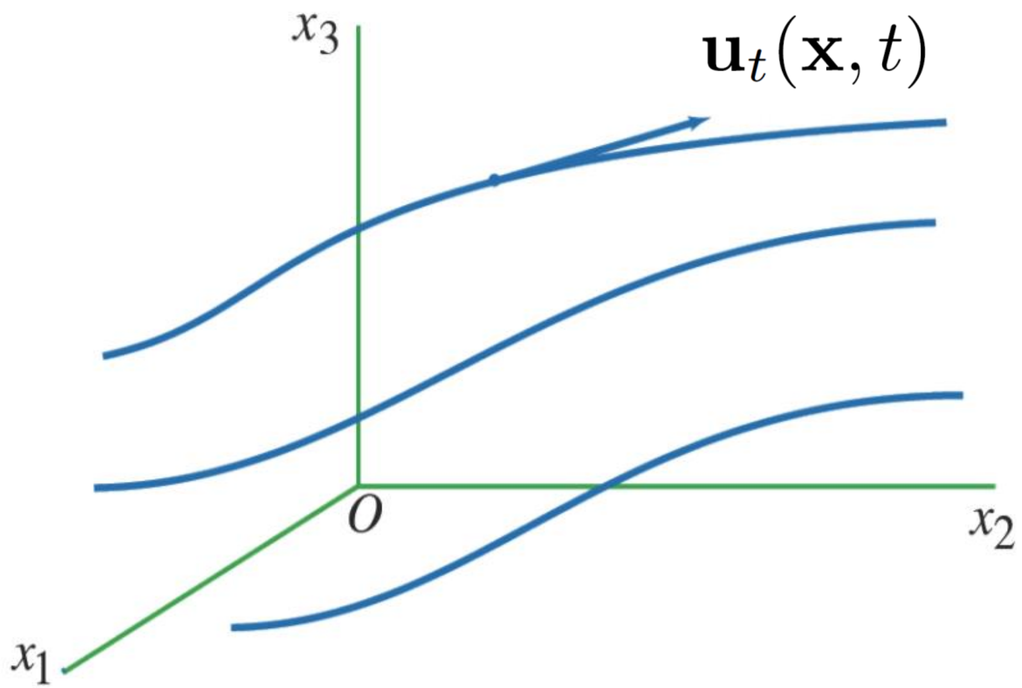
Tag (label)

Trajectory: $$\mathbf{x} = \boldsymbol{\Phi}(\mathbf{X}, t)$$

Field: $$B = B(\mathbf{X}, t)$$

Velocity: $$\mathbf{U}(\mathbf{X}, t) = \frac{\partial \boldsymbol{\Phi}}{\partial t}(\mathbf{X}, t)$$

Eulerian description



Trajectory: $$\frac{d\mathbf{x}}{dt} = \mathbf{u}\ (\mathbf{x}, t)$$

Field: $$B = b(\mathbf{x}, t)$$

$$\mathbf{x}(t = 0) = \mathbf{X}$$

↑
Location

Total derivative

$$B(\mathbf{X}, t) = b(\mathbf{x}, t) = b[\mathbf{\Phi}(\mathbf{X}, t), t]$$

$$\dot{B} = \frac{\partial B}{\partial t} = \frac{\partial b}{\partial t} + \nabla b \cdot \frac{\partial \mathbf{\Phi}}{\partial t}$$

$$\boxed{\dot{B} = \frac{db}{dt} = \frac{\partial b}{\partial t} + \nabla b \cdot \mathbf{u}}$$

Total derivative

Local derivative

Convective derivative

Special cases

Uniform flow

$$\nabla \mathbf{u} = \begin{pmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} & \frac{\partial u_x}{\partial z} \\ \frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial y} & \frac{\partial u_y}{\partial z} \\ \frac{\partial u_z}{\partial x} & \frac{\partial u_z}{\partial y} & \frac{\partial u_z}{\partial z} \end{pmatrix} = 0$$

Stationary flow

$$\frac{\partial \mathbf{u}}{\partial t} = 0$$

Total derivative (material derivative)

- In the Eulerian description, one aims at quantifying the temporal variations of a quantity associated to a fluid parcel

$$\boxed{\frac{db}{dt} = \frac{\partial b}{\partial t} + u_x \frac{\partial b}{\partial x} + u_y \frac{\partial b}{\partial y} + u_z \frac{\partial b}{\partial z}}$$

Total derivative (material derivative)

$$\boxed{\frac{db}{dt}} = \boxed{\frac{\partial b}{\partial t}} + \boxed{u_x \frac{\partial b}{\partial x} + u_y \frac{\partial b}{\partial y} + u_z \frac{\partial b}{\partial z}}$$

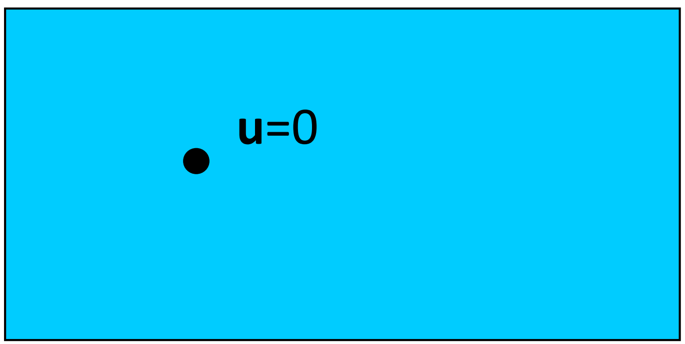
Convective derivative, i.e. temporal variations of b in the fluid parcel due to the transport (advection) of the inhomogeneous field b at the velocity U into from the fluid parcel

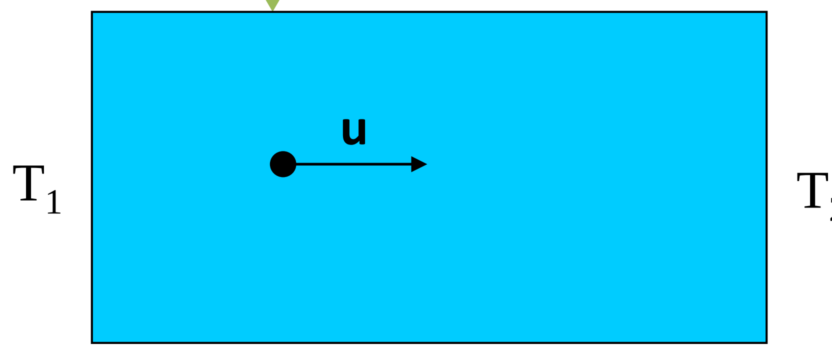
Local derivative, i.e. temporal variation of b at the location of the fluid parcel, i.e. at a geometric fixed location M0

Material derivative, i.e. temporal variation of b inside a fluid parcel

Total derivative (material derivative)

$$\frac{db}{dt} = \boxed{\frac{\partial b}{\partial t}} + \boxed{u_x \frac{\partial b}{\partial x} + u_y \frac{\partial b}{\partial y} + u_z \frac{\partial b}{\partial z}}$$





Example: I am floating in a heated pool i.e. T(t)

$$\frac{\partial T}{\partial t} \neq 0$$

Example: I am floating in pool where T=T(x,y,z)

$$\frac{\partial T}{\partial t} = 0 \quad \text{but} \quad \frac{dT}{dt} \neq 0$$

Lagrange/Euler?

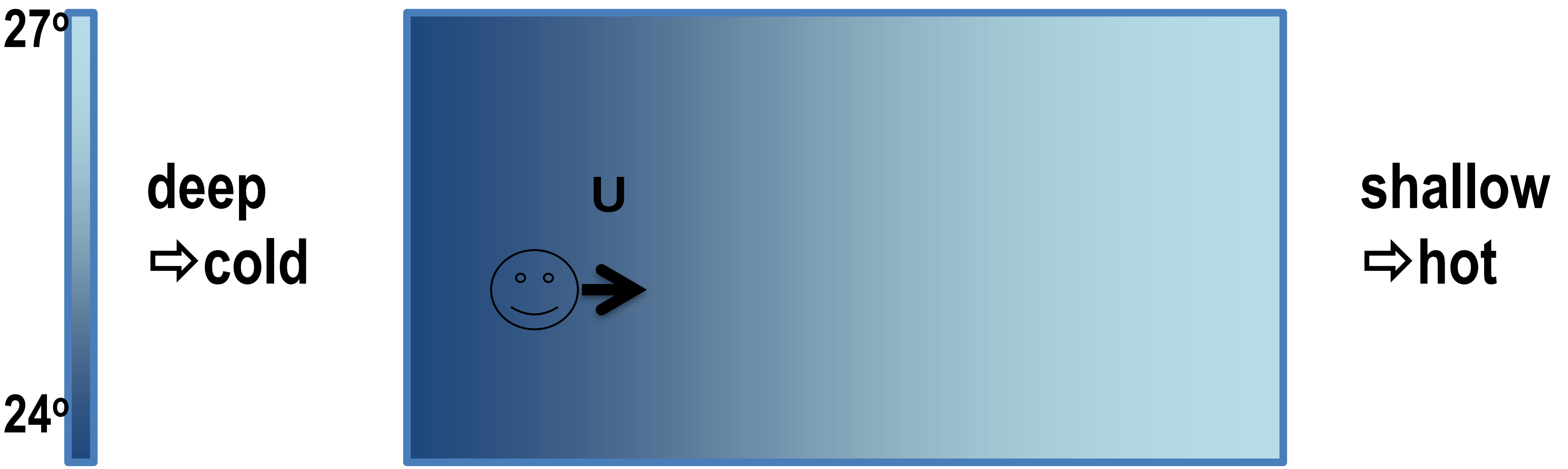
Ex: felt temperature by a swimmer in a swimming pool with varying depth and therefore temperature

The swimmer is immobile. The temperature does not change with time

$$\boxed{\frac{DT}{Dt} = 0}$$

Lagrange/Euler?

The swimmer now swims at U



The temperature felt by the swimmer increases with time $\frac{DT}{Dt} > 0$

despite the fact that from an Eulerian point of view $\frac{\partial T}{\partial t} = 0$

$$\boxed{\frac{DT}{Dt} = U\frac{\partial T}{\partial x}}$$

Lagrange/Euler?

The swimmer is at rest again, but the sun shines hard

The temperature felt by the swimmer increases with time because it increases point wise. There is no motion, so that Euler and Lagrange have the same point of view. $\frac{DT}{Dt} > 0$

$$\boxed{\frac{DT}{Dt} = \frac{\partial T}{\partial t}}$$

Lagrange/Euler?

The swimmer is at rest again, but the sun shines hard

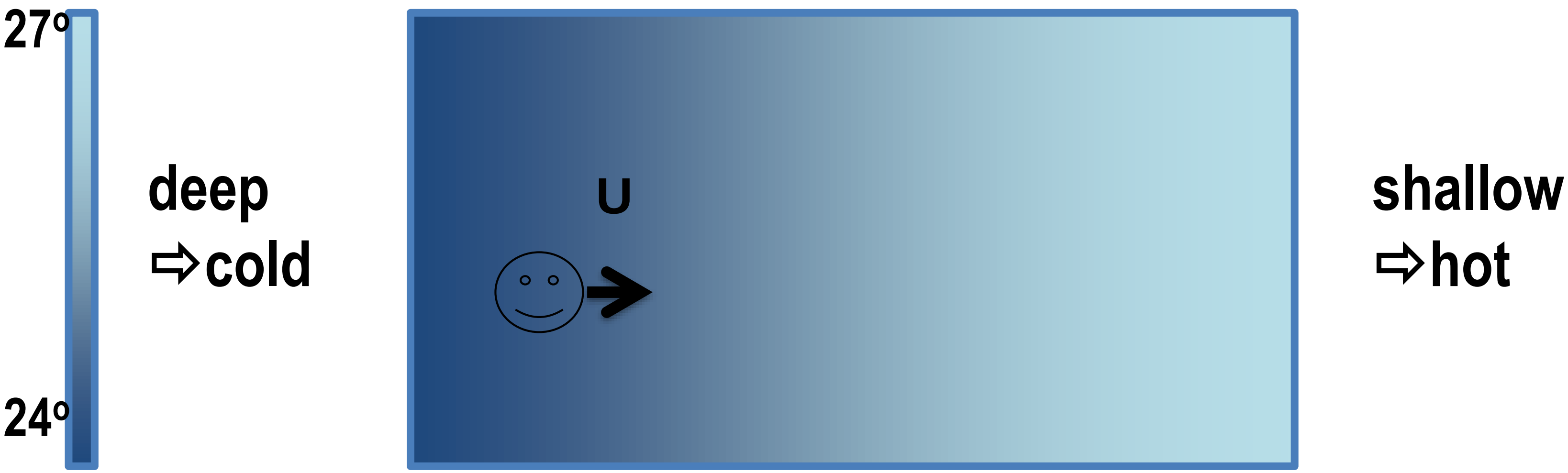
The temperature felt by the swimmer increases with time because it increases point wise. There is no motion, so that Euler and Lagrange have the same point of view.

$$\frac{DT}{Dt} > 0$$

$$\boxed{\frac{DT}{Dt} = \frac{\partial T}{\partial t}}$$

Lagrange/Euler?

The swimmer starts swimming again and clouds arrive…



$$\boxed{\frac{DT}{Dt}} = \boxed{\frac{\partial T}{\partial t}} + \boxed{u_x \frac{\partial T}{\partial x} + u_y \frac{\partial T}{\partial y} + u_z \frac{\partial T}{\partial z}}$$

Lagrangienne derivative
Total derivative

Advective derivative

Eulerian derivative

Acceleration

The acceleration is the particular derivative of the velocity

Local acceleration

Convective acceleration

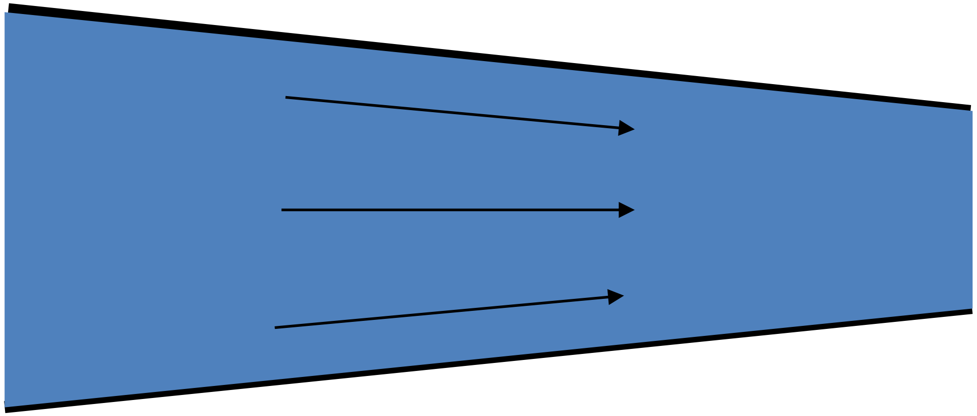
$$a_x = \underbrace{\boxed{\frac{\partial u_x}{\partial t}}}_{\text{Local acceleration}} + \underbrace{\boxed{u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} + u_z \frac{\partial u_x}{\partial z}}}_{\text{Convective acceleration}}$$

$$a_y = \ldots$$

$$a_z = \ldots$$

Acceleration

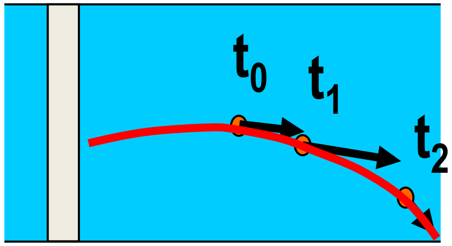
- **Stationary** flow in a **convergent** pipe



The acceleration is not zero (= convective acceleration)

Trajectory

A trajectory is the path of a particle



ODE

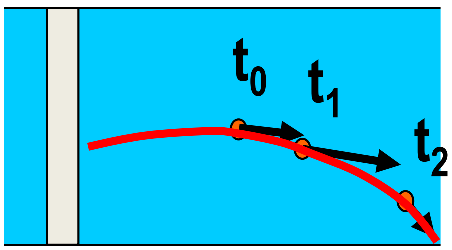
$$\frac{d\mathbf{X}}{dt} = \mathbf{u}(\mathbf{X}, t)$$

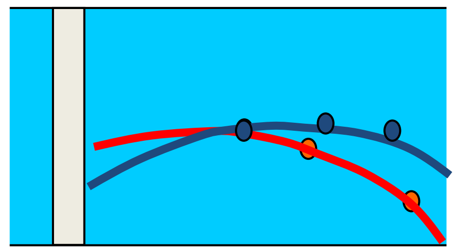
Initial condition

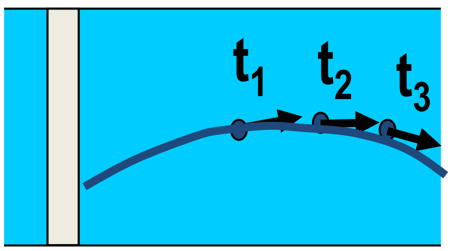
$$\mathbf{X}(t_0) = \mathbf{X}_0$$

Trajectories can cross

In an usteady flow, trajectories can cross

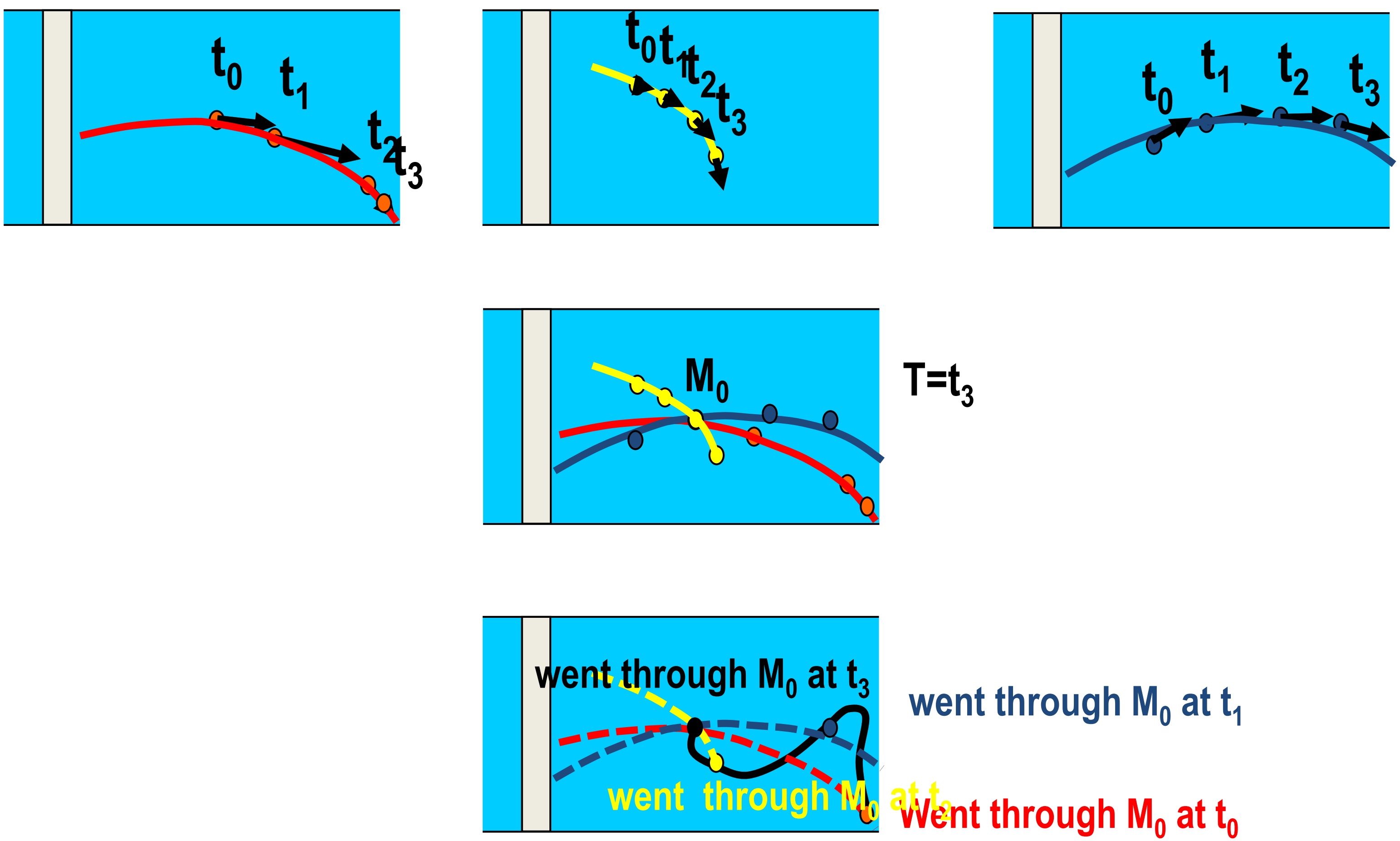






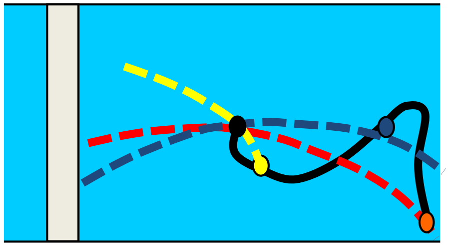
Path lines

Collection of locations of particles at t=T, that went through M_0 at t<T



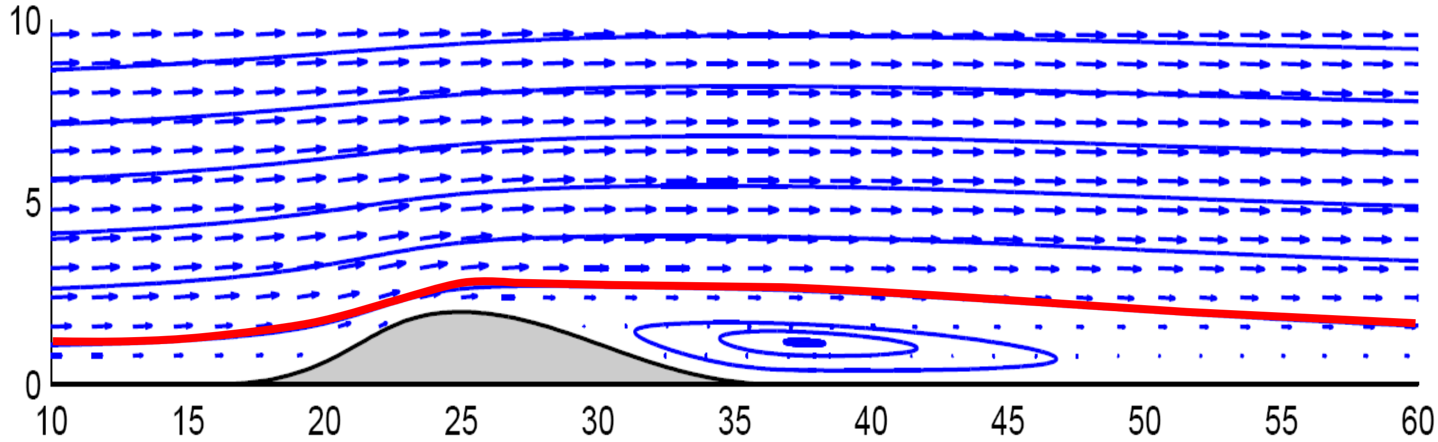
Trajectories and path lines

In an unsteady flow, trajectories and path lines are not superimposed



Streamlines

Eulerian concept : curve everywhere tangent to the velocity field

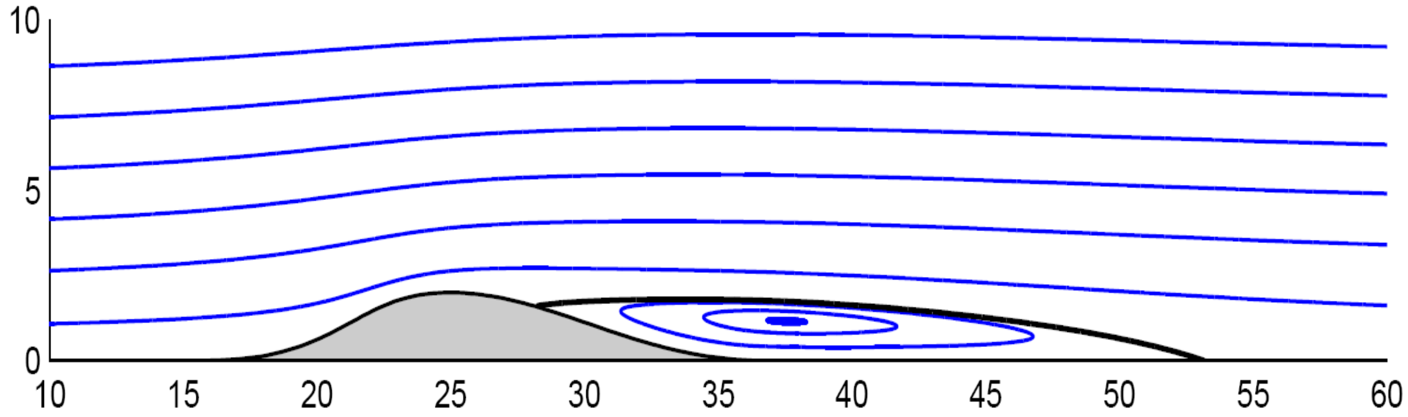


This is a geometric property at a given time t

Streamlines

A streamline does not touch walls

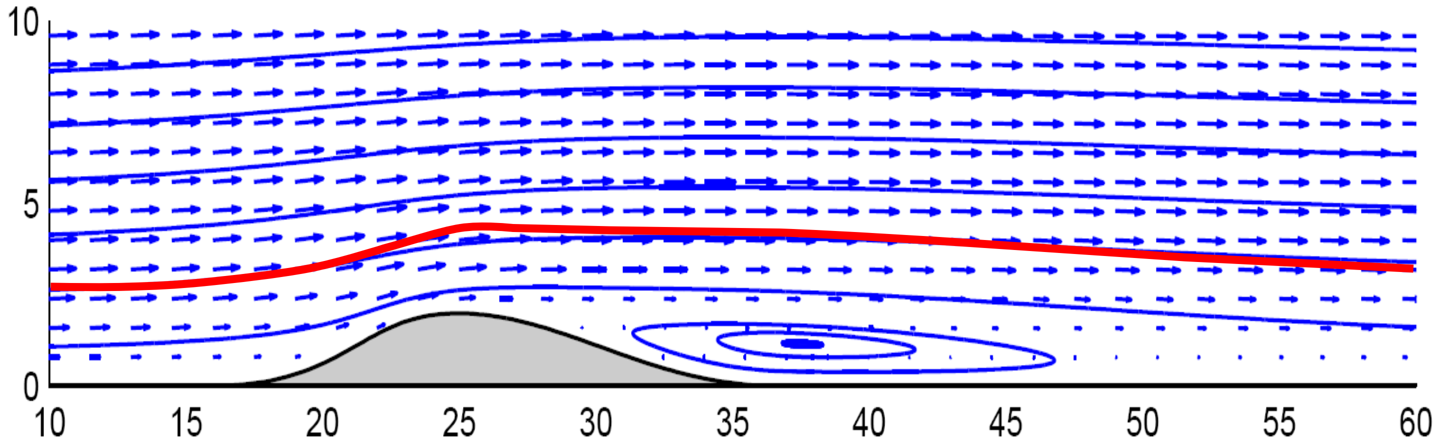
Unless at a stagnation point*, where a separatrix emanates



***where the wall shear stress is zero**

Streamline equation

Curve everywhere tangent to the flow field

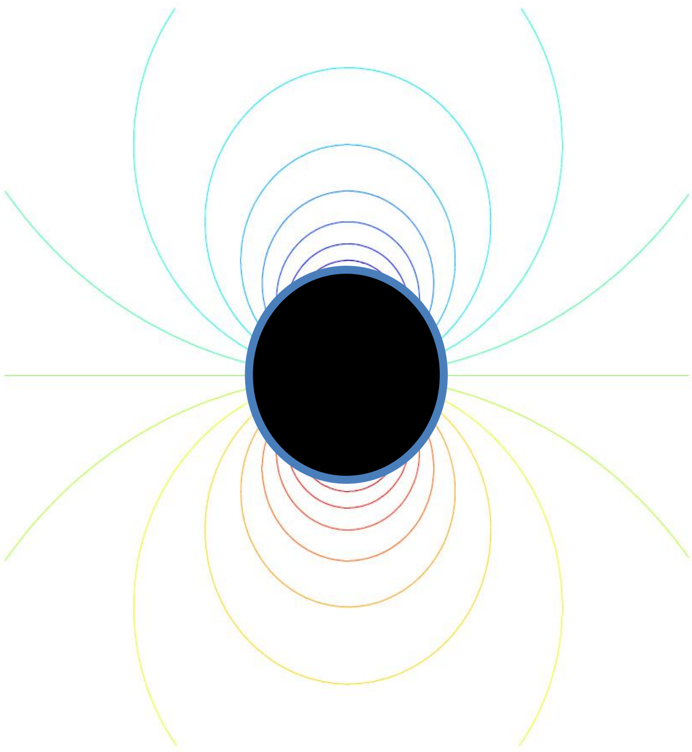


Differential equation

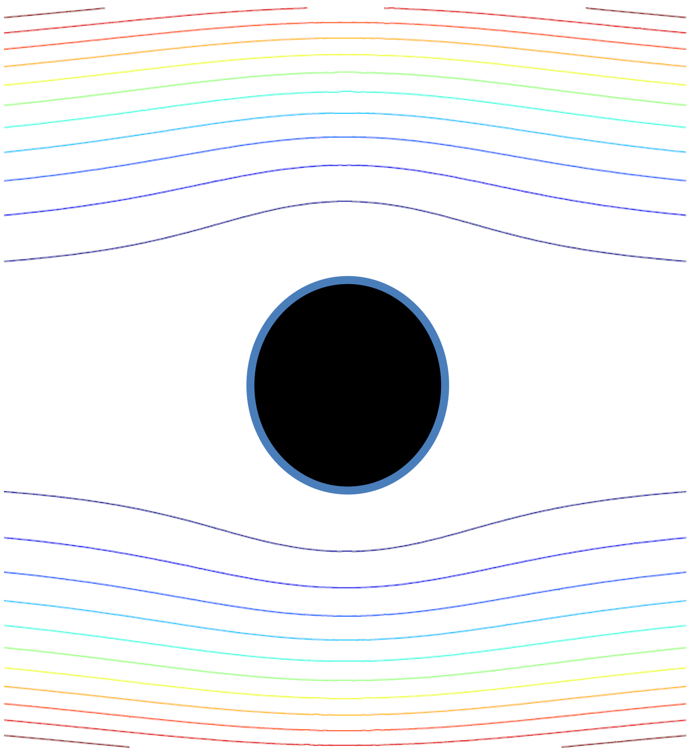
$$\mathbf{u} \wedge d\mathbf{x} = 0$$

Beware of the reference frame!

A cylinder moves at constant velocity in a very viscous fluid

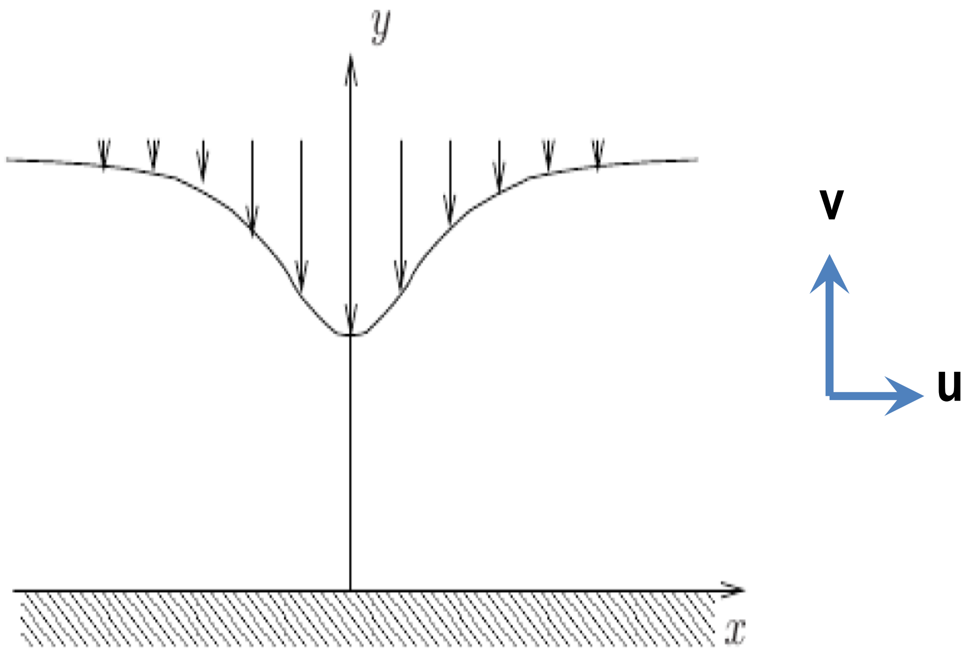


Lab reference frame



Cylinder reference frame

Impacting jet

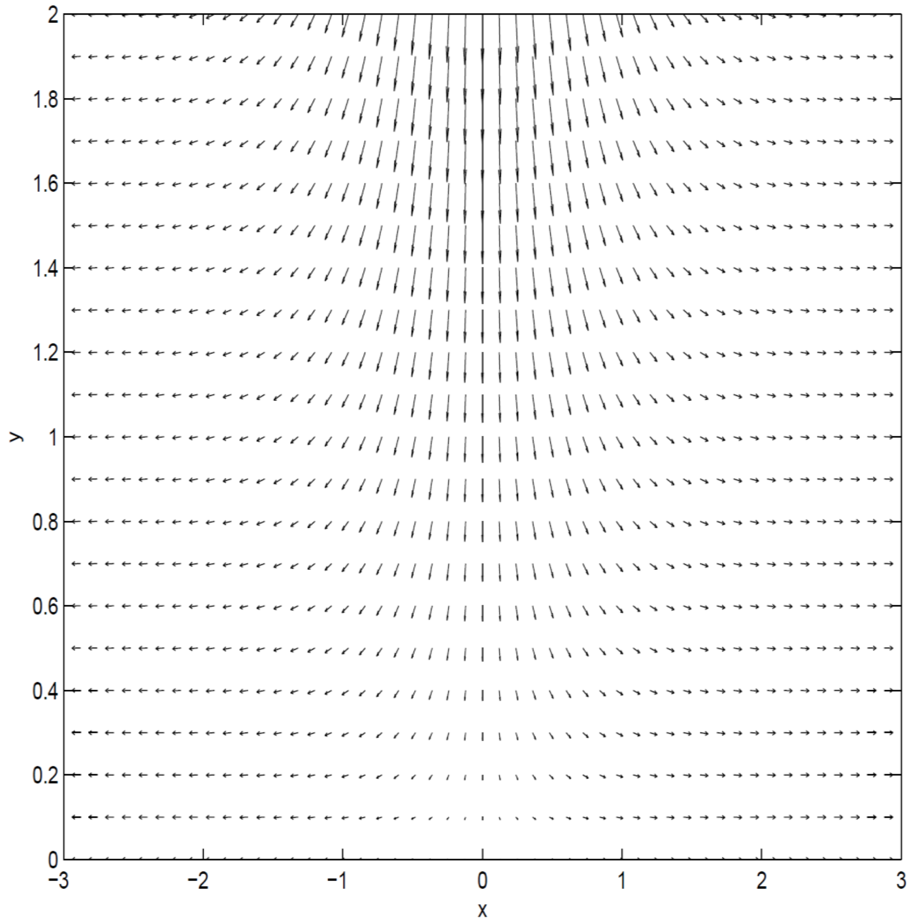


$$v = -V_0 y \cosh^{-2}(x - x_0)$$

$$u = V_0 \tanh(x - x_0)$$

Stationary jet

Flow field



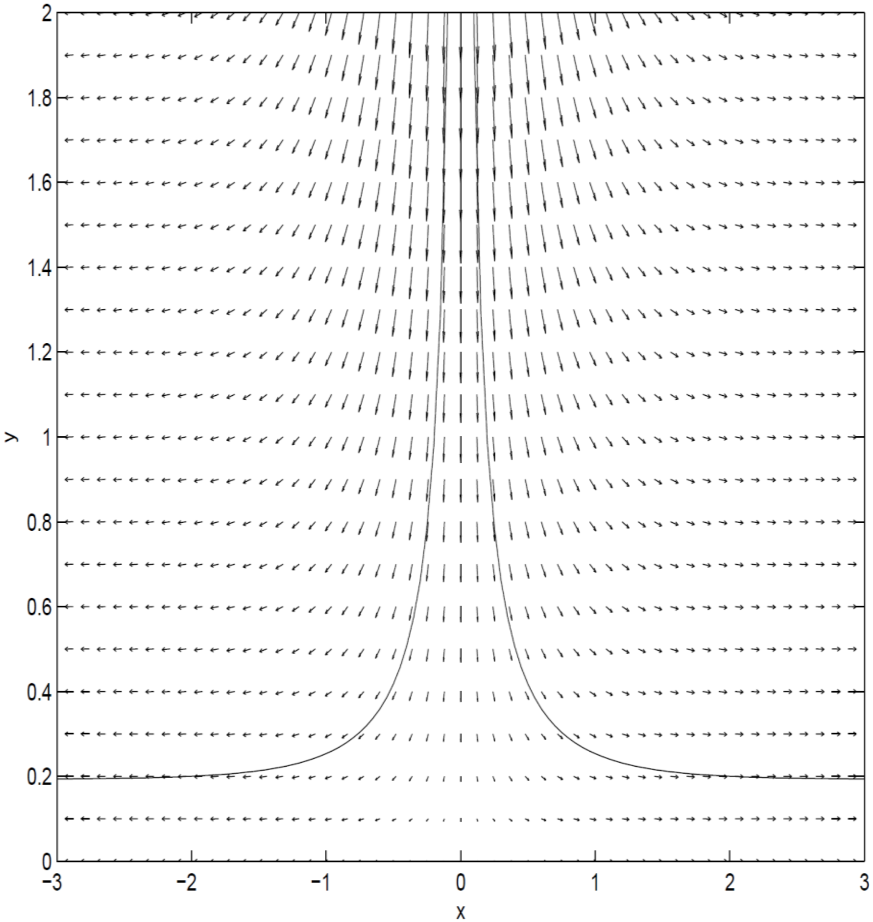
$$v = -V_0 y \cosh^{-2}(x - x_0)$$

$$u = V_0 \tanh(x - x_0)$$

$$x_0 = 0$$

$$V_0 = 1$$

Streamline



$$\frac{dx}{dy} = \frac{u}{v}$$

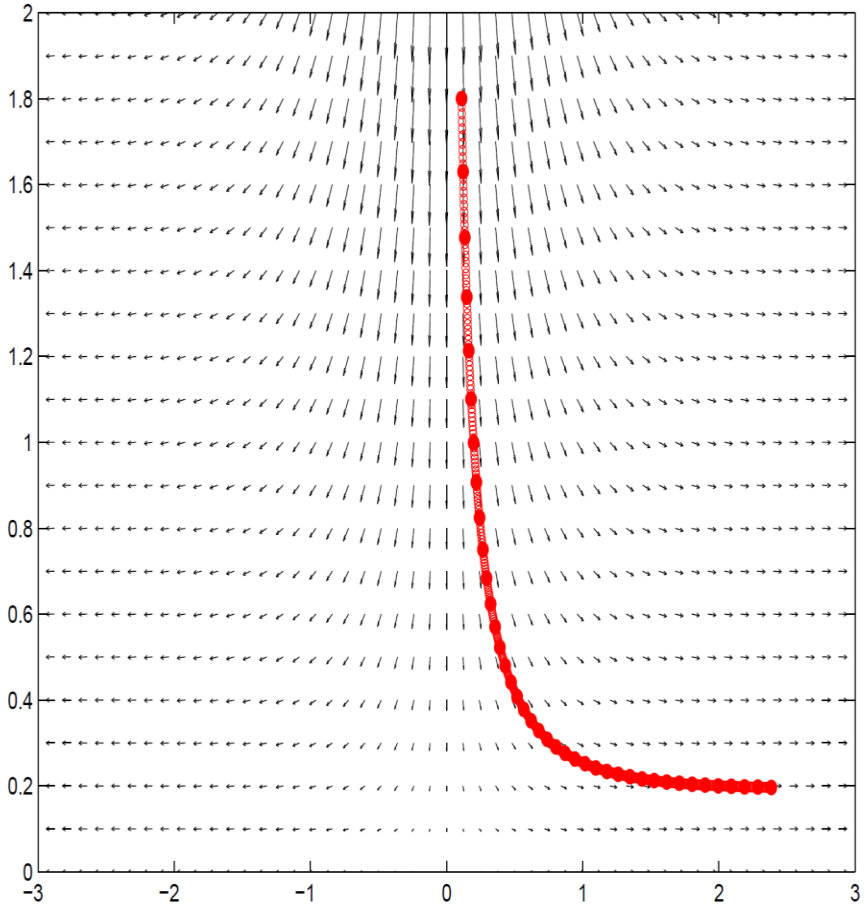
$$u = V_0 \tanh(x - x_0)$$

$$v = -V_0 y \cosh^{-2}(x - x_0)$$

$$\frac{dx}{\cosh(x - x_0)\sinh(x - x_0)} = \frac{dy}{y}$$

$$\boxed{y = \left| \frac{k}{\tanh(x - x_0)} \right|}$$

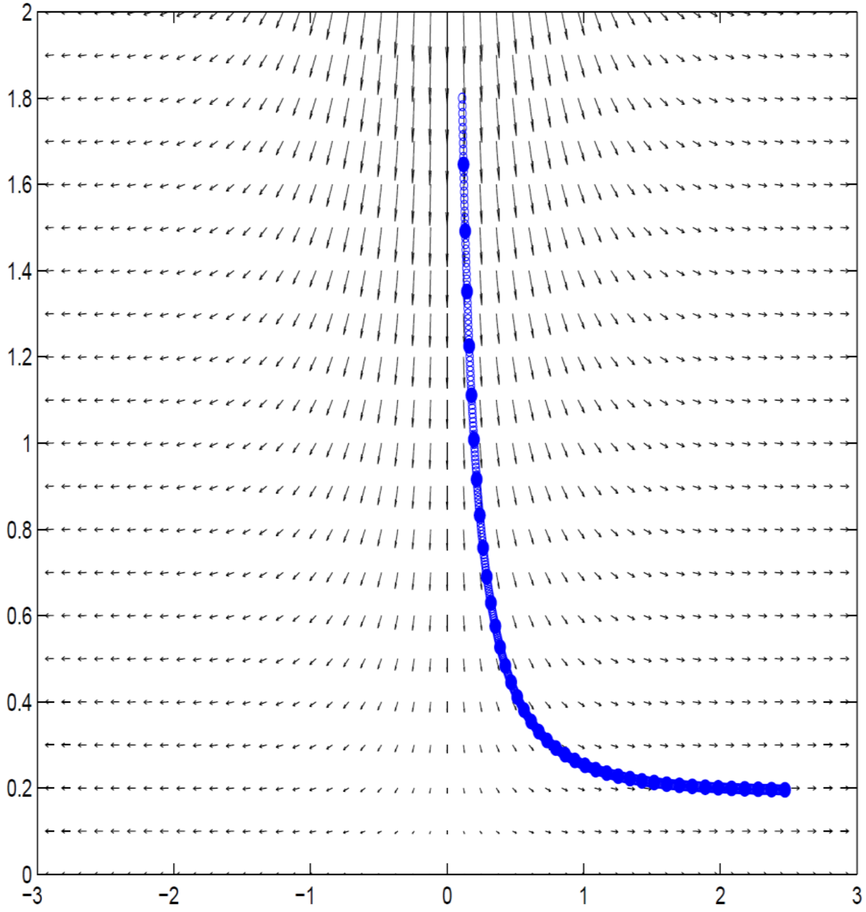
Trajectory (till T=4)



Particle position every dt

Particle position avery 10dt

Streakline



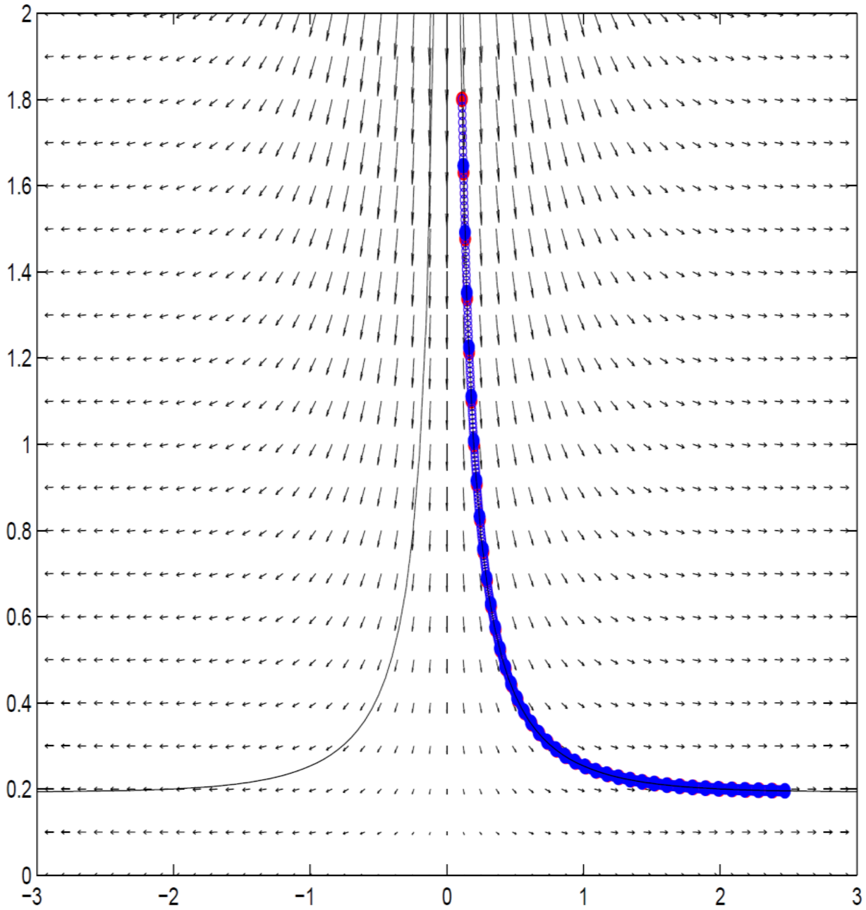
Particule released every dt ○

Particule released every 10dt ●

$$u = V_0 \tanh(x - x_0)$$

$$v = -V_0 y \cosh^{-2}(x - x_0)$$

Trajectory = Streakline = Streamline



Particle position every dt

Particle position avery 10dt

Particule released every dt

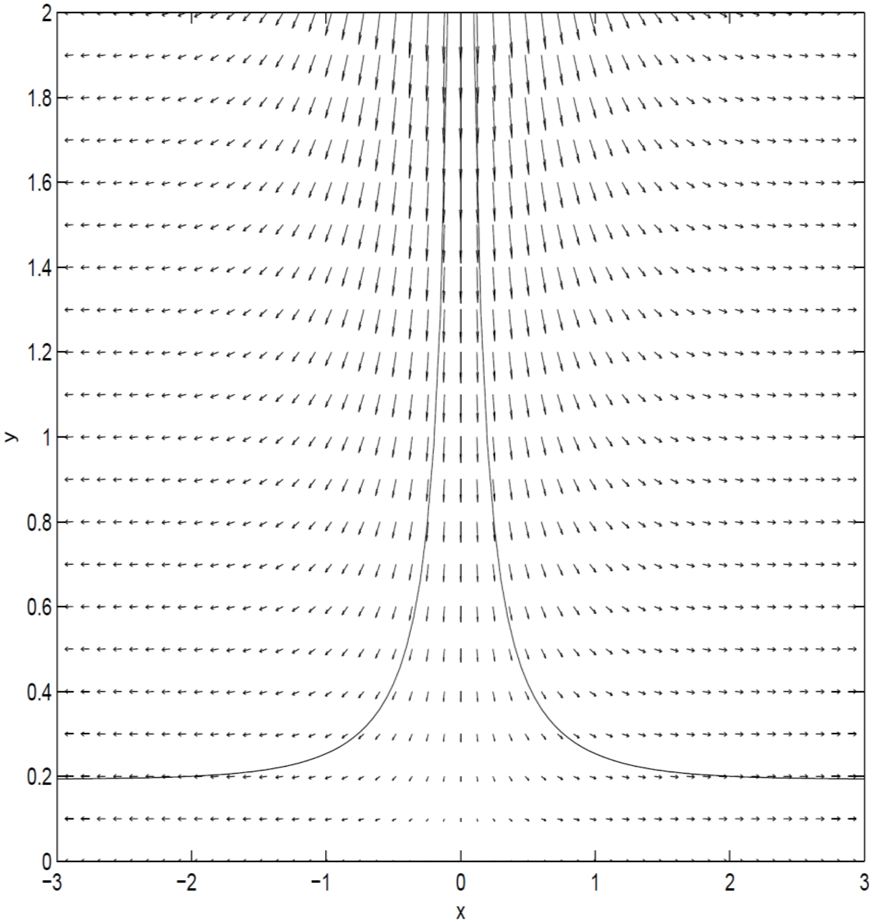
Particule released every 10dt

$$u = V_0 \tanh(x - x_0)$$

$$v = -V_0 y \cosh^{-2}(x - x_0)$$

$$\boxed{y = \left| \frac{k}{\tanh(x - x_0)} \right|}$$

Oscillating jet : instantaneous streamline



$$x_0 = \sin(2t) \;\; V_0 = 1$$

$$\frac{dx}{dy} = \frac{u}{v}$$

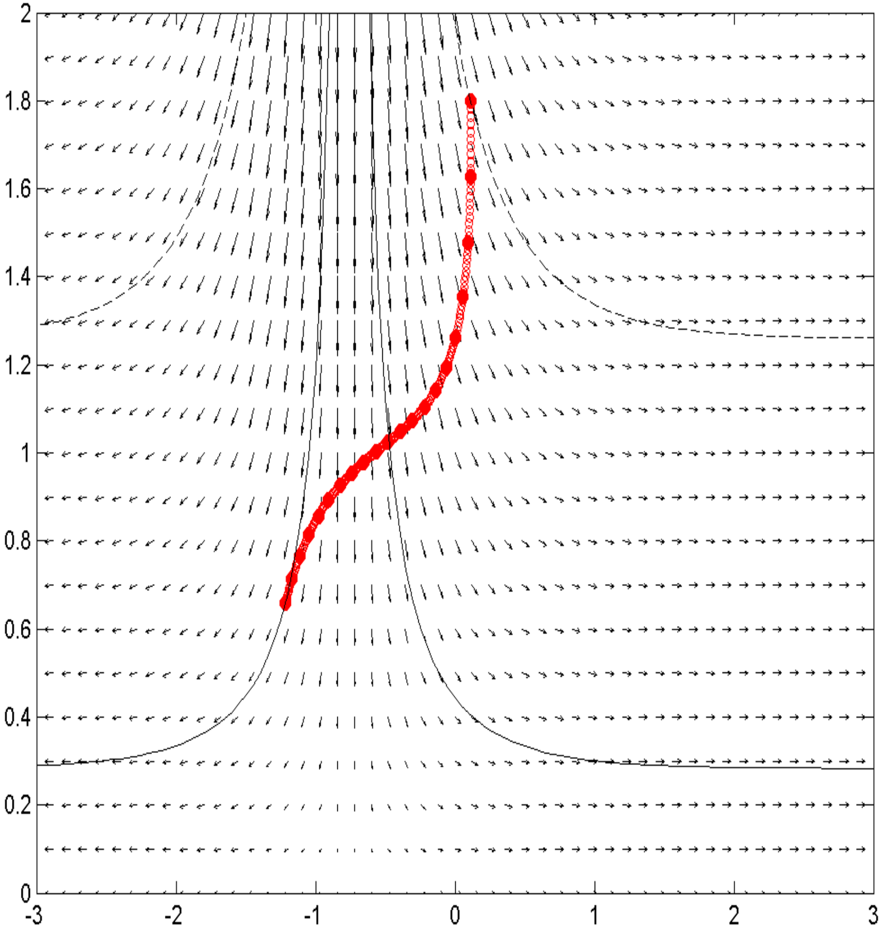
$$u = V_0 \tanh(x - x_0)$$

$$v = -V_0 y \cosh^{-2}(x - x_0)$$

$$\frac{dx}{\cosh(x - x_0)\sinh(x - x_0)} = \frac{dy}{y}$$

$$\boxed{y = \left| \frac{k}{\tanh(x - x_0)} \right|}$$

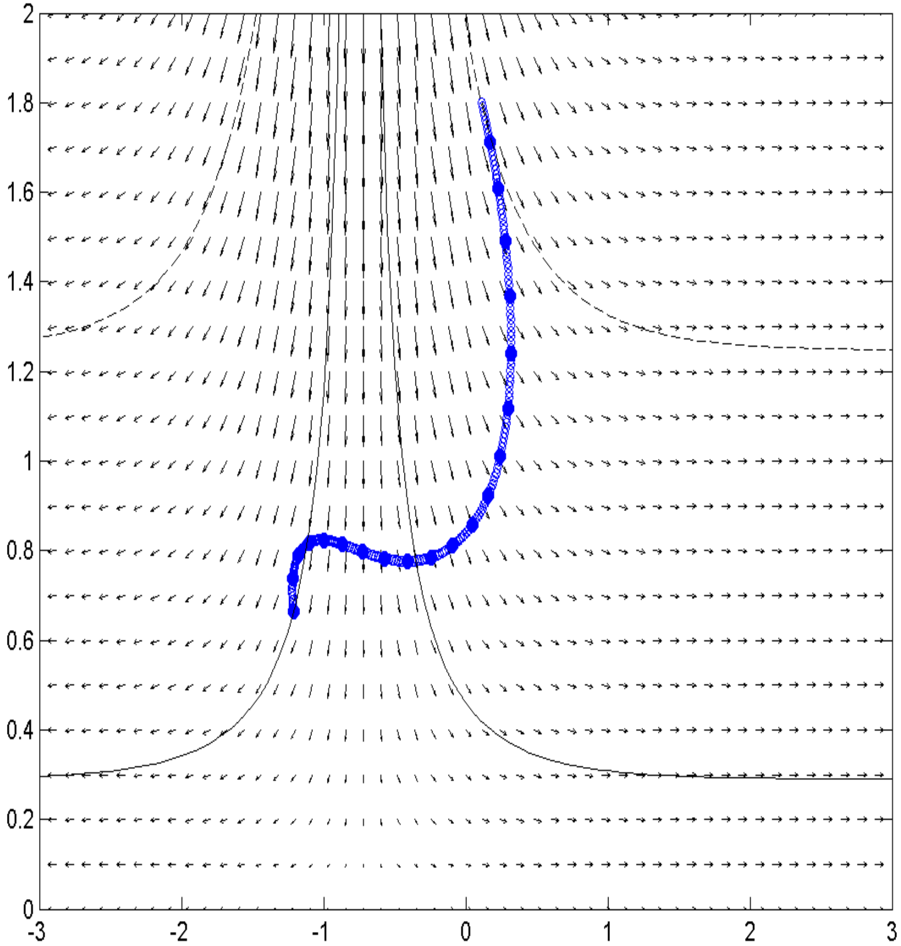
Trajectory (till T=2)



Particle position every dt

Particle position avery 10dt

Streakline



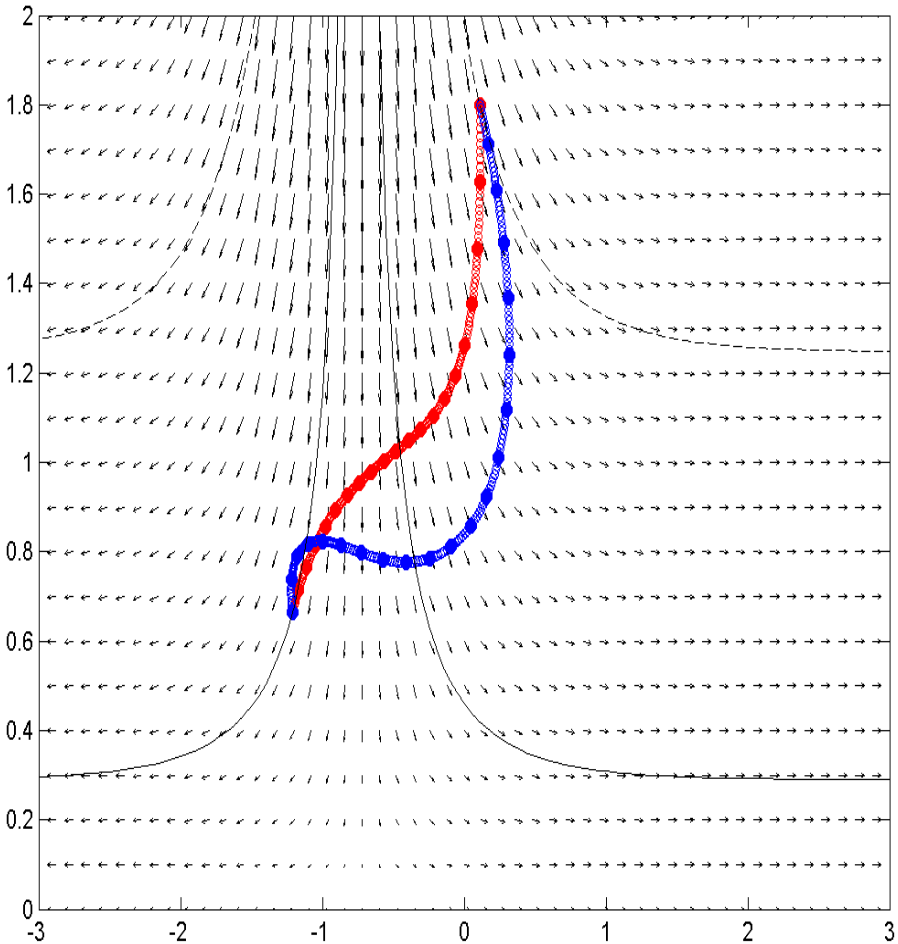
Particule released every dt

Particule released every 10dt

$$u = V_0 \tanh(x - x_0)$$

$$v = -V_0 y \cosh^{-2}(x - x_0)$$

Trajectory ≠ Streakline ≠ Streamline



Particle position every dt

Particle position avery 10dt

Particule released every dt

Particule released every 10dt

$$u = V_0 \tanh(x - x_0)$$

$$v = -V_0 y \cosh^{-2}(x - x_0)$$

$$\boxed{y = \left| \frac{k}{\tanh(x - x_0)} \right|}$$

Outline

1. Introduction
2. Fluid: Definition, models and classifications
3. Navier-Stokes

   What do we need?

   1. F=ma and Lavoisier
   2. Fluid Kinematics, Euler-Lagrange, transport theorem
   3. A constitutive model
   4. Differential operators

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Divergence

Vectoriel field (ex : velocity) $\overrightarrow{U}(x,y,z) = U_x \overrightarrow{e_x} + U_y \overrightarrow{e_y} + U_z \overrightarrow{e_z}$

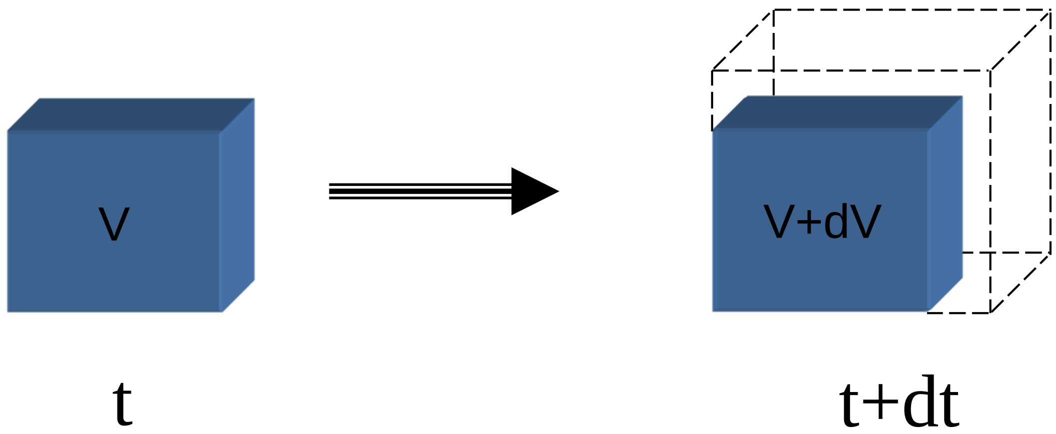
$$div\,\overrightarrow{U} = \overrightarrow{\nabla}.\overrightarrow{U} = \frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y} + \frac{\partial U_z}{\partial z}$$

→ Divergence of vector = scalar

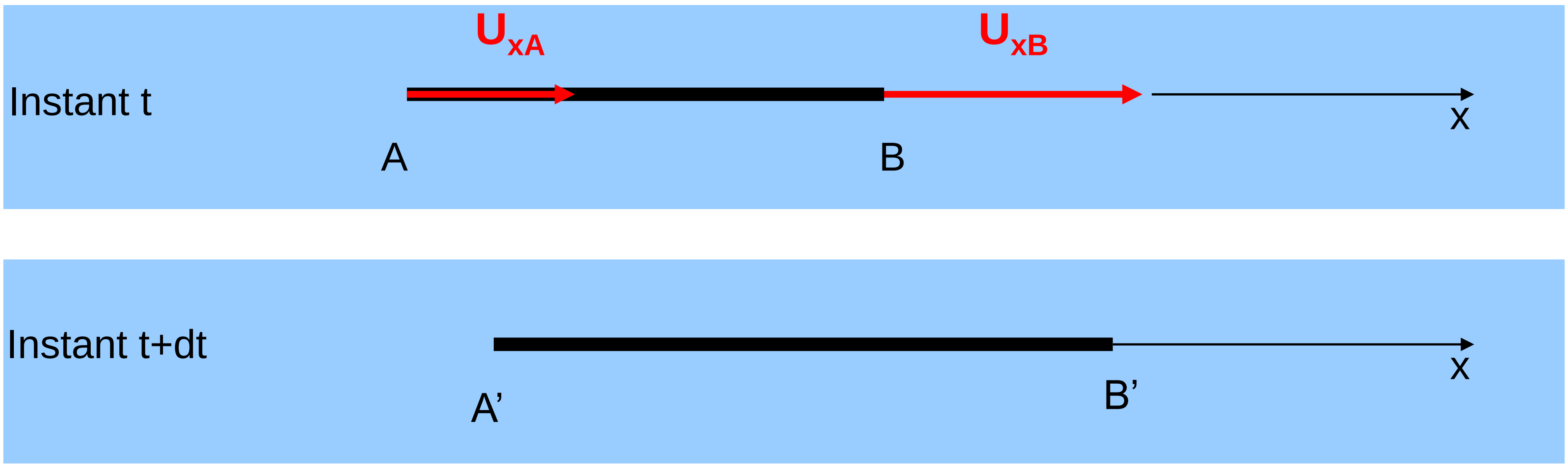
Divergence : physical interpretation

The divergence of the velocity field corresponds to the volumetric dilatation rate of an infinitesimal fluid volume

$$div\,\vec{U} = \frac{1}{V}\frac{dV}{dt}$$

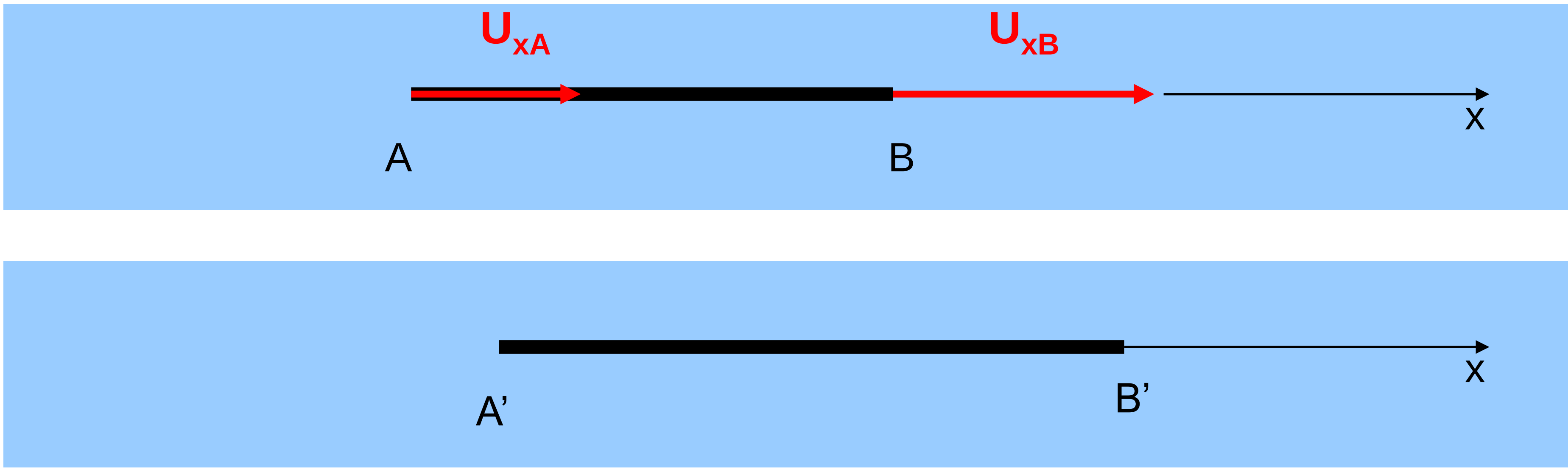


Stretching



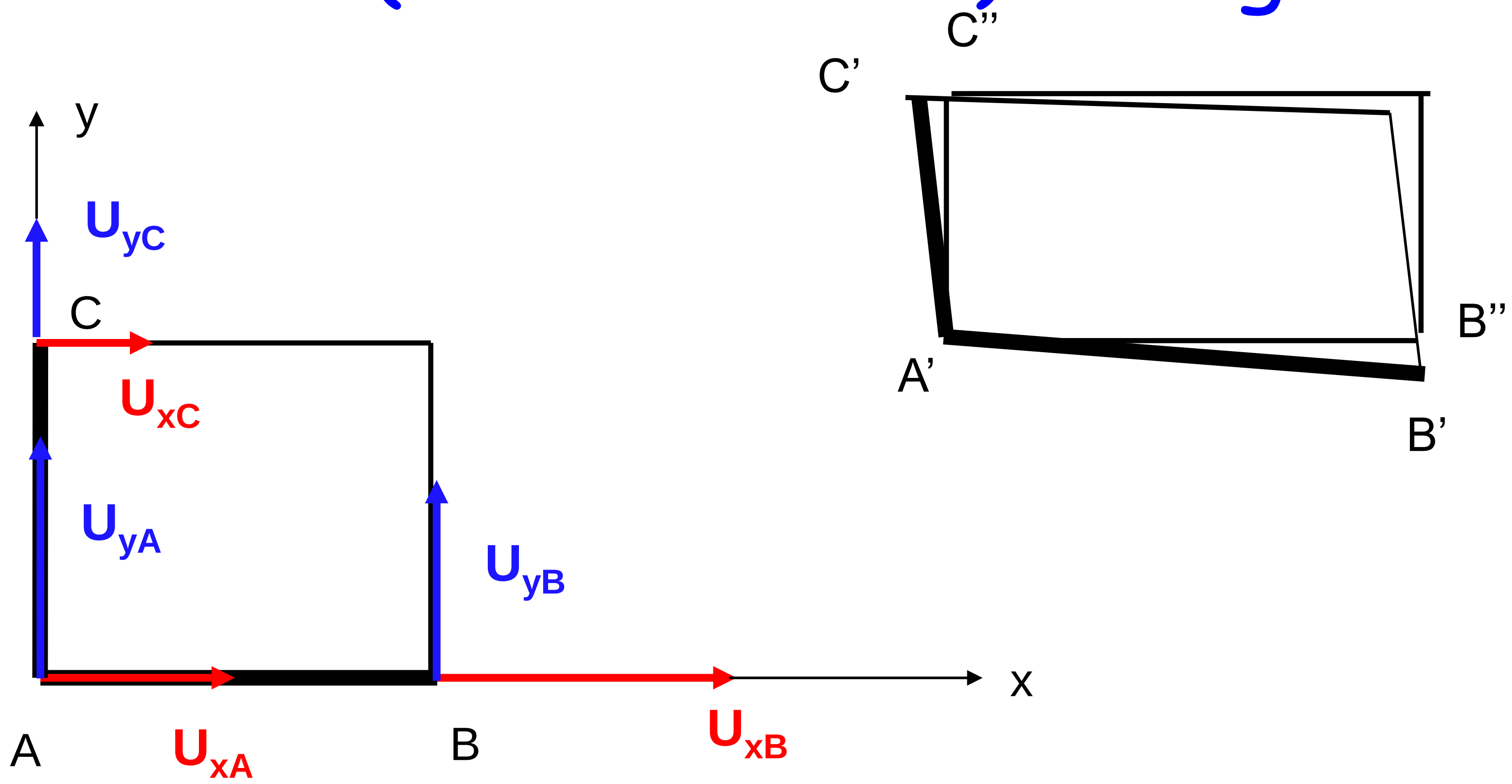
AB is a line of fluid particles in a flow such that $U_{xA}<U_{xB}$. Since the velocity is higher in B than in A, the segment AB is stretched in the x direction. This **deformation** is called a **stretching**. The relevant quantity is the derivative of the velocity with respect to the direction tangential to this velocity

Stretching



$$A'B' - AB = (U_{xB} - U_{xA})dt = \boxed{\frac{\partial U_x}{\partial x}} dx\,dt$$

Volume (surface area) change



$$(A''B'')(A''C'')-(AB)(AC)=(AB)(AC)\left(\frac{\partial U_x}{\partial x}+\frac{\partial U_y}{\partial y}\right)dt$$

Gradient of a scalar

Scalar field $p(x, y, z)$
(ex : pressure)

$$\overrightarrow{grad}\ p = \vec{\nabla} p = \begin{pmatrix} \dfrac{\partial p}{\partial x} \\ \dfrac{\partial p}{\partial y} \\ \dfrac{\partial p}{\partial z} \end{pmatrix}$$

$\vec{\nabla}$ Gradient operateur (nabla)

→ Gradient of scalar = vector

Gradient of a vector: application to the velocity field

Taylor expansion of the velocity field

$$\mathbf{u}(\mathbf{x}+\delta\mathbf{x}) = \mathbf{u}(\mathbf{x}) + \nabla\mathbf{u}\;\delta\mathbf{x}$$

$$\boxed{\nabla\mathbf{u}} = \boxed{\mathbf{D}} + \boxed{\boldsymbol{\Omega}}$$

Velocity gradient **Symetric part** **Antisymetric part**

$$\boxed{\mathbf{D} = \frac{1}{2}\left((\nabla\mathbf{u}) + (\nabla\mathbf{u})^T\right)}$$

$$\boxed{\Omega = \frac{1}{2}\left((\nabla\mathbf{u}) - (\nabla\mathbf{u})^T\right)}$$

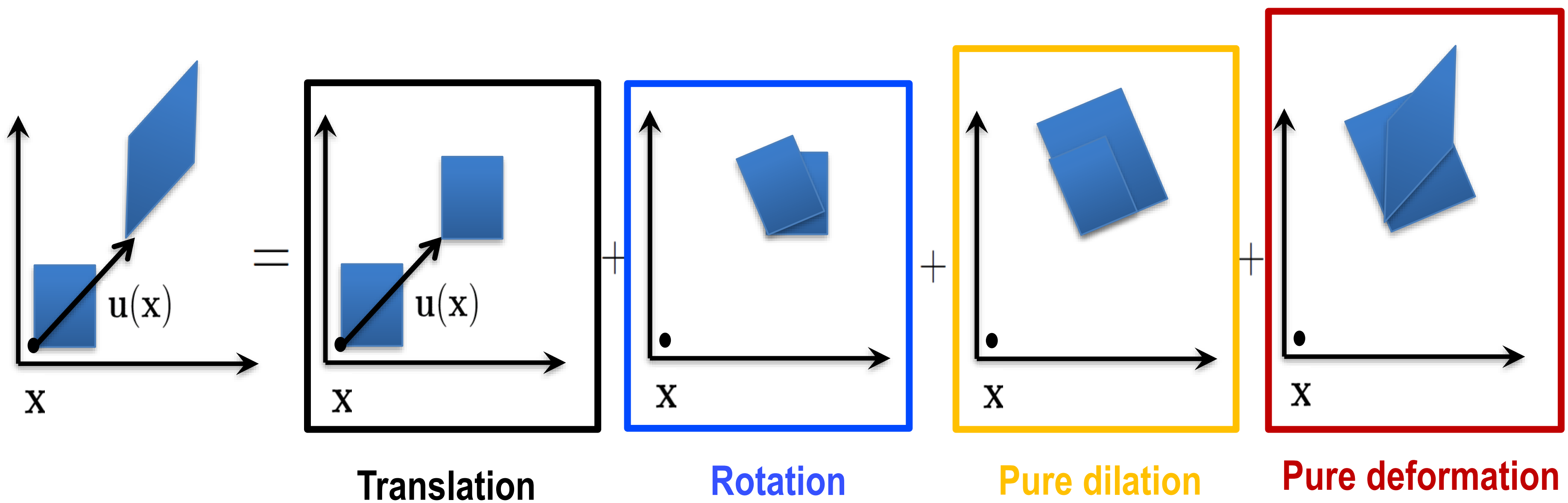
Deformation of a fluid parcel centered in $\mathbf{x}$

Taylor expansion of the velocity field

$$\mathbf{u}(\mathbf{x}+\delta\mathbf{x}) = \boxed{\mathbf{u}(\mathbf{x})} + \nabla\mathbf{u}\,\delta\mathbf{x}$$

$$\nabla\mathbf{u} = \boxed{\mathbf{S}} + \boxed{\mathbf{T}} + \boxed{\mathbf{\Omega}}$$

diagonal trace-free antisymmetric



Gradient of a vector: application to the velocity field

Taylor expansion of the velocity field

$$\mathbf{u}(\mathbf{x}+\delta\mathbf{x}) = \mathbf{u}(\mathbf{x}) + \nabla\mathbf{u}\;\delta\mathbf{x}$$

$$\nabla\mathbf{u} = \boxed{\mathbf{S}} + \boxed{\mathbf{T}} + \boxed{\mathbf{\Omega}}$$

diagonal trace free antisymmetric

Rotation and vorticity

The action of the antisymetric part of the velocity gradient can be reexpressed as a vectorial product

$$\boxed{\Omega = \frac{1}{2}\left((\nabla \mathbf{u}) - (\nabla \mathbf{u})^T\right)}$$

$$\boxed{\frac{1}{2}\begin{pmatrix} 0 & \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) & \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \\ \left(-\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & 0 & \left(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\right) \\ \left(-\frac{\partial u}{\partial z} - \frac{\partial w}{\partial z}\right) & \left(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\right) & 0 \end{pmatrix}} \begin{pmatrix} \delta x \\ \delta y \\ \delta z \end{pmatrix} = \frac{1}{2} \boxed{\begin{pmatrix} -\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\ +\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \\ -\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{pmatrix}} \wedge \begin{pmatrix} \delta x \\ \delta y \\ \delta z \end{pmatrix}$$

$$\boxed{\Omega} \quad \delta \mathbf{x} = \frac{1}{2} \boxed{\omega} \wedge \delta \mathbf{x}$$

vorticity

$$\boxed{\omega = \nabla \wedge \mathbf{u}}$$

Rotational

Vectorial field
(ex : velocity)

$$\overrightarrow{U}(x,y,z) = U_x \overrightarrow{e_x} + U_y \overrightarrow{e_y} + U_z \overrightarrow{e_z}$$

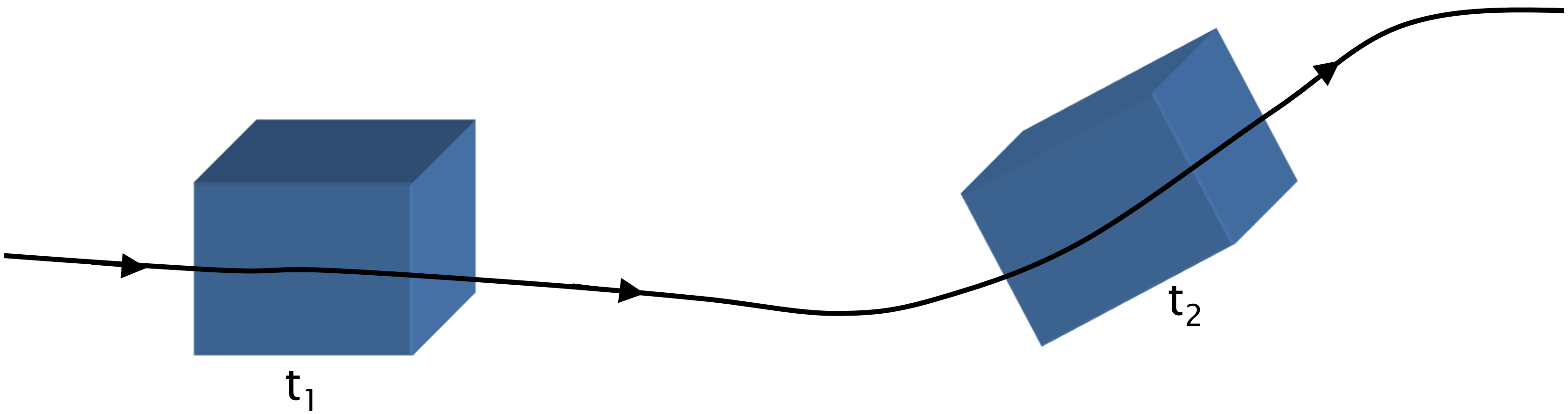
$$\overrightarrow{rot}\,\overrightarrow{U} = \overrightarrow{\nabla} \wedge \overrightarrow{U} = \begin{pmatrix} \dfrac{\partial U_z}{\partial y} - \dfrac{\partial U_y}{\partial z} \\ \dfrac{\partial U_x}{\partial z} - \dfrac{\partial U_z}{\partial x} \\ \dfrac{\partial U_y}{\partial x} - \dfrac{\partial U_x}{\partial y} \end{pmatrix}$$

→ Rotational of vector = vector

$\overrightarrow{\Omega} = \overrightarrow{rot}\ \ \overrightarrow{U}$ vorticity

Rotational : physical interpretation

The vorticity characterizres the instantaneous rotation of a parcel of fluid around its center

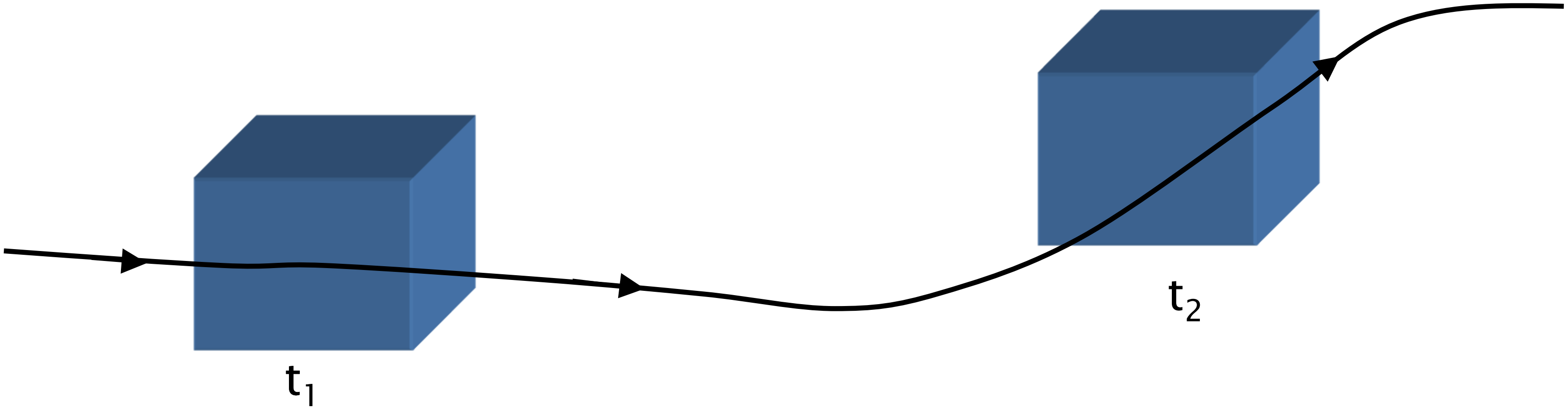


Rotational motion

$$\overrightarrow{rot}\,\overrightarrow{U} \neq 0$$

Rotational : physical interpretation

The vorticity characterizres the instantaneous rotation of a parcel of fluid around its center



Irrotational motion

$$\overrightarrow{rot}\,\overrightarrow{U} = 0$$

Laplacian

Scalar field $U_x(x, y, z)$
(ex : one component of the velocity field)

$$\Delta U_x = \frac{\partial^2 U_x}{\partial x^2} + \frac{\partial^2 U_x}{\partial y^2} + \frac{\partial^2 U_x}{\partial z^2}$$

$\rightarrow$ Laplacien of scalar = scalar

Laplacian

Vector field
(ex : velocity)

$\vec{U}(x, y, z)$

$$\Delta\vec{U} = \begin{pmatrix} \Delta U_x \\ \Delta U_y \\ \Delta U_z \end{pmatrix}$$

$\rightarrow$ Laplacian of vector = vector

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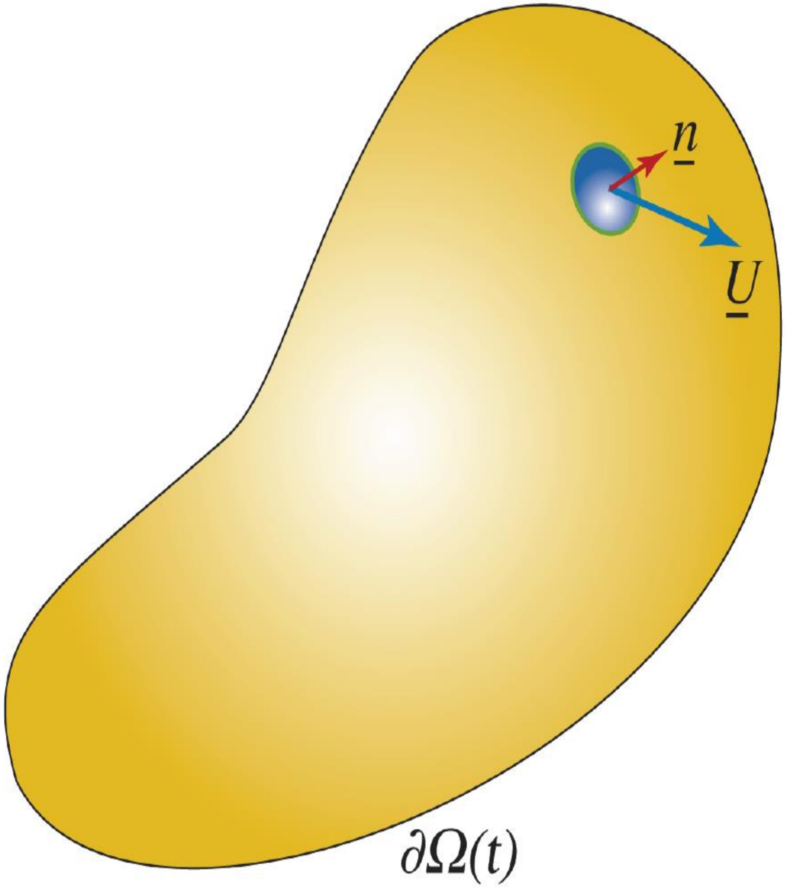
Material Volume



Transport theorem

$$\frac{d}{dt}\int_{\Omega(t)} b(\underline{x},t)\, d\Omega(t) \qquad \textbf{?}$$

Transport theorem

$$\frac{d}{dt}\int_{\Omega(t)} b(\underline{x},t)\,d\Omega(t) = \int_{\Omega(t)} \left(\frac{db}{dt}\,d\Omega(t) + b\,\dot{\widehat{d\Omega(t)}} \right)$$

Transport theorem

$$\frac{d}{dt}\int_{\Omega(t)} b(\underline{x},t)\, d\Omega(t) = \int_{\Omega(t)} \left(\frac{db}{dt} d\Omega(t) + b\, \dot{\overparen{d\Omega(t)}} \right)$$

$$\dot{\overparen{d\Omega(t)}} = d\Omega(t)\, \mathrm{div}\, \underline{U}$$

Transport theorem

$$\frac{d}{dt}\int_{\Omega(t)} b(\underline{x},t)\,d\Omega(t) = \int_{\Omega(t)}\left(\frac{db}{dt}\,d\Omega(t) + b\,\dot{\widehat{d\Omega(t)}}\right)$$

$$\dot{\widehat{d\Omega(t)}} = d\Omega(t)\,\mathrm{div}\,\underline{U}$$

$$\frac{d}{dt}\int_{\Omega(t)} b(\underline{x},t)\,d\Omega(t) = \int_{\Omega(t)}\left(\frac{db}{dt} + b\,\mathrm{div}\,\underline{U}\right)d\Omega(t)$$

Material derivative

$$B\left(\underline{X},t\right)=b\left[\underline{\phi}\left(\underline{X},t\right),t\right]$$

$$\dot{\mathcal{B}}=\frac{\partial B}{\partial t}=\frac{\partial b}{\partial t}+\operatorname{grad} b.\frac{\partial \underline{\phi}}{\partial t}$$

$$\dot{\mathcal{B}}\equiv\frac{db}{dt}=\frac{\partial b}{\partial t}+\operatorname{grad} b.\underline{U}$$

Material derivative | Local derivative | Convective derivative

Transport theorem

$$\frac{d}{dt}\int_{\Omega(t)} b(\underline{x},t)\,d\Omega(t) = \int_{\Omega(t)}\left(\frac{db}{dt}d\Omega(t) + b\,\dot{\overbrace{d\Omega(t)}}\right)$$

$$\dot{\overbrace{d\Omega(t)}} = d\Omega(t)\,\mathrm{div}\,\underline{U}$$

$$\frac{d}{dt}\int_{\Omega(t)} b(\underline{x},t)\,d\Omega(t) = \int_{\Omega(t)}\left(\frac{db}{dt} + b\,\mathrm{div}\,\underline{U}\right)d\Omega(t)$$

$$= \boxed{\int_{\Omega(t)}\left(\frac{\partial b}{\partial t} + \mathrm{div}\left(b\quad\underline{U}\right)\right)d\Omega(t)}$$

volumetric form

Transport theorem

$$\frac{d}{dt}\int_{\Omega(t)} b(\underline{x},t)\, d\Omega(t) = \int_{\Omega(t)} \left(\frac{db}{dt} d\Omega(t) + b\, \dot{\overbrace{d\Omega(t)}} \right)$$

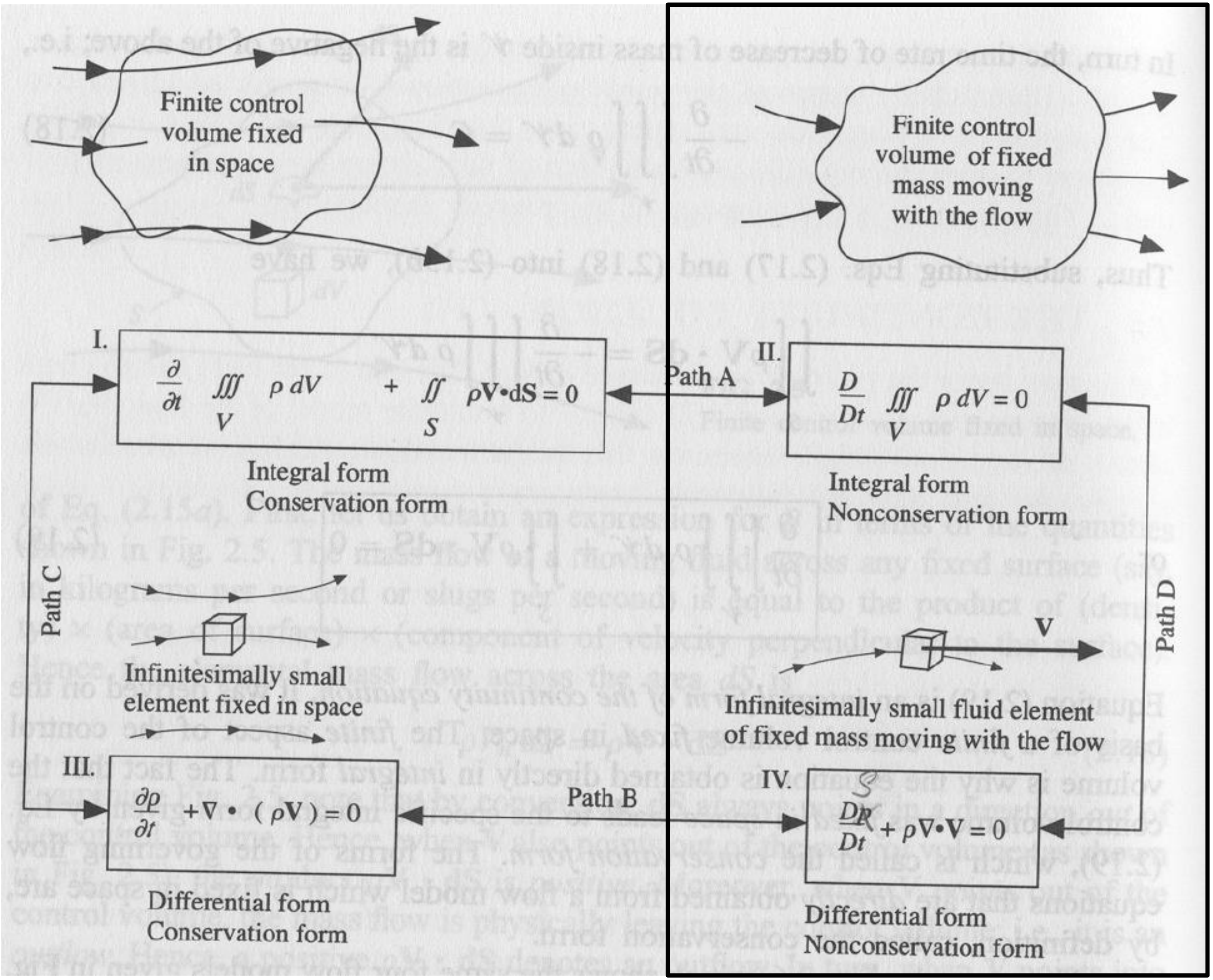
$$\dot{\overbrace{d\Omega(t)}} = d\Omega(t)\,\mathrm{div}\,\underline{U}$$

$$\frac{d}{dt}\int_{\Omega(t)} b(\underline{x},t)\, d\Omega(t) = \int_{\Omega(t)} \left(\frac{db}{dt} + b\,\mathrm{div}\,\underline{U} \right) d\Omega(t)$$

$$= \int_{\Omega(t)} \left(\frac{\partial b}{\partial t} + \mathrm{div}\left(b \quad \underline{U}\right) \right) d\Omega(t)$$

$$\frac{d}{dt}\int_{\Omega(t)} b(\underline{x},t)\, d\Omega(t) = \int_{\Omega(t)} \frac{\partial b}{\partial t} d\Omega(t) + \int_{\partial\Omega(t)} b\left(\underline{U}.\underline{n}\right) da(t)$$

Surface flux expression



Finite control volume fixed in space
Finite control volume of fixed mass moving with the flow
I.
$\frac{\partial}{\partial t} \iiint_V \rho \, dV + \iint_S \rho \mathbf{V} \cdot d\mathbf{S} = 0$
Path A
II.
$\frac{D}{Dt} \iiint_V \rho \, dV = 0$
Integral form
Conservation form
Integral form
Nonconservation form
Path C
Path D
V
Infinitesimally small element fixed in space
Infinitesimally small fluid element of fixed mass moving with the flow
III.
$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0$
Path B
IV.
$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{V} = 0$
Differential form
Conservation form
Differential form
Nonconservation form

Fundamental laws

Balance	$b(\underline{x},t)$
Mass	ρ
Momentum	$\rho\,\underline{U}$
Angular Momentum	$\rho\ \underline{OM} \wedge \underline{U}$
Energy	$\rho\ e + U^2/2$

Mass conservation of a fluid element

$$\frac{d\,M(t)}{dt} = \int_{\omega(t)} \left[\frac{D\rho}{Dt} + \rho \operatorname{div} \boldsymbol{v} \right] dV$$

$$= \int_{\omega(t)} \left[\frac{\partial\rho}{\partial t} + \operatorname{div}(\rho \boldsymbol{v}) \right] dV$$

$$\boxed{\frac{D\rho}{Dt} = - \rho \operatorname{div} \boldsymbol{v}}$$

Continuity equation

$$\boxed{\frac{D\rho}{Dt} = -\,\rho\,\mathrm{div}\,\boldsymbol{v}}$$

Incompressible flow

$$\boxed{\mathrm{div}\,\boldsymbol{v} = 0}$$

The density is constant on a trajectory

Fundamental laws
Local forms

Conservative form

$$\frac{\partial}{\partial t}(\rho b) + \operatorname{div}(\rho b \; \underline{U}) = Q + \operatorname{div} A$$

Volume sources

Surface fluxes

Fundamental laws
Local forms

Conservative form

$$\frac{\partial}{\partial t}(\rho\, b) + \mathrm{div}(\rho\, b\ \ \underline{U}) = Q + \mathrm{div}\, A$$

Continuity equation

Non conservative form

$$\rho \frac{db}{dt} = Q + \mathrm{div}\, A$$

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Conservation of momentum

Newton's second law

$$m\vec{a} = \sum \vec{F}$$

Force balance :

- pressure forces
- viscous forces
- volumetric forces f

$$\frac{d}{dt}\int_{\omega(t)} \rho\,\boldsymbol{v}\,dV = \int_{\partial\omega(t)} \boldsymbol{t}\,dS + \int_{\omega(t)} \rho \boldsymbol{f}\,dV$$

What type of stresses?

- Volumetric stresses, associated to a volumetric force distribution

  → gravity, electro-magnetic force (conducting fluid)…

- Surface stresses, applying at the surface of a continuum parcel

  → friction, pressure, surface tension,…

Momentum conservation

$$\frac{d}{dt}\int_{\omega(t)} \rho\,\boldsymbol{v}\,dV = \boxed{\int_{\partial\omega(t)} \boldsymbol{t}\,dS} + \boxed{\int_{\omega(t)} \rho\,\boldsymbol{f}\,dV}$$

Volumetric forces

A theorem due to Cauchy, using small tetrahedra of arbitrary orientation, shows that the surface force is linear with the normal to the surface and allows us to represent the cohesion forces by a stress tensor

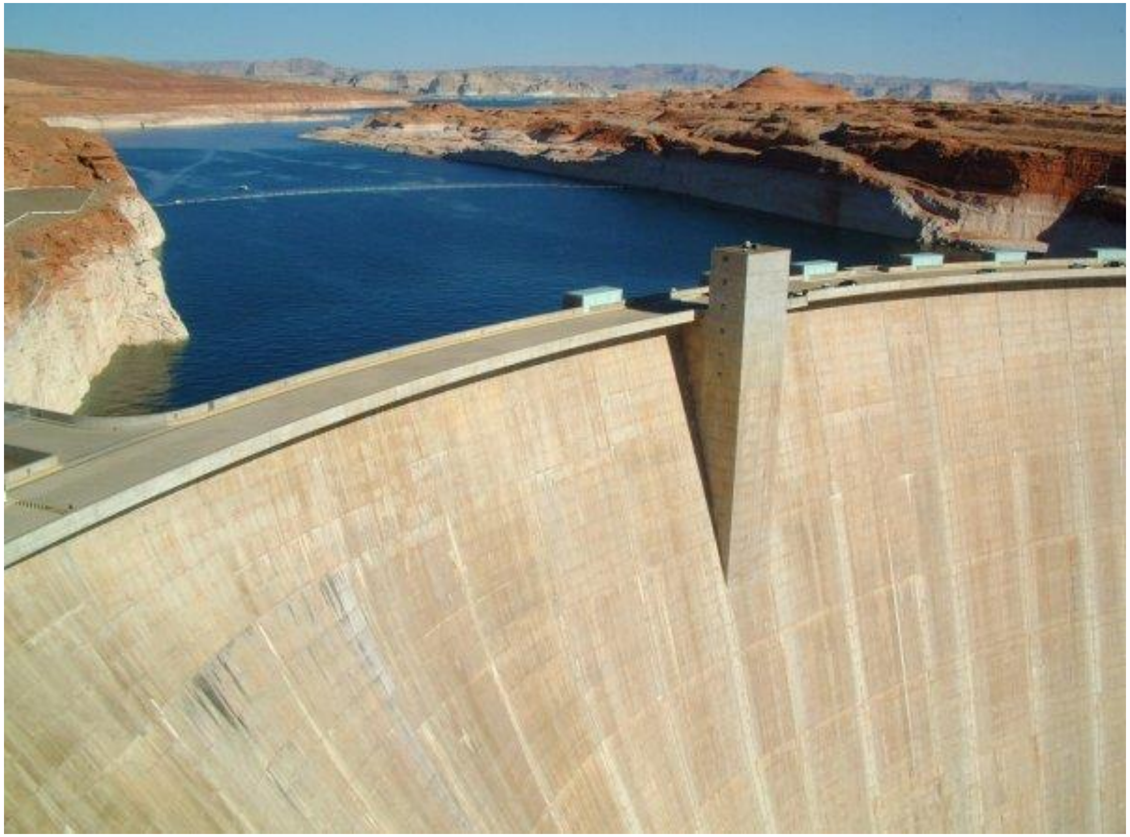
$$\boldsymbol{t} = \boldsymbol{\sigma}\boldsymbol{n}$$

Normal

Surface forces

Cauchy stress tensor

Hydrostatics

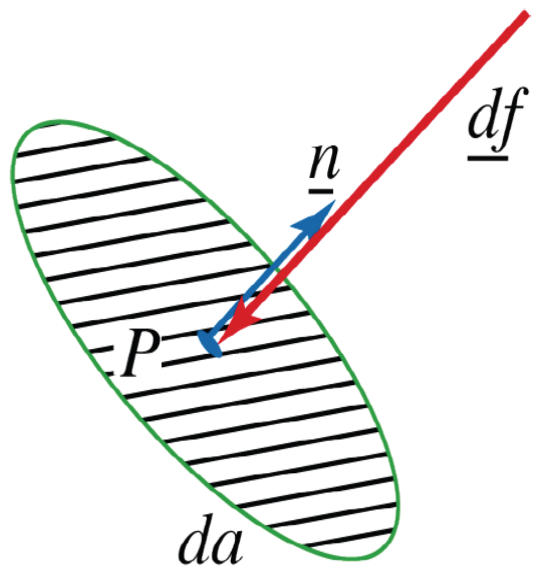


Pressure

Stresses in a fluid at rest : pressure

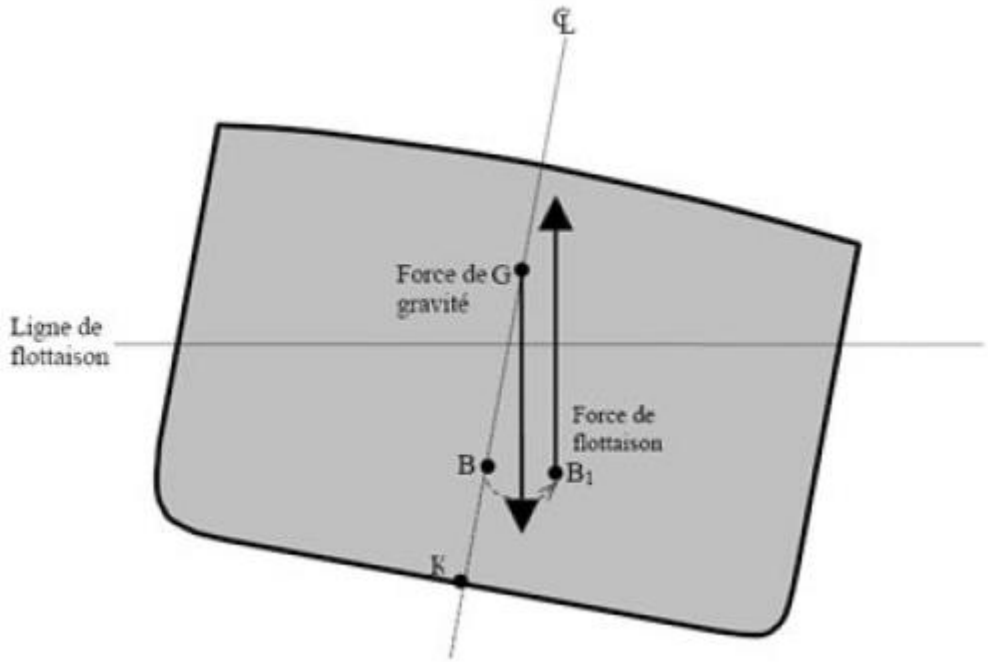
A fluid at rest is subjected to isotropic normal forces!

$$\underline{\underline{\sigma}} = -p\left(\underline{x}\right)\underline{\underline{1}}$$

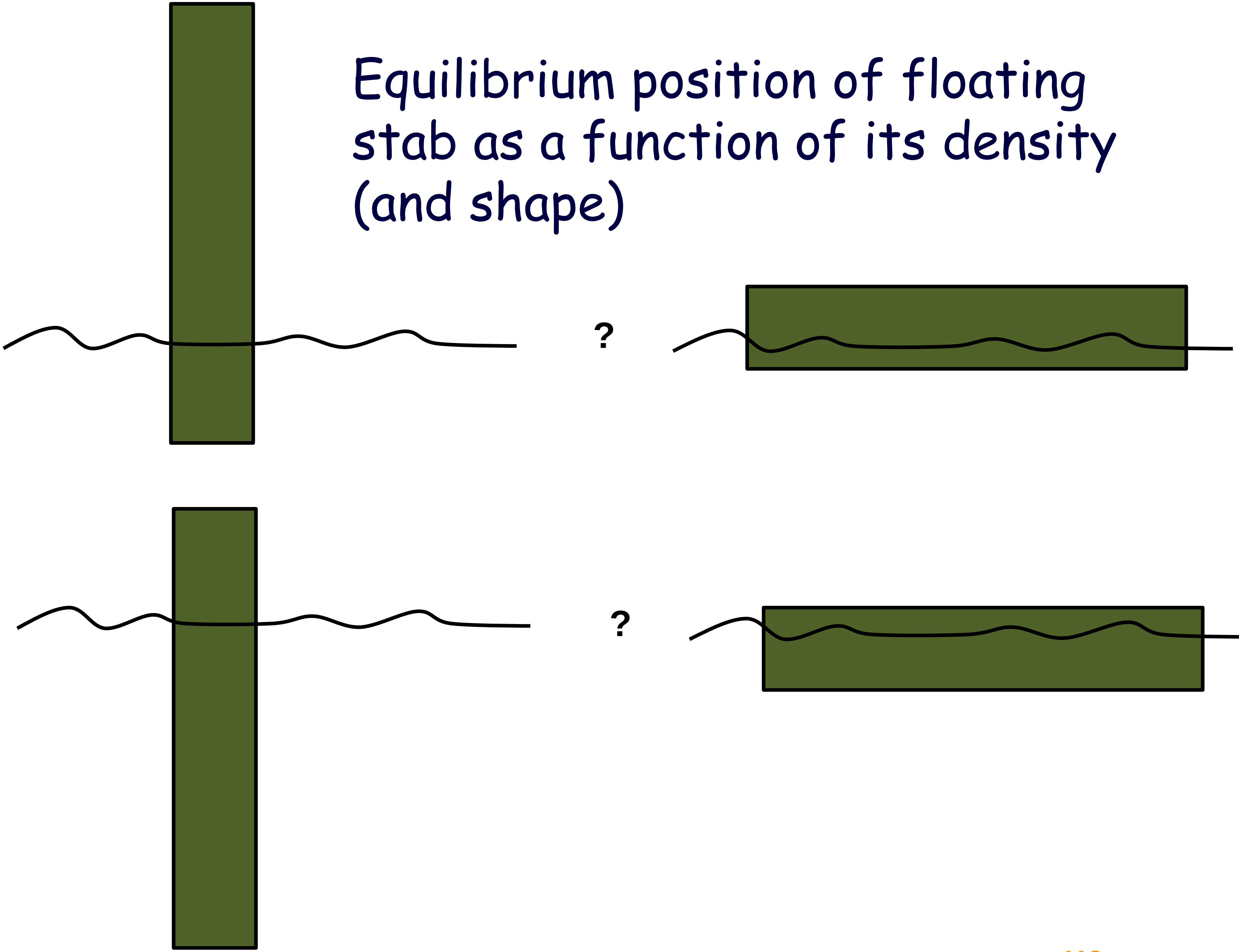


$$\underline{df} = \underline{T}\left(\underline{x},\underline{n}\left(\underline{x}\right)\right)da = -p\left(\underline{x}\right)\underline{n}\left(\underline{x}\right)da$$

Archimedes law


CL
Force de G
gravité
Ligne de
flottaison
Force de
flottaison
B
B1
K

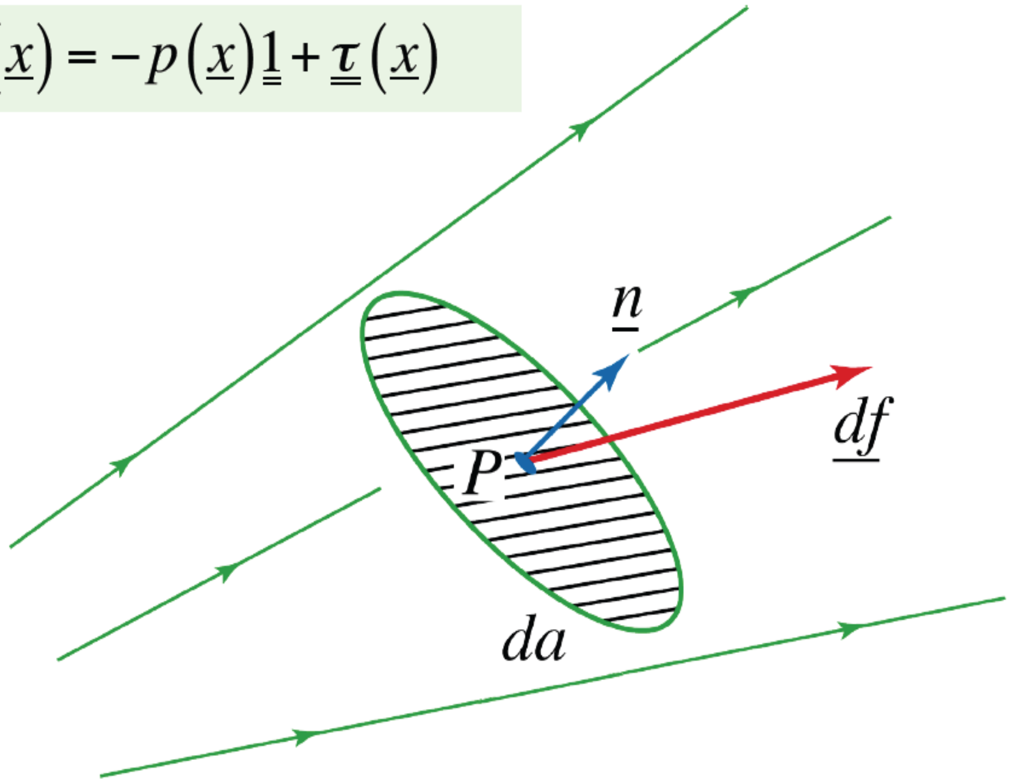
Equilibrium position of floating stab as a function of its density (and shape)



Stresses in a moving fluid

In addition to the normal isotropic pressure force, the fluid element feels both normal and tangential viscous forces

$$\underline{\underline{\sigma}}(\underline{x}) = -p(\underline{x})\underline{\underline{1}} + \underline{\underline{\tau}}(\underline{x})$$



$$\underline{df} = \underline{T}(\underline{x}, \underline{n}(\underline{x}))\,da = -p(\underline{x})\underline{n}(\underline{x})\,da + \underline{\underline{\tau}}(\underline{x})\cdot\underline{n}(\underline{x})\,da$$

$p(\underline{x})$: pression

$\underline{\underline{\tau}}(\underline{x})$: tenseur des contraintes d'origine visqueuse

Viscous stress tensor

$$\boxed{\overrightarrow{df} = \left(-p\vec{n} + \vec{\vec{\tau}}.\vec{n} \right) dS}$$

$$\begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix} . \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix} = \begin{pmatrix} \tau_{xx}n_x + \tau_{xy}n_y + \tau_{xz}n_z \\ \tau_{yx}n_x + \tau_{yy}n_y + \tau_{yz}n_z \\ \tau_{zx}n_x + \tau_{zy}n_y + \tau_{zz}n_z \end{pmatrix}$$

Stress in a moving fluid: viscous stress tensor

Newtonian fluid model :

- The stresses do not apply in preferential direction
- The intensity of the stress is a linear function of the velocity gradient.

Most usual fluids (water, air, quicksilver…) are well approximated by the Newtonian fluid model.

Viscous stress tensor

Newtonian fluid model

$$\begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix}$$

$$\tau_{xx} = -\frac{2}{3}\mu\left(\frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y} + \frac{\partial U_z}{\partial z}\right) + 2\boxed{\mu}\frac{\partial U_x}{\partial x}$$

$$\tau_{yy} = -\frac{2}{3}\mu\left(\frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y} + \frac{\partial U_z}{\partial z}\right) + 2\mu\frac{\partial U_y}{\partial y}$$

$$\tau_{zz} = -\frac{2}{3}\mu\left(\frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y} + \frac{\partial U_z}{\partial z}\right) + 2\mu\frac{\partial U_z}{\partial z}$$

μ: dynamic viscosity (kg/m/s)

$\nu = \mu/\rho$:
Kinematic viscosity (m^2/s)

Viscous stress tensor

$$\tau_{xx} = -\frac{2}{3}\mu\left(\frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y} + \frac{\partial U_z}{\partial z}\right) + 2\mu\frac{\partial U_x}{\partial x}$$

$$\tau_{yy} = -\frac{2}{3}\mu\left(\frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y} + \frac{\partial U_z}{\partial z}\right) + 2\mu\frac{\partial U_y}{\partial y}$$

$$\tau_{zz} = -\frac{2}{3}\mu\left(\frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y} + \frac{\partial U_z}{\partial z}\right) + 2\mu\frac{\partial U_z}{\partial z}$$

$$\begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix}$$

Divergence of velocity field,
i.e. volumnetric dilatation

Viscous stress tensor

$$\tau_{xx} = -\frac{2}{3}\mu\cancel{\left(\frac{\partial U_x}{\partial x}+\frac{\partial U_y}{\partial y}+\frac{\partial U_z}{\partial z}\right)}+2\mu\frac{\partial U_x}{\partial x}$$

$$\tau_{yy} = -\frac{2}{3}\mu\cancel{\left(\frac{\partial U_x}{\partial x}+\frac{\partial U_y}{\partial y}+\frac{\partial U_z}{\partial z}\right)}+2\mu\frac{\partial U_y}{\partial y}$$

$$\tau_{zz} = -\frac{2}{3}\mu\cancel{\left(\frac{\partial U_x}{\partial x}+\frac{\partial U_y}{\partial y}+\frac{\partial U_z}{\partial z}\right)}+2\mu\frac{\partial U_z}{\partial z}$$

$$\begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix}$$

$\rightarrow$ =0 if flow is incompressible

Viscous stress tensor

$$\tau_{xx} = 2\mu \boxed{\frac{\partial U_x}{\partial x}}$$

$$\tau_{yy} = 2\mu \boxed{\frac{\partial U_y}{\partial y}}$$

$$\tau_{zz} = 2\mu \boxed{\frac{\partial U_z}{\partial z}}$$

Stretching terms

$$\begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix}$$

Viscous stress tensor

$$\begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix}$$

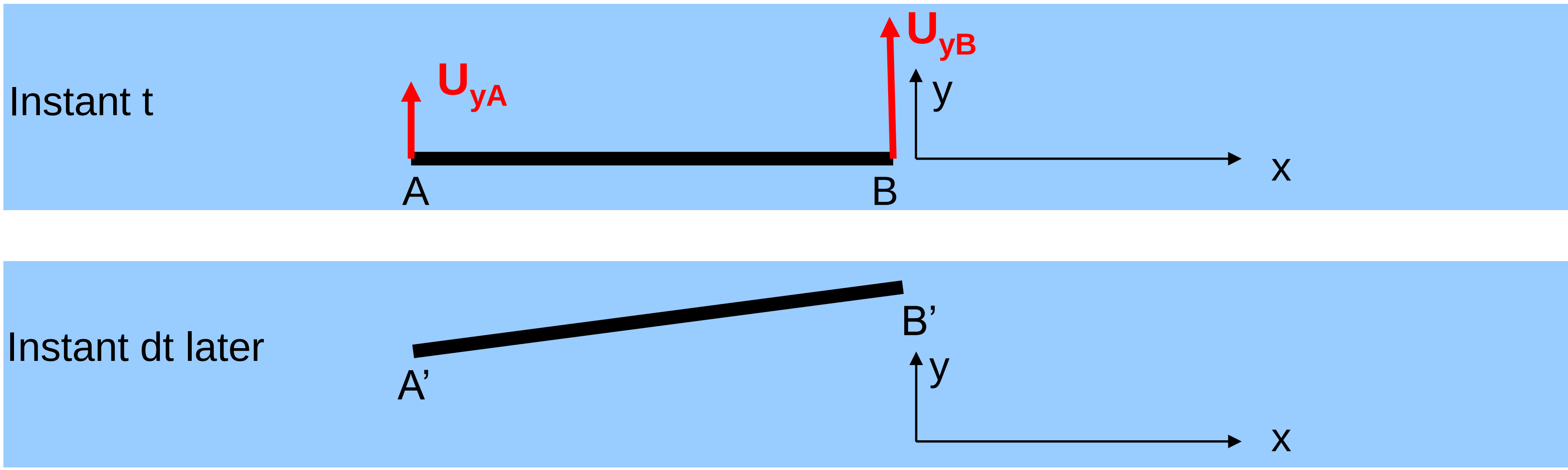
$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial U_x}{\partial y} + \frac{\partial U_y}{\partial x}\right)$$

$$\tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial U_z}{\partial x} + \frac{\partial U_x}{\partial z}\right)$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial U_y}{\partial z} + \frac{\partial U_z}{\partial y}\right)$$

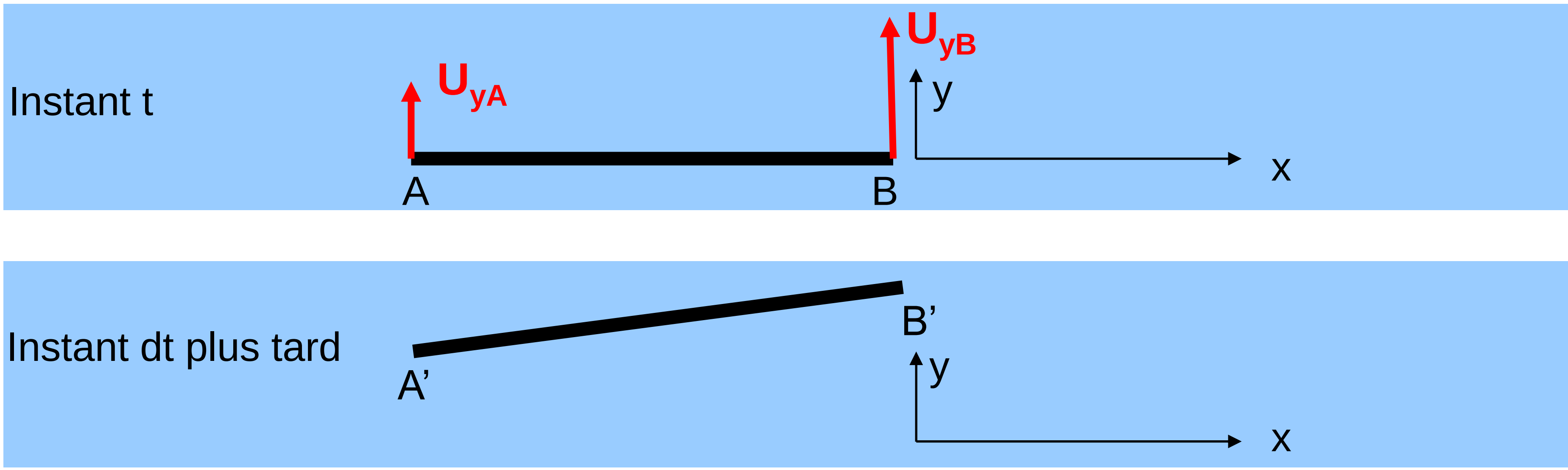
Shear terms

Stresses : shear



AB is a line of fluid particles in a flow such that U_{yA}<U_{yB}. Since the velocity is higher in B than in A, the segment AB will rotate. This **deformation** is called **shear**. The relevant quantity is the derivative of the velocity with respect to the direction **normal** to this velocity

Stresses : shear



$$\boxed{\frac{\partial U_y}{\partial x}}$$

Outline

1. Introduction
2. Fluid: Definition, models and classifications
3. Navier-Stokes

   What do we need?

   1. F=ma and Lavoisier
   2. Fluid Kinematics, Euler-Lagrange, transport theorem
   3. A constitutive model
   4. Differential operators

Momentum conservation for incompresible fluid

Newton's second law

$$\frac{d}{dt}\int_{\omega(t)} \rho\, \boldsymbol{v}\, dV = \int_{\partial\omega(t)} \boldsymbol{t}\, dS + \int_{\omega(t)} \rho \boldsymbol{f}\, dV$$

Transport theorem

$$\frac{d}{dt}\int_{\omega(t)} \rho\, \boldsymbol{v}\, dV = \int_{\omega(t)} \rho\, \frac{D\boldsymbol{v}}{Dt}\, dV$$

Green's identity

$$\int_{\omega(t)} \left[\rho\, \frac{D\boldsymbol{v}}{Dt} - \boldsymbol{div}\, \boldsymbol{\sigma} - \rho \boldsymbol{f}\right] dV = 0$$

$$\boxed{\rho \frac{D\boldsymbol{v}}{Dt} = \boldsymbol{div}\, \boldsymbol{\sigma} + \rho \boldsymbol{f}}$$

Newtonian incompressible fluid

$$\boxed{\rho \frac{D\boldsymbol{v}}{Dt} = -\boldsymbol{\nabla} p + \mu \, \Delta \boldsymbol{v} + \rho \boldsymbol{f}}$$

$$\boxed{\operatorname{div} \boldsymbol{v} = 0}$$

Navier-Stokes equations

$$\left(m\vec{a} = \sum \vec{F}\right).\vec{e_x}$$

$$\rho \boxed{\frac{dU_x}{dt}} = \rho f_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 U_x}{\partial x^2} + \frac{\partial^2 U_x}{\partial y^2} + \frac{\partial^2 U_x}{\partial z^2}\right)$$

acceleration
(total derivative)

Navier-Stokes equations

$$\left(m\vec{a} = \sum \vec{F}\right) . \vec{e_x}$$

$$\rho \frac{dU_x}{dt} = \boxed{\rho f_x} - \frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 U_x}{\partial x^2} + \frac{\partial^2 U_x}{\partial y^2} + \frac{\partial^2 U_x}{\partial z^2} \right)$$

Volumnar forces

Navier-Stokes equations

$$\left(m\vec{a} = \sum \vec{F}\right).\vec{e_x}$$

$$\rho \frac{dU_x}{dt} = \rho f_x - \boxed{\frac{\partial p}{\partial x}} + \mu\left(\frac{\partial^2 U_x}{\partial x^2} + \frac{\partial^2 U_x}{\partial y^2} + \frac{\partial^2 U_x}{\partial z^2}\right)$$

Pressure gradient

Navier-Stokes equations

$$\left(m\vec{a} = \sum \vec{F}\right).\vec{e_x}$$

$$\rho \frac{dU_x}{dt} = \rho f_x - \frac{\partial p}{\partial x} + \boxed{\mu\left(\frac{\partial^2 U_x}{\partial x^2} + \frac{\partial^2 U_x}{\partial y^2} + \frac{\partial^2 U_x}{\partial z^2}\right)}$$

Viscous stress

Navier-Stokes equations

$$\rho\left(\frac{\partial U_x}{\partial t}+U_x\frac{\partial U_x}{\partial x}+U_y\frac{\partial U_x}{\partial y}+U_z\frac{\partial U_x}{\partial z}\right)=\rho f_x-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 U_x}{\partial x^2}+\frac{\partial^2 U_x}{\partial y^2}+\frac{\partial^2 U_x}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial U_y}{\partial t}+U_x\frac{\partial U_y}{\partial x}+U_y\frac{\partial U_y}{\partial y}+U_z\frac{\partial U_y}{\partial z}\right)=\rho f_y-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 U_y}{\partial x^2}+\frac{\partial^2 U_y}{\partial y^2}+\frac{\partial^2 U_y}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial U_z}{\partial t}+U_x\frac{\partial U_z}{\partial x}+U_y\frac{\partial U_z}{\partial y}+U_z\frac{\partial U_z}{\partial z}\right)=\rho f_z-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 U_z}{\partial x^2}+\frac{\partial^2 U_z}{\partial y^2}+\frac{\partial^2 U_z}{\partial z^2}\right)$$

Newtonian incompressible fluid

$$\boxed{\rho \frac{D\boldsymbol{v}}{Dt} = -\boldsymbol{\nabla} p + \mu \, \Delta \boldsymbol{v} + \rho \boldsymbol{f}}$$

$$\boxed{\operatorname{div} \boldsymbol{v} = 0}$$