Hydrodynamics

Self-similar solution to the diffusion equation- First Stokes problem

After non-dimensionalization, we are left to the following equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial y^2}, \tag{1}$$

$$u(t > 0, y = 0) = 1, \tag{2}$$

$$u(t, y \to \infty) = 0, \tag{3}$$

$$u(t = 0, y) = 0. \tag{4}$$

1 Necessary conditions on the dilation coefficients for invariance

From the implicit function theorem, we know that there exists a function $F(t, y, u)$ which is zero when $u(t, y)$ is solution of the above differential equations including their boundary and initial conditions. Now we introduce a set of dilations for all variables

$$u = u^* \hat{u}, \tag{5}$$

$$t = t^* \hat{t} \tag{6}$$

$$y = y^* \hat{y} \tag{7}$$

Here the * are (positive) and non null dilation coefficients, and the $\hat{}$ are the stretched variables. The key idea here is to find under which conditions between these coefficients, the stretched (hat) problem is exactly the same as the initial problem. The two problems are then invariant under dilation, if these relations between the coefficients hold. The hat problem writes

$$\frac{u^*}{t^*} \frac{\partial \hat{u}}{\partial \hat{t}} = \frac{u^*}{y^{*2}} \frac{\partial^2 \hat{u}}{\partial \hat{y}^2}, \tag{8}$$

$$u^* u(t^* \hat{t} > 0, y^* \hat{y} = 0) = 1, \tag{9}$$

$$u^* u(t^* \hat{t}, y^* \hat{y} \to \infty) = 0, \tag{10}$$

$$u^* u(t^* \hat{t} = 0, y^* \hat{y}) = 0. \tag{11}$$

It is clear that if

$$t^* = y^{*2}, u^* = 1, \tag{12}$$

hold, then the hat-problem becomes

$$\frac{\partial \hat{u}}{\partial \hat{t}} = \frac{\partial^2 \hat{u}}{\partial \hat{y}^2}, \tag{13}$$

$$u(\hat{t} > 0, \hat{y} = 0) = 1, \tag{14}$$

$$u(\hat{t}, \hat{y}) \to \infty = 0, \tag{15}$$

$$u(\hat{t} = 0, \hat{y}) = 0. \tag{16}$$

and is therefore exactly similar to the initial problem.

2 Determination of a self-similar variable

The next step is to identify new variables that are non stretched when the stretching is applied. There are multiple choices so we have to pick up a convenient one (actually there is no rule). Here one can chose u and $\eta = \frac{y}{\sqrt{t}}$. It is then clear that $\hat{u} = u$ and $\hat{\eta} = \eta$. We therefore make the following change of variables $(t, y, u) \leftrightarrow (t, \eta, u)$ and introduce the changed implicit function

$$G(t, \eta, u) = 0. \tag{17}$$

If one now streches the three variables with stretching coefficients that follow the necessary conditions, then

$$G(t^*\hat{t}, \hat{\eta}, \hat{u}) = 0. \tag{18}$$

It should be clear here observing both equations that G CANNOT depend on t. A function of t cannot be the same function of t^*t, unless it does not depend on t. If the function is for example $t^2 - \eta u = 0$ than it is impossible to have simultaneously $100t^2 - \eta u = 0$ (I chose randomly $t^* = 10$) unless $t = 0$ and $\eta u = 0$. Therefore actually there exists a function $H(\eta, u) = 0$, or applying the implicit function theorem

$$u = f(\eta). \tag{19}$$

We have thereby succeeded in finding a pair of self-similar variables η and u (this one turns out to be equal to an initial variable, but this is not always the case), which combine the initial variables in a smart way and help reducing the number of variables.

3 Rewriting the initial PDE (partial differential equation) into an ODE (ordinary differential equation) with help of the self-similar variables

We now use the chain rule to express the partial derivatives. For instance,

$$\frac{\partial u}{\partial t} = \frac{df}{d(\eta)}\frac{\partial \eta}{\partial t}. \tag{20}$$

With the definition of $\eta = \frac{y}{\sqrt{t}}$,

$$\frac{\partial \eta}{\partial t} = -\frac{y}{2t\sqrt{t}}, \tag{21}$$

which yields

$$\frac{\partial u}{\partial t} = -\frac{y}{2t\sqrt{t}}\frac{df}{d(\eta)}. \tag{22}$$

Similarly

$$\frac{\partial \eta}{\partial y} = \frac{1}{\sqrt{t}}, \tag{23}$$

therefore

$$\frac{\partial u}{\partial y} = \frac{df}{d(\eta)} \frac{\partial \eta}{\partial y} = \frac{df}{d(\eta)} \frac{1}{\sqrt{t}}. \tag{24}$$

Therefore, derivating again (observe that it was smart to take a self-similar variable linear in the variable with the highest partial derivative, but this is no always also so simple)

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{\sqrt{t}} \frac{d^2 f}{d(\eta^2)} \frac{\partial \eta}{\partial y} = \frac{1}{t} \frac{d^2 f}{d(\eta^2)}. \tag{25}$$

The diffusion equation becomes

$$-\frac{y}{2t\sqrt{t}} \frac{df}{d(\eta)} = \frac{1}{t} \frac{d^2 f}{d(\eta^2)}. \tag{26}$$

or in other words

$$-\eta \frac{df}{d(\eta)} = \frac{d^2 f}{d(\eta^2)}. \tag{27}$$

There is a consistency check here, only η should appear, not the initial variables. The boundary conditions have also to be expressed in η. When $y = 0$, $\eta = 0$. When $y \to \infty$ or $t = 0$, $\eta \to$, yielding

$$f(0) = 1; f(\infty) = 0. \tag{28}$$

4 Solution of the ODE

It turns out that the ODE which has been obtained can be solved analytically, but this is not always the case. Anyways, we have won a lot, because we have transformed a PDE (which would need to be discretized in space AND time) into an ODE (which is discretized in $\eta \in [0\infty)$). The price to pay is to make a constant coefficient PDE into a non-constant coefficient ODE. There is no free lunch! Let us solve it in the present case, using obvious notations

$$-\eta f' = f'' \tag{29}$$

can be rewritten

$$-\eta = \frac{f''}{f'}. \tag{30}$$

Integrate once to give

$$\frac{-\eta^2}{2} + A = ln(f'), \tag{31}$$

and, after taking the exponential

$$A \exp^{-\frac{\eta^2}{2}} = f'. \tag{32}$$

This means that f' is a Gaussian, a function which appears again and again in diffusion processes. The integral of Gaussian has a name, the error function

$$erf(x) = \int_0^x \exp^{-x'^2} dx'. \tag{33}$$

A last change of variable yields

$$f(\eta) = 1 - erf(\eta/2). \tag{34}$$