Exercise 1

Propagation and amplification of a tsunami

We consider the motion of the sea above a sea floor $z = f(x)$. The sea is assumed to be an inviscid incompressible fluid of density ρ and subject to gravitational acceleration $\mathbf{g} = -g\mathbf{e_z}$.

The free surface of the sea, defined by $z = h(x,t)$ is characterized by a uniform pressure p_0. Let $u(x,z,t)$ and $w(x,z,t)$ denote the horizontal and vertical components of the velocity field and $p(x,z,t)$ the local pressure.

1. Write and name the governing equations
2. Show that the boundary condition at the seafloor writes
$$w = uf' \quad \text{at} \quad z = f(x) \tag{1}$$
3. What is the physical meaning of p_0? What assumption is hidden behind the boundary condition $p(x,h,t) = p_0$? Write the dynamic and kinematic boundary condition on the interface.
4. Let H be the maximum see level and L a characteristic horizontal length of the basin and let us assume $\delta = H/L \ll 1$, an assumption called shallow water assumption. Introduce the gauges $z = H\tilde{z}$, $x = L\tilde{x}$, $t = \tau\tilde{t}$, $u = U\tilde{u}$, $w = W\tilde{w}$ and $p = P\tilde{p}$. We further assume that $\tilde{u}(\tilde{x},\tilde{t})$ only. Determine the right gauges to obtain in the limit $\delta \ll 1$
$$\frac{\partial \tilde{u}}{\partial \tilde{t}} + \frac{\partial}{\partial \tilde{x}}\left(\frac{\tilde{u}^2}{2} + \tilde{p}\right) = 0 \tag{2}$$
$$\frac{\partial}{\partial \tilde{z}}\left(\tilde{p} + \frac{\tilde{z}}{Fr^2}\right) = 0 \tag{3}$$
$$\frac{\partial}{\partial \tilde{z}}\left(\tilde{w} + \tilde{z}\frac{\partial \tilde{u}}{\partial \tilde{x}}\right) = 0. \tag{4}$$
5. Let us also introduce the rescaled quantities
$$f(x) = H\,\tilde{f}(\tilde{x}),\ h(x,t) = H\,\tilde{h}(\tilde{x},\tilde{t}). \tag{5}$$
Rewrite the boundary conditions.
6. Establish $\tilde{p}$ as a function of $\tilde{z}$ and $\tilde{h}$
7. Integrate the continuity equation between the sea floor and the sea surface to write an equation for $\tilde{h}$.
8. Deduce with the previous results the following two equation, (written without tildes for ease of notation):
$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left(u(h-f)\right) = 0 \tag{6}$$
$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2} + \frac{h}{Fr^2}\right) = 0 \tag{7}$$

9. We assume that the floor is flat. We assume also that the waves are of small amplitude in comparison to the mean water level H. We therefore assume that $h(x,t) = 1 + \epsilon h'$ and $u(x,t) = \epsilon u'$. Linearize the above equations and show that

$$\frac{\partial^2 h'}{\partial t^2} - c_0^2 \frac{\partial^2 h'}{\partial x^2} = 0, \tag{8}$$

   and give the expression of c_0.

10. Assume now $h'(x,t) = \cos(kx - \omega t)$. Determine ω as a function of k. How do you name such a relation? What is the physical interpretation of ω, k, c_0? Is the shallow water model dispersive?

    Apply these results to a tsunami generated at 4000km from the coast by a coherent earthquake of size $> 100km$ and in a sea of depth $4km$. How long does it take for the tsunami to reach the coast?

11. We assume now that in vicinity of the shore the floor has a constant slope $f = ax$ with $a > 0$. Write the equation for h'. Assuming $a \ll 1$, look for a solution of the form $h' = \exp(i(kx - \omega t))$ with complex $k = k_r + ik_i$ and real ω. Determine k_r and k_i and show that the amplitude of the waves increase as one gets closer to the shore.