Chapter 2

1. Vector analysis
2. Transport theorem
3. Stress modeling
4. Navier-Stokes

Vector

Analysis

Divergence

Vectoriel field (ex : velocity) $\overrightarrow{U}(x,y,z) = U_x \overrightarrow{e_x} + U_y \overrightarrow{e_y} + U_z \overrightarrow{e_z}$

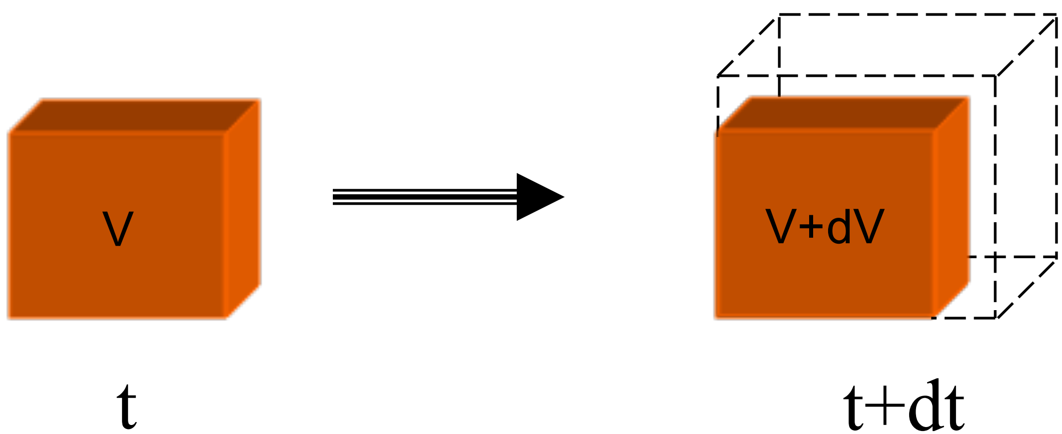
$$div\,\overrightarrow{U} = \overrightarrow{\nabla}.\overrightarrow{U} = \frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y} + \frac{\partial U_z}{\partial z}$$

→ Divergence of vector = scalar

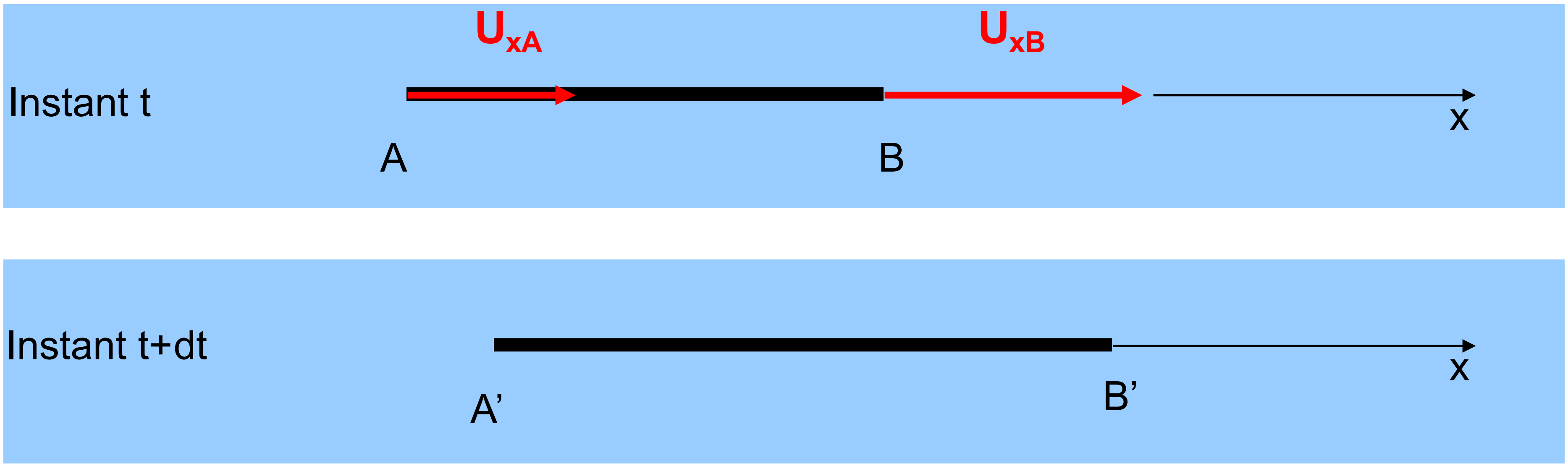
Divergence : physical interpretation

The divergence of the velocity field corresponds to the volumetric dilatation rate of a fluid volume

$$div\,\vec{U} = \frac{1}{V}\frac{dV}{dt}$$

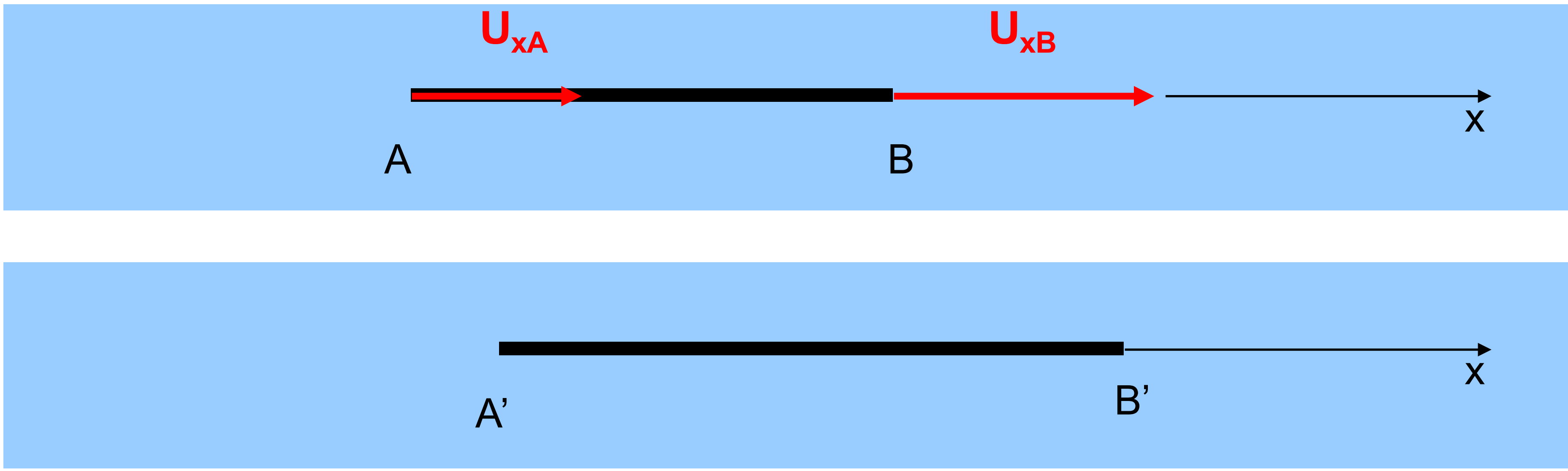


Stretching



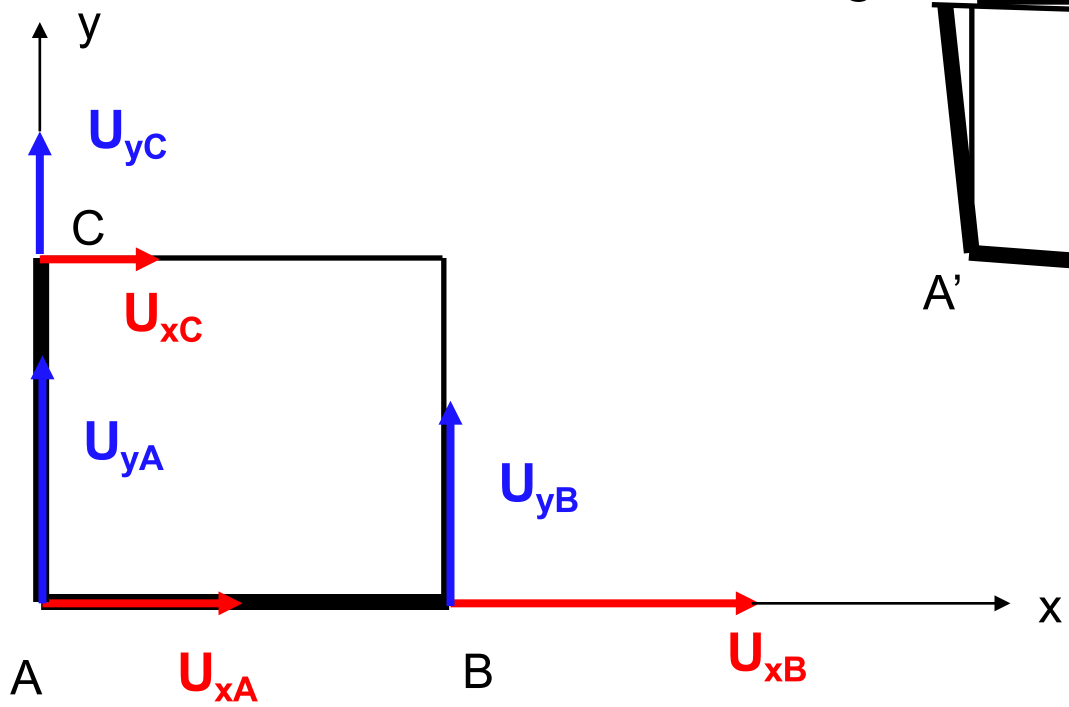
AB is a line of fluid particles in a flow such that $U_{xA}<U_{xB}$. Since the velocity is higher in B than in A, the segment AB is stretched in the x direction. This **deformation** is called a **stretching**. The relevant quantity is the derivative of the velocity with respect to the direction tangential to this velocity

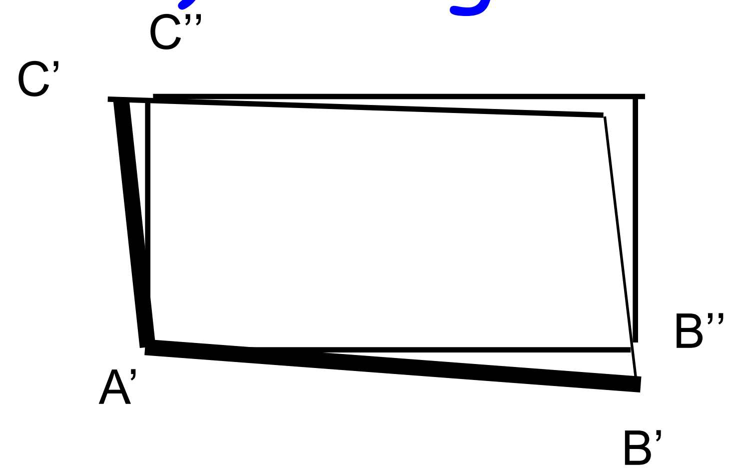
Stretching



$$A'B'-AB=(U_{xB}-U_{xA})dt=\boxed{\frac{\partial U_x}{\partial x}}dxdt$$

Volume (surface area) change





$$(A''B'')(A''C'')-(AB)(AC)=(AB)(AC)\left(\frac{\partial U_x}{\partial x}+\frac{\partial U_y}{\partial y}\right) dt$$

Gradient of a scalar

Scalar field $p(x,y,z)$
(ex : pressure)

$$\overrightarrow{grad}\ p = \underbrace{\vec{\nabla}}_{} p = \begin{pmatrix} \frac{\partial p}{\partial x} \\ \frac{\partial p}{\partial y} \\ \frac{\partial p}{\partial z} \end{pmatrix}$$

$\vec{\nabla}$ Gradient operateur (nabla)

→ Gradient of scalar = vector

Gradient of a vector: application to the velocity field

Taylor expansion of the velocity field

$$\mathbf{u}(\mathbf{x} + \delta\mathbf{x}) = \mathbf{u}(\mathbf{x}) + \nabla\mathbf{u}\ \delta\mathbf{x}$$

$$\boxed{\nabla\mathbf{u}} = \boxed{\mathbf{D}} + \boxed{\mathbf{\Omega}}$$

Velocity gradient **Symetric part** **Antisymetric part**

$$\boxed{\mathrm{D} = \frac{1}{2}\left((\nabla\mathbf{u}) + (\nabla\mathbf{u})^T\right)}$$

$$\boxed{\Omega = \frac{1}{2}\left((\nabla\mathbf{u}) - (\nabla\mathbf{u})^T\right)}$$

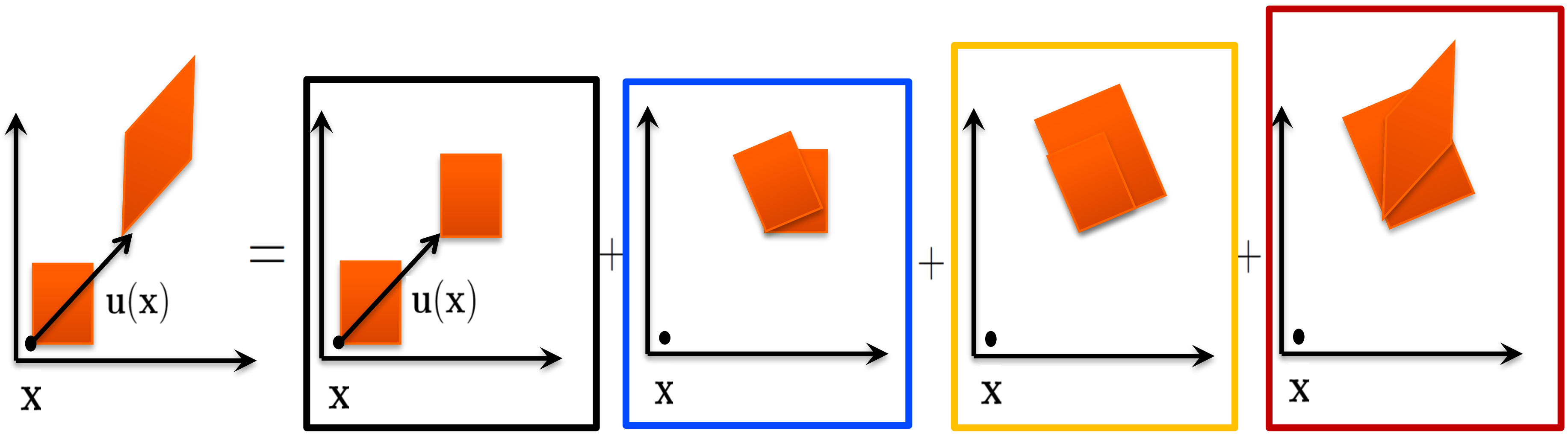
Deformation of a fluid parcel centered in $\mathbf{x}$

Taylor expansion of the velocity field

$$\mathbf{u}(\mathbf{x}+\delta\mathbf{x}) = \boxed{\mathbf{u}(\mathbf{x})} + \nabla\mathbf{u}\;\delta\mathbf{x}$$

$$\nabla\mathbf{u} = \boxed{\mathbf{S}} + \boxed{\mathbf{T}} + \boxed{\mathbf{\Omega}}$$

diagonal trace-free antisymmetric



Translation Rotation Pure dilation Pure deformation

Gradient of a vector: application to the velocity field

Taylor expansion of the velocity field

$$\mathbf{u}(\mathbf{x}+\delta\mathbf{x}) = \mathbf{u}(\mathbf{x}) + \nabla\mathbf{u}\ \delta\mathbf{x}$$

$$\nabla\mathbf{u} = \boxed{\mathbf{S}} + \boxed{\mathbf{T}} + \boxed{\mathbf{\Omega}}$$

diagonal trace free antisymmetric

Rotation and vorticity

The action of the antisymetric part of the velocity gradient can be reexpressed as a vectorial product

$$\boxed{\Omega = \frac{1}{2}\left((\nabla \mathbf{u}) - (\nabla \mathbf{u})^T\right)}$$

$$\boxed{\frac{1}{2}\begin{pmatrix} 0 & \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) & \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \\ \left(-\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & 0 & \left(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\right) \\ \left(-\frac{\partial u}{\partial z} - \frac{\partial w}{\partial z}\right) & \left(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\right) & 0 \end{pmatrix}} \begin{pmatrix} \delta x \\ \delta y \\ \delta z \end{pmatrix} = \frac{1}{2} \boxed{\begin{pmatrix} -\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\ +\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \\ -\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{pmatrix}} \wedge \begin{pmatrix} \delta x \\ \delta y \\ \delta z \end{pmatrix}$$

$$\boxed{\Omega} \qquad \delta \mathbf{x} = \frac{1}{2} \underbrace{\boxed{\omega}}_{\text{vorticity}} \wedge \delta \mathbf{x}$$

$$\boxed{\omega = \nabla \wedge \mathbf{u}}$$

Rotational

Vectorial field
(ex : velocity)

$$\overrightarrow{U}(x,y,z) = U_x \overrightarrow{e_x} + U_y \overrightarrow{e_y} + U_z \overrightarrow{e_z}$$

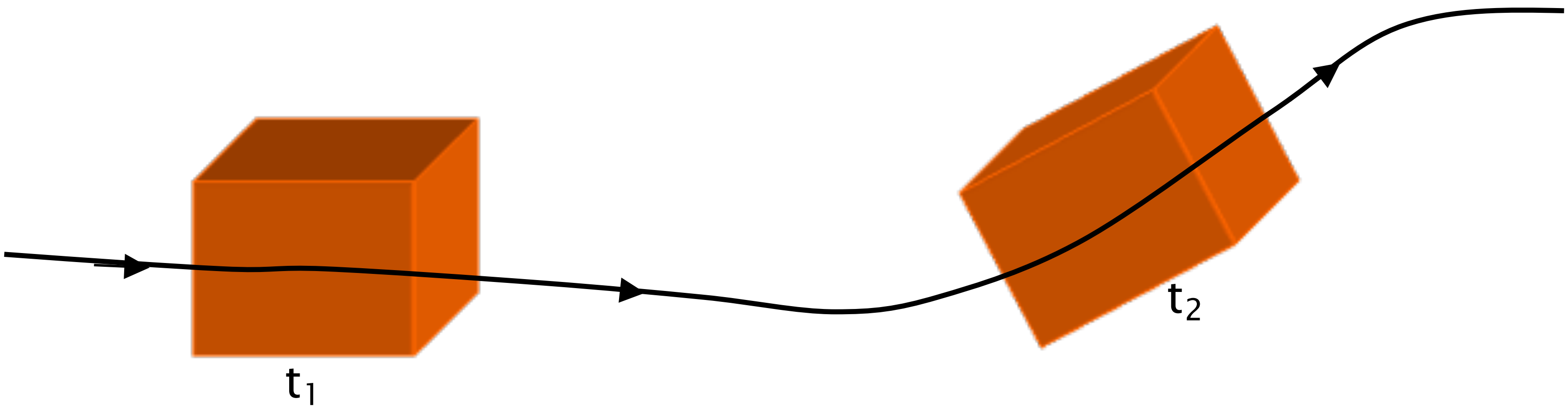
$$\overrightarrow{rot}\,\overrightarrow{U} = \overrightarrow{\nabla} \wedge \overrightarrow{U} = \begin{pmatrix} \dfrac{\partial U_z}{\partial y} - \dfrac{\partial U_y}{\partial z} \\ \dfrac{\partial U_x}{\partial z} - \dfrac{\partial U_z}{\partial x} \\ \dfrac{\partial U_y}{\partial x} - \dfrac{\partial U_x}{\partial y} \end{pmatrix}$$

→ Rotational of vector = vector

$\overrightarrow{\Omega} = \overrightarrow{rot}\ \ \overrightarrow{U}$ vorticity

Rotational : physical interpretation

The vorticity characterizres the instantaneous rotation of a parcel of fluid around its center

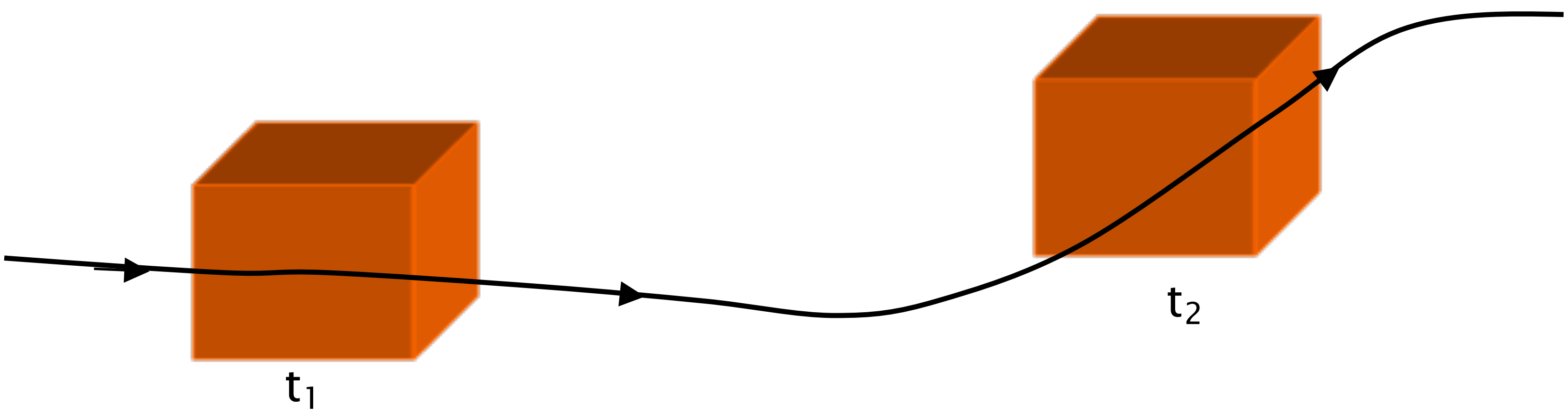


Rotational motion

$$\overrightarrow{rot}\,\vec{U} \neq 0$$

Rotational : physical interpretation

The vorticity characterizres the instantaneous rotation of a parcel of fluid around its center



Irrotational motion

$$\overrightarrow{rot}\,\overrightarrow{U} = 0$$

Laplacian

Scalar field $U_x(x, y, z)$
(ex : one component of the velocity field)

$$\Delta U_x = \frac{\partial^2 U_x}{\partial x^2} + \frac{\partial^2 U_x}{\partial y^2} + \frac{\partial^2 U_x}{\partial z^2}$$

→ Laplacien of scalar = scalar

Laplacian

Vector field $\vec{U}(x,y,z)$
(ex : velocity)

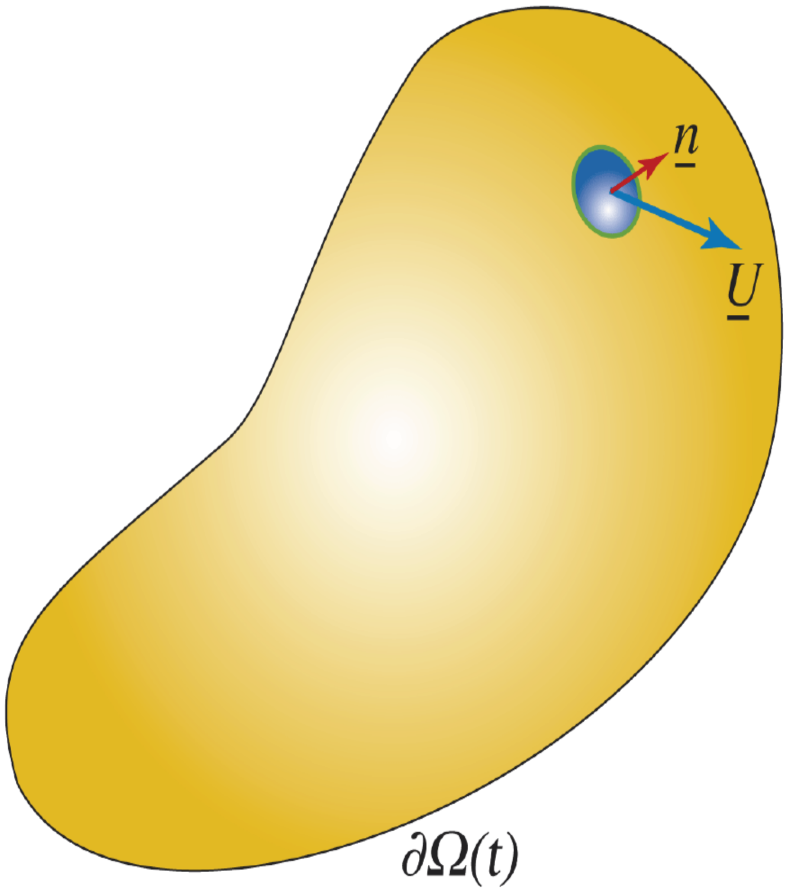
$$\Delta\vec{U} = \begin{pmatrix} \Delta U_x \\ \Delta U_y \\ \Delta U_z \end{pmatrix}$$

→ Laplacian of vector = vector

Navier-Stokes equations

Transport theorem

Material Volume



Transport theorem

$$\frac{d}{dt}\int_{\Omega(t)} b(\underline{x},t)\, d\Omega(t) \qquad ?$$

Transport theorem

$$\frac{d}{dt}\int_{\Omega(t)} b(\underline{x},t)\, d\Omega(t) = \int_{\Omega(t)} \left(\frac{db}{dt} d\Omega(t) + b\, \dot{\widehat{d\Omega(t)}} \right)$$

Transport theorem

$$\frac{d}{dt}\int_{\Omega(t)} b\left(\underline{x},t\right) d\Omega\left(t\right) = \int_{\Omega(t)} \left(\frac{db}{dt} d\Omega\left(t\right) + b\,\dot{\widehat{d\Omega\left(t\right)}}\right)$$

$$\dot{\widehat{d\Omega\left(t\right)}} = d\Omega\left(t\right) \operatorname{div} \underline{U}$$

Transport theorem

$$\frac{d}{dt}\int_{\Omega(t)} b(\underline{x},t)\,d\Omega(t) = \int_{\Omega(t)} \left(\frac{db}{dt}\,d\Omega(t) + b\,\dot{\widehat{d\Omega(t)}}\right)$$

$$\dot{\widehat{d\Omega(t)}} = d\Omega(t)\,\mathrm{div}\,\underline{U}$$

$$\frac{d}{dt}\int_{\Omega(t)} b(\underline{x},t)\,d\Omega(t) = \int_{\Omega(t)} \left(\frac{db}{dt} + b\,\mathrm{div}\,\underline{U}\right) d\Omega(t)$$

Material derivative

$$B\left(\underline{X},t\right)=b\left[\underline{\phi}\left(\underline{X},t\right),t\right]$$

$$\dot{\mathcal{B}}=\frac{\partial B}{\partial t}=\frac{\partial b}{\partial t}+\text{grad}\,b.\frac{\partial \underline{\phi}}{\partial t}$$

$$\dot{\mathcal{B}}\equiv\frac{db}{dt}=\frac{\partial b}{\partial t}+\text{grad}\,b.\underline{U}$$

Material derivative	Local derivative	Convective derivative

Transport theorem

$$\frac{d}{dt}\int_{\Omega(t)} b(\underline{x},t)\, d\Omega(t) = \int_{\Omega(t)} \left(\frac{db}{dt} d\Omega(t) + b\, \dot{\widehat{d\Omega(t)}} \right)$$

$$\dot{\widehat{d\Omega(t)}} = d\Omega(t)\,\mathrm{div}\,\underline{U}$$

$$\frac{d}{dt}\int_{\Omega(t)} b(\underline{x},t)\, d\Omega(t) = \int_{\Omega(t)} \left(\frac{db}{dt} + b\,\mathrm{div}\,\underline{U} \right) d\Omega(t)$$

$$= \int_{\Omega(t)} \left(\frac{\partial b}{\partial t} + \mathrm{div}\left(b \quad \underline{U}\right) \right) d\Omega(t)$$

volumetric form

Transport theorem

$$\frac{d}{dt}\int_{\Omega(t)} b(\underline{x},t)\, d\Omega(t) = \int_{\Omega(t)} \left(\frac{db}{dt} d\Omega(t) + b\, \dot{\overbrace{d\Omega(t)}} \right)$$

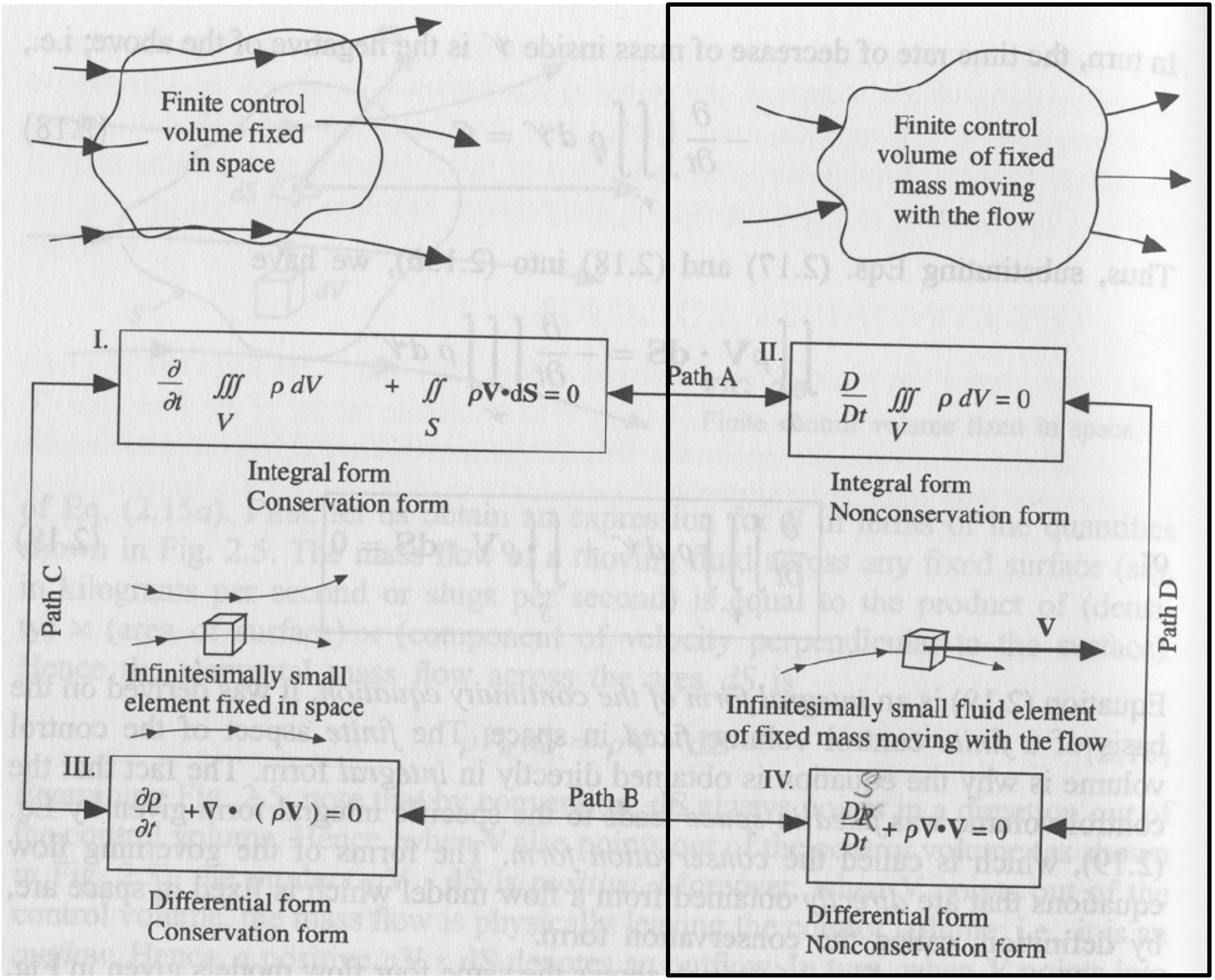
$$\dot{\overbrace{d\Omega(t)}} = d\Omega(t)\,\mathrm{div}\,\underline{U}$$

$$\frac{d}{dt}\int_{\Omega(t)} b(\underline{x},t)\, d\Omega(t) = \int_{\Omega(t)} \left(\frac{db}{dt} + b\,\mathrm{div}\,\underline{U} \right) d\Omega(t)$$

$$= \int_{\Omega(t)} \left(\frac{\partial b}{\partial t} + \mathrm{div}\left(b \quad \underline{U}\right) \right) d\Omega(t)$$

$$\frac{d}{dt}\int_{\Omega(t)} b(\underline{x},t)\, d\Omega(t) = \int_{\Omega(t)} \frac{\partial b}{\partial t} d\Omega(t) + \int_{\partial\Omega(t)} b\left(\underline{U}.\underline{n}\right) da(t)$$

Surface flux expression


Finite control volume fixed in space
Finite control volume of fixed mass moving with the flow
I.
$\frac{\partial}{\partial t} \iiint_V \rho \, dV + \iint_S \rho \mathbf{V} \cdot d\mathbf{S} = 0$
Integral form
Conservation form
Path A
II.
$\frac{D}{Dt} \iiint_V \rho \, dV = 0$
Integral form
Nonconservation form
Path C
Path D
Infinitesimally small element fixed in space
V
Infinitesimally small fluid element of fixed mass moving with the flow
III.
$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0$
Differential form
Conservation form
Path B
IV.
$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{V} = 0$
Differential form
Nonconservation form

Fundamental laws

Balance	$b(\underline{x},t)$
Mass	ρ
Momentum	$\rho\,\underline{U}$
Angular Momentum	$\rho\;\underline{OM}\wedge\underline{U}$
Energy	$\rho\;e + U^2/2$

Mass conservation of a fluid element

$$\frac{d\,M(t)}{dt} = \int_{\omega(t)} \left[\frac{D\rho}{Dt} + \rho\,\mathrm{div}\,\boldsymbol{v} \right] dV$$

$$= \int_{\omega(t)} \left[\frac{\partial\rho}{\partial t} + \mathrm{div}\,(\rho\,\boldsymbol{v}) \right] dV$$

$$\boxed{\frac{D\rho}{Dt} = -\,\rho\,\mathrm{div}\,\boldsymbol{v}}$$

Continuity equation

$$\boxed{\frac{D\rho}{Dt} = -\rho\, \mathrm{div}\, \boldsymbol{v}}$$

Incompressible flow

$$\boxed{\mathrm{div}\, \boldsymbol{v} = 0}$$

The density is constant on a trajectory

Fundamental laws
Local forms

Conservative form

$$\frac{\partial}{\partial t}(\rho\, b) + \mathrm{div}(\rho\, b\;\underline{U}) = Q + \mathrm{div}\, A$$

Volume sources

Surface fluxes

Fundamental laws
Local forms

Conservative form

$$\frac{\partial}{\partial t}(\rho\, b) + \mathrm{div}(\rho\, b\, \underline{U}) = Q + \mathrm{div}\, A$$

Continuity equation

Non conservative form

$$\rho \frac{db}{dt} = Q + \mathrm{div}\, A$$

Conservation of momentum

Newton's second law

$$m\vec{a} = \sum \vec{F}$$

Force balance :

- pressure forces
- viscous forces
- volumetric forces f

$$\frac{d}{dt}\int_{\omega(t)} \rho\, \boldsymbol{v}\, dV = \int_{\partial\omega(t)} \boldsymbol{t}\, dS + \int_{\omega(t)} \rho \boldsymbol{f}\, dV$$

What type of stresses?

- Volumetric stresses, associated to a volumetric force distribution
  → gravity, electro-magnetic force (conducting fluid)…

- Surface stresses, applying at the surface of a continuum parcel
  → friction, pressure, surface tension,…

Momentum conservation

$$\frac{d}{dt}\int_{\omega(t)} \rho\, \boldsymbol{v}\, dV = \boxed{\int_{\partial\omega(t)} \boldsymbol{t}\, dS} + \boxed{\int_{\omega(t)} \rho \boldsymbol{f}\, dV}$$

Volumetric forces

A theorem due to Cauchy, using small tetrahedra of arbitrary orientation, shows that the surface force is linear with the normal to the surface and allows us to represent the cohesion forces by a stress tensor

$$\boldsymbol{t} = \boldsymbol{\sigma} \boldsymbol{n}$$

Normal

Surface forces

Cauchy stress tensor

Stress modeling

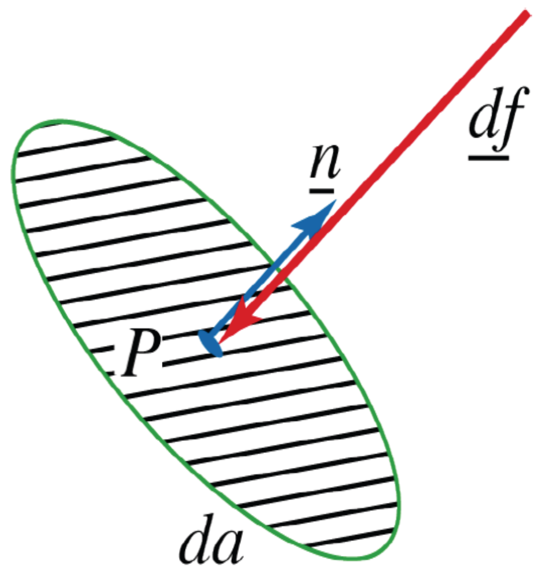
Hydrostatics

$\boxed{\text{Pressure}}$

Stresses in a fluid at rest : pressure

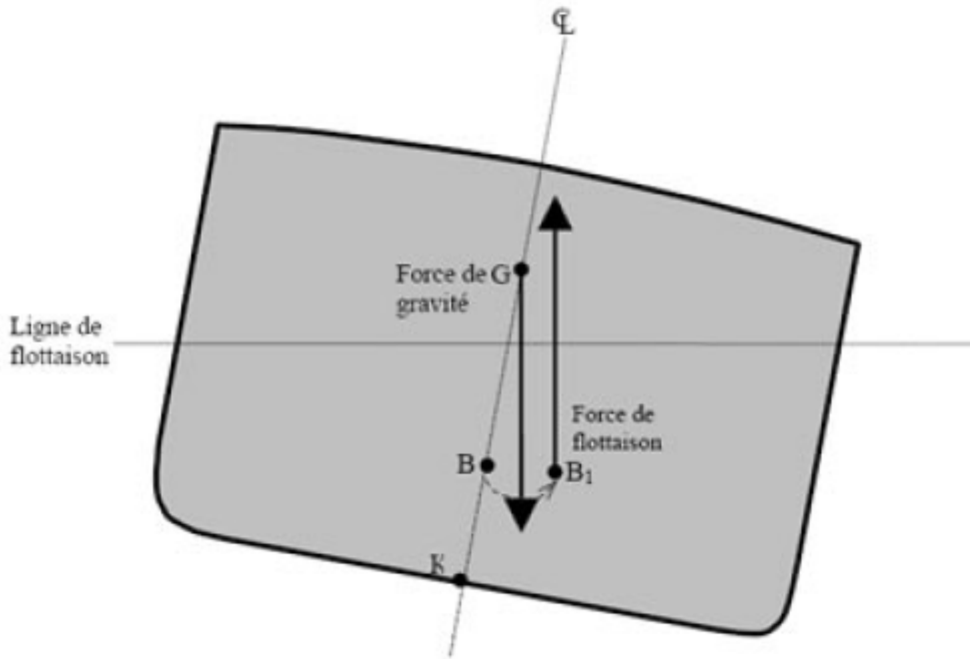
A fluid at rest is subjected to isotropic normal forces!

$$\underline{\underline{\sigma}} = -p(\underline{x})\underline{\underline{1}}$$

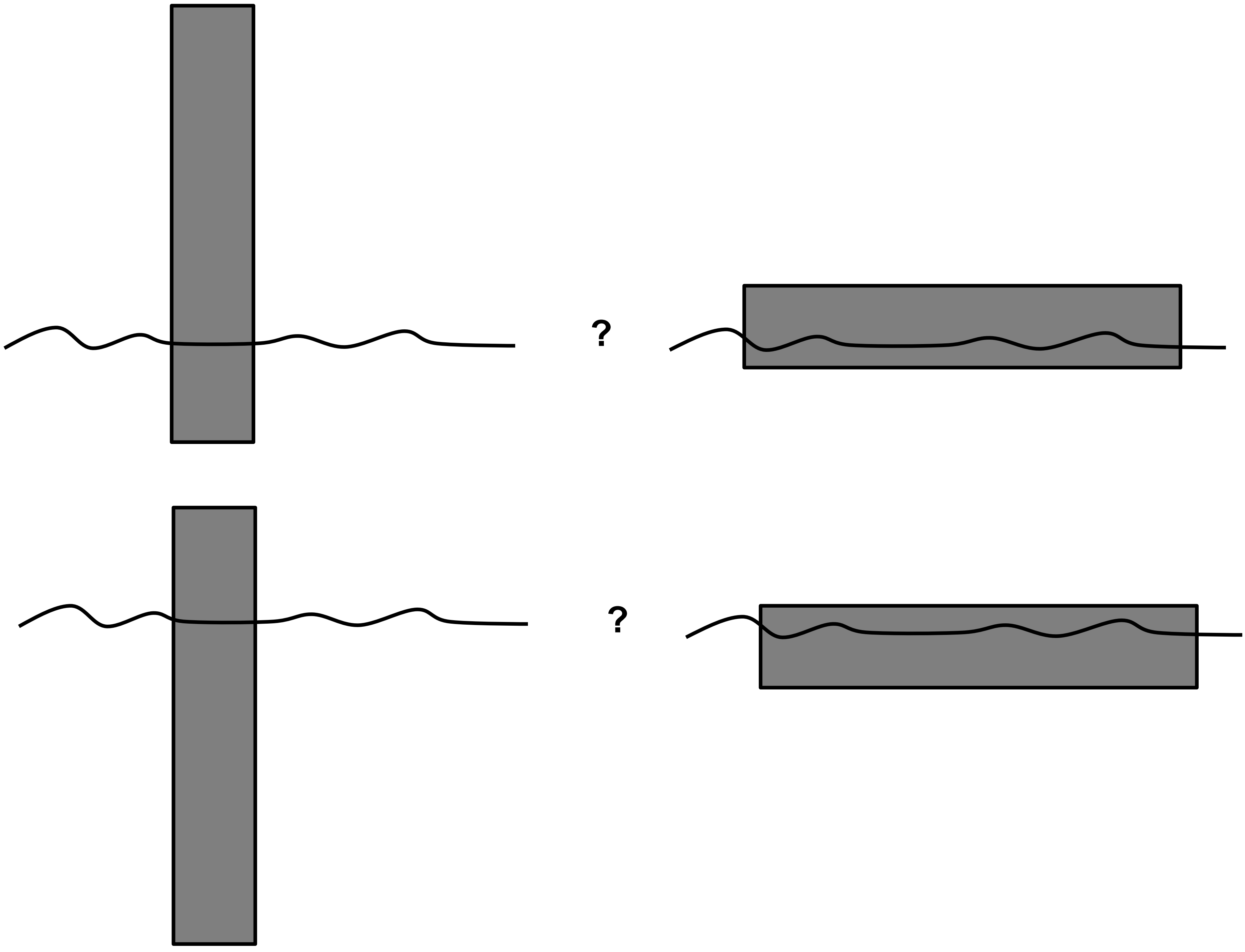


$$\underline{df} = \underline{T}(\underline{x},\underline{n}(\underline{x}))\,da = -p(\underline{x})\,\underline{n}(\underline{x})\,da$$

Archimedes law



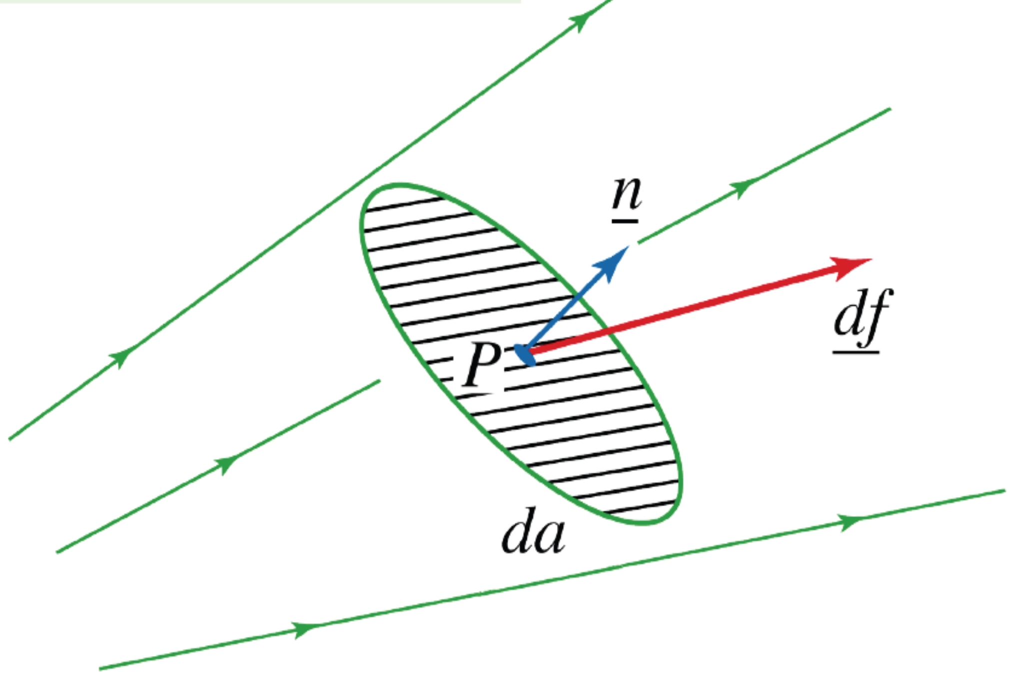
CL
Force de G
gravité
Ligne de
flottaison
Force de
flottaison
B
B1
K



Stresses in a moving fluid

In addition to the normal isotropic pressure force, the fluid element feels both normal and tangential viscous forces

$$\underline{\underline{\sigma}}(\underline{x}) = -p(\underline{x})\underline{\underline{1}} + \underline{\underline{\tau}}(\underline{x})$$



$$\underline{df} = \underline{T}(\underline{x},\underline{n}(\underline{x}))\,da = -p(\underline{x})\,\underline{n}(\underline{x})\,da + \underline{\underline{\tau}}(\underline{x})\cdot\underline{n}(\underline{x})\,da$$

$p(\underline{x})$: pression

$\underline{\underline{\tau}}(\underline{x})$: tenseur des contraintes d'origine visqueuse

Viscous stress tensor

Newtonian fluid model

$$\boxed{\overrightarrow{df} = \left(-p\vec{n} + \vec{\vec{\tau}}.\vec{n} \right) dS}$$

$$\begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix} . \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix} = \begin{pmatrix} \tau_{xx} n_x + \tau_{xy} n_y + \tau_{xz} n_z \\ \tau_{yx} n_x + \tau_{yy} n_y + \tau_{yz} n_z \\ \tau_{zx} n_x + \tau_{zy} n_y + \tau_{zz} n_z \end{pmatrix}$$

Stress in a moving fluid: viscous stress tensor

Newtonian fluid model :

- The stresses do not apply in preferential direction
- The intensity of the stress is a linear function of the velocity gradient.

Most usual fluids (water, air, quicksilver…) are newtonian fluids.

Viscous stress tensor

Newtonian fluid model

$$\begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix}$$

$$\tau_{xx} = -\frac{2}{3}\mu\left(\frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y} + \frac{\partial U_z}{\partial z}\right) + 2\boxed{\mu}\frac{\partial U_x}{\partial x}$$

$$\tau_{yy} = -\frac{2}{3}\mu\left(\frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y} + \frac{\partial U_z}{\partial z}\right) + 2\mu\frac{\partial U_y}{\partial y}$$

$$\tau_{zz} = -\frac{2}{3}\mu\left(\frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y} + \frac{\partial U_z}{\partial z}\right) + 2\mu\frac{\partial U_z}{\partial z}$$

μ: dynamic viscosity (kg/m/s)

$\nu=\mu/\rho$:
Kinematic viscosity
(m^2/s)

Viscous stress tensor

$$\tau_{xx} = -\frac{2}{3}\mu\left(\frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y} + \frac{\partial U_z}{\partial z}\right) + 2\mu\frac{\partial U_x}{\partial x}$$

$$\tau_{yy} = -\frac{2}{3}\mu\left(\frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y} + \frac{\partial U_z}{\partial z}\right) + 2\mu\frac{\partial U_y}{\partial y}$$

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$$\begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix}$$

Divergence of velocity field,
i.e. volumnetric dilatation

Viscous stress tensor

$$\begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix}$$

$$\tau_{xx} = -\frac{2}{3}\mu\left(\frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y} + \frac{\partial U_z}{\partial z}\right) + 2\mu\frac{\partial U_x}{\partial x}$$

$$\tau_{yy} = -\frac{2}{3}\mu\left(\frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y} + \frac{\partial U_z}{\partial z}\right) + 2\mu\frac{\partial U_y}{\partial y}$$

$$\tau_{zz} = -\frac{2}{3}\mu\left(\frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y} + \frac{\partial U_z}{\partial z}\right) + 2\mu\frac{\partial U_z}{\partial z}$$

→ =0 if flow is incompressible

Viscous stress tensor

$$\begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix}$$

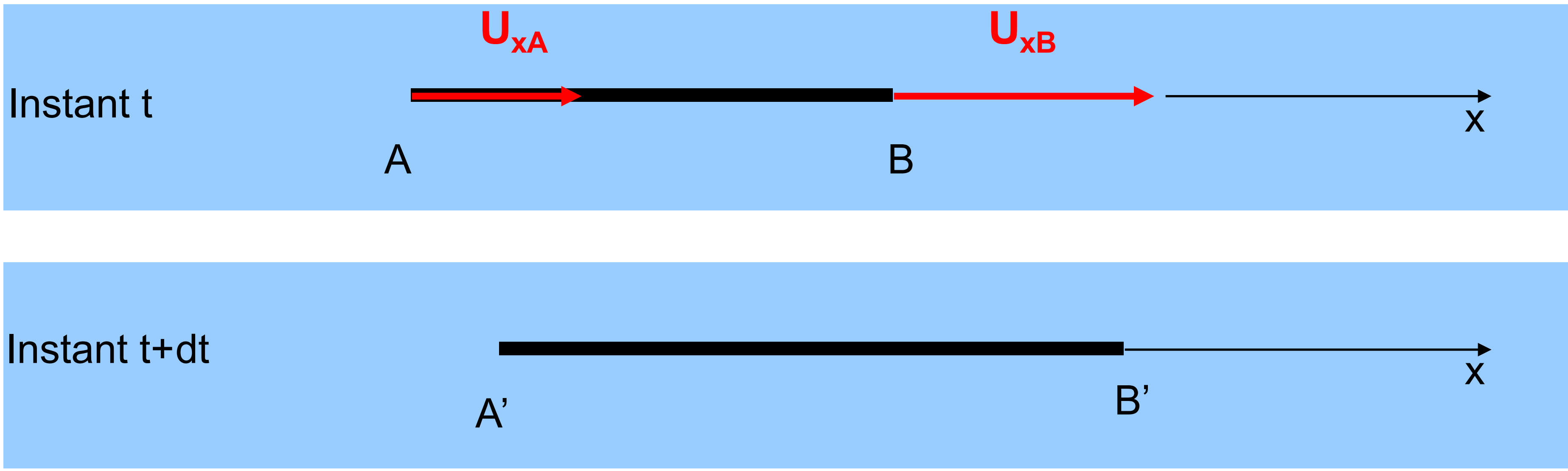
$$\tau_{xx} = 2\mu \frac{\partial U_x}{\partial x}$$

$$\tau_{yy} = 2\mu \frac{\partial U_y}{\partial y}$$

$$\tau_{zz} = 2\mu \frac{\partial U_z}{\partial z}$$

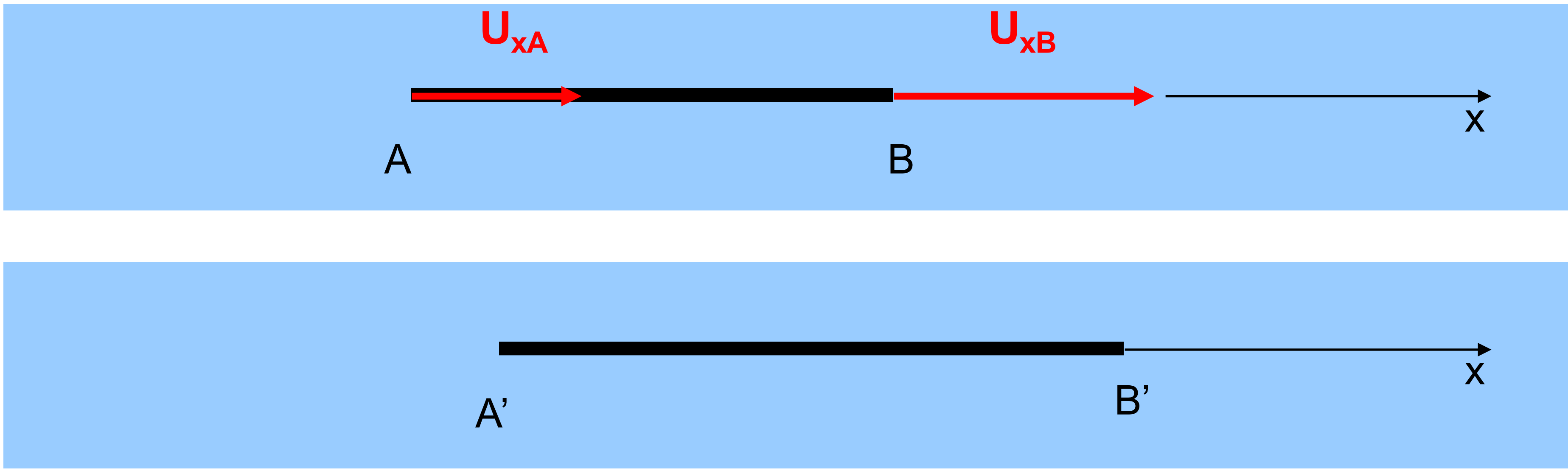
Stretching terms

Stresses : stretching



AB is a line of fluid particles in a flow such that $U_{xA}<U_{xB}$. Since the velocity is higher in B than in A, the segment AB is stretched in the x direction. This **deformation** is called a **stretching**. The relevant quantity is the derivative of the velocity with respect to the direction tangential to this velocity

Stresses : stretching



$$A'B'-AB=(U_{xB}-U_{xA})dt=\boxed{\frac{\partial U_x}{\partial x}}dxdt$$

Viscous stress tensor

$$\begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix}$$

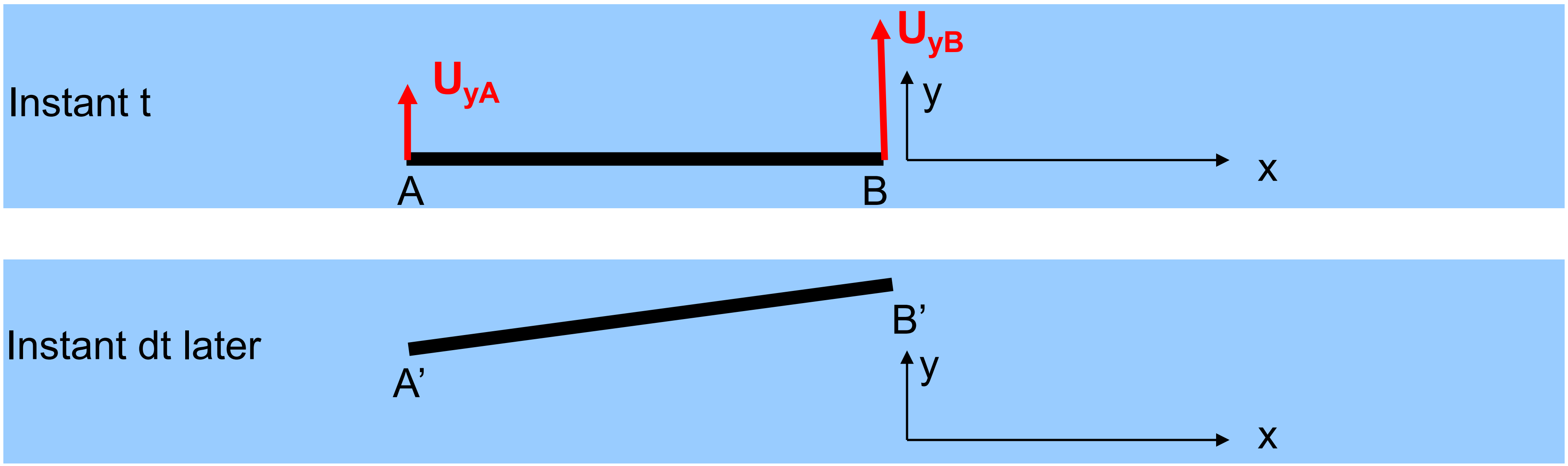
$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial U_x}{\partial y} + \frac{\partial U_y}{\partial x}\right)$$

$$\tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial U_z}{\partial x} + \frac{\partial U_x}{\partial z}\right)$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial U_y}{\partial z} + \frac{\partial U_z}{\partial y}\right)$$

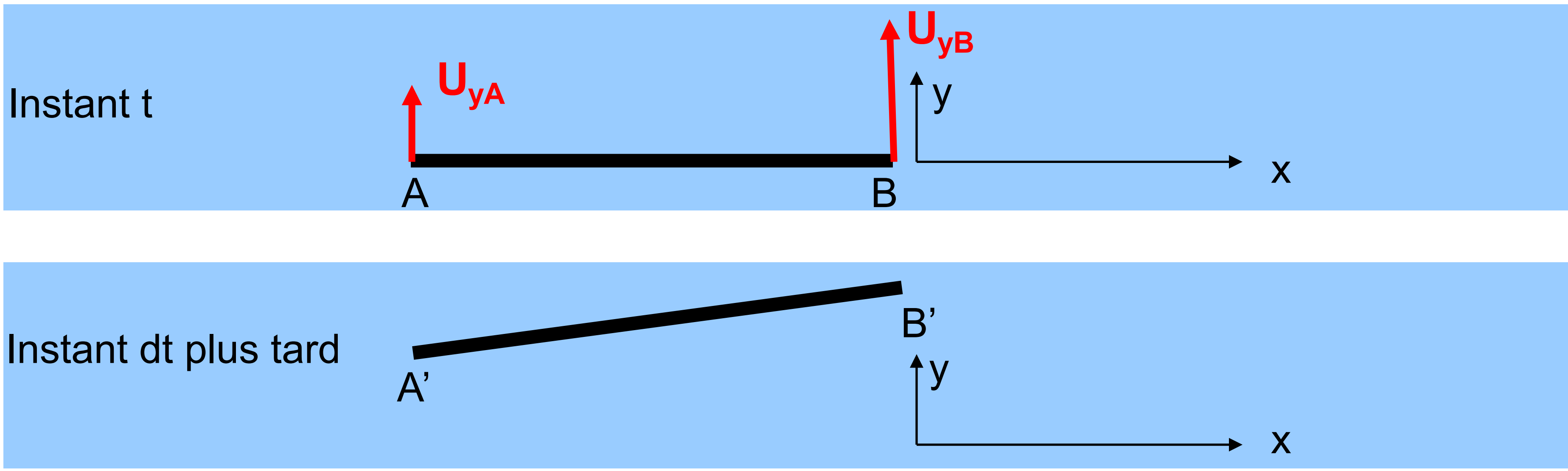
Shear terms

Stresses : shear



AB is a line of fluid particles in a flow such that $U_{yA}<U_{yB}$. Since the velocity is higher in B than in A, the segment AB will rotate. This **deformation** is called **shear**. The relevant quantity is the derivative of the velocity with respect to the direction **normal** to this velocity

Stresses : shear



$$\boxed{\frac{\partial U_y}{\partial x}}$$

The equations of fluid mechanics

Momentum conservation for incompresible fluid

Newton's second law

$$\frac{d}{dt}\int_{\omega(t)} \rho\, \boldsymbol{v}\, dV = \int_{\partial\omega(t)} \boldsymbol{t}\, dS + \int_{\omega(t)} \rho \boldsymbol{f}\, dV$$

Transport theorem

$$\frac{d}{dt}\int_{\omega(t)} \rho\, \boldsymbol{v}\, dV = \int_{\omega(t)} \rho \frac{D\boldsymbol{v}}{Dt}\, dV$$

Green's identity

$$\int_{\omega(t)} \left[\rho \frac{D\boldsymbol{v}}{Dt} - \boldsymbol{div}\, \boldsymbol{\sigma} - \rho \boldsymbol{f}\right] dV = 0$$

$$\boxed{\rho \frac{D\boldsymbol{v}}{Dt} = \boldsymbol{div}\, \boldsymbol{\sigma} + \rho \boldsymbol{f}}$$

Newtonian incompressible fluid

$$\boxed{\rho \frac{D\boldsymbol{v}}{Dt} = -\boldsymbol{\nabla} p + \mu \Delta \boldsymbol{v} + \rho \boldsymbol{f}}$$

$$\boxed{\operatorname{div} \boldsymbol{v} = 0}$$

Navier-Stokes equations

$$\left(m\vec{a} = \sum \vec{F}\right).\vec{e_x}$$

$$\rho \boxed{\frac{dU_x}{dt}} = \rho f_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 U_x}{\partial x^2} + \frac{\partial^2 U_x}{\partial y^2} + \frac{\partial^2 U_x}{\partial z^2}\right)$$

acceleration
(total derivative)

Navier-Stokes equations

$$\left(m\vec{a} = \sum \vec{F}\right)\cdot\vec{e_x}$$

$$\rho \frac{dU_x}{dt} = \boxed{\rho f_x} - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 U_x}{\partial x^2} + \frac{\partial^2 U_x}{\partial y^2} + \frac{\partial^2 U_x}{\partial z^2}\right)$$

Volumnar forces

Navier-Stokes equations

$$\left(m\vec{a} = \sum \vec{F}\right).\vec{e_x}$$

$$\rho \frac{dU_x}{dt} = \rho f_x - \boxed{\frac{\partial p}{\partial x}} + \mu\left(\frac{\partial^2 U_x}{\partial x^2} + \frac{\partial^2 U_x}{\partial y^2} + \frac{\partial^2 U_x}{\partial z^2}\right)$$

Pressure gradient

Navier-Stokes equations

$$\left(m\vec{a} = \sum \vec{F}\right).\vec{e_x}$$

$$\rho \frac{dU_x}{dt} = \rho f_x - \frac{\partial p}{\partial x} + \boxed{\mu\left(\frac{\partial^2 U_x}{\partial x^2} + \frac{\partial^2 U_x}{\partial y^2} + \frac{\partial^2 U_x}{\partial z^2}\right)}$$

Viscous stress

Navier-Stokes equations

$$\rho\left(\frac{\partial U_x}{\partial t}+U_x\frac{\partial U_x}{\partial x}+U_y\frac{\partial U_x}{\partial y}+U_z\frac{\partial U_x}{\partial z}\right)=\rho f_x-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 U_x}{\partial x^2}+\frac{\partial^2 U_x}{\partial y^2}+\frac{\partial^2 U_x}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial U_y}{\partial t}+U_x\frac{\partial U_y}{\partial x}+U_y\frac{\partial U_y}{\partial y}+U_z\frac{\partial U_y}{\partial z}\right)=\rho f_y-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 U_y}{\partial x^2}+\frac{\partial^2 U_y}{\partial y^2}+\frac{\partial^2 U_y}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial U_z}{\partial t}+U_x\frac{\partial U_z}{\partial x}+U_y\frac{\partial U_z}{\partial y}+U_z\frac{\partial U_z}{\partial z}\right)=\rho f_z-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 U_z}{\partial x^2}+\frac{\partial^2 U_z}{\partial y^2}+\frac{\partial^2 U_z}{\partial z^2}\right)$$

Newtonian incompressible fluid

$$\boxed{\rho \frac{D\boldsymbol{v}}{Dt} = -\boldsymbol{\nabla} p + \mu \, \Delta \boldsymbol{v} + \rho \boldsymbol{f}}$$

$$\boxed{\operatorname{div} \boldsymbol{v} = 0}$$

Boundary conditions

1. Liquid/solid interfaces

   the solid is viewed as undeformable

   NO SLIP BOUNDARY CONDITION

   $\mathbf{U}_{liq}=\mathbf{U}_{sol}$

   sometimes relaxed to

   NON PENETRATION (FREE SLIP)

   $\mathbf{U}_{liq}.n=\mathbf{U}_{sol}.n$

2. Liquid/liquid interfaces

   $\mathbf{U}_1=\mathbf{U}_2$ Continuity of velocity

   $\mathbf{U}_1.\mathbf{n}= \mathbf{U}_2.\mathbf{n}=\mathbf{V}_{interface}$ Non penetration (free slip)

   $\boldsymbol{\sigma}_1.\mathbf{n}= \boldsymbol{\sigma}_2.\mathbf{n}$ Continuity of stress

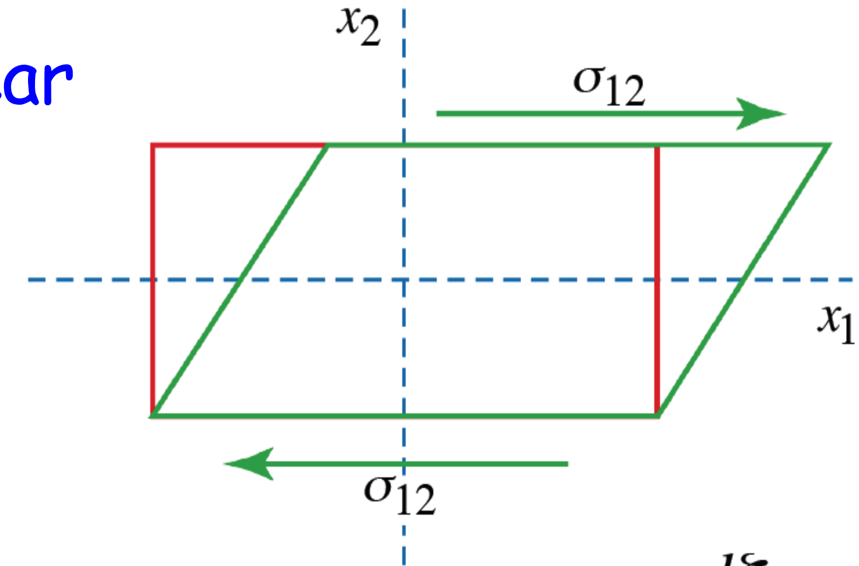
Boundary conditions

3. Free surface (liquid/gas interface)

$\mathbf{U}.\mathbf{n}=\mathbf{V}_{surface}$	Non penetration (free slip)
$\boldsymbol{\sigma}.\mathbf{n}=\mathbf{0}$	No stress

Solid or fluid?

Pure shear



Elastic solid $$\sigma_{12} = \mu_s \frac{d\xi_1}{dx_2} = \rho\, c_t^2 \frac{d\xi_1}{dx_2}$$

Newtonian fluid $$\sigma_{12} = \mu \frac{dU_1}{dx_2}$$

Characteristic time $$\mu_s = \frac{\mu}{\tau} \quad \Rightarrow \quad \boxed{\tau = \frac{\mu}{\mu_s} = \frac{\mu}{\rho\, c_t^2}}$$

Viscoelastic medium

$$\frac{\partial \sigma_{12}}{\partial t} + \frac{1}{\tau}\sigma_{12} = \mu_s \frac{\partial U_1}{\partial x_2}$$

- $T << \tau$ Elastic solid

$$\frac{\partial \sigma_{12}}{\partial t} \sim \mu_s \frac{\partial U_1}{\partial x_2} \Longrightarrow \sigma_{12} \sim \mu_s \frac{\partial \xi_1}{\partial x_2}$$

- $T >> \tau$ Newtonian fluid

$$\frac{1}{\tau}\sigma_{12} \sim \mu_s \frac{\partial U_1}{\partial x_2} \Longrightarrow \sigma_{12} \sim \mu \frac{\partial U_1}{\partial x_2}$$

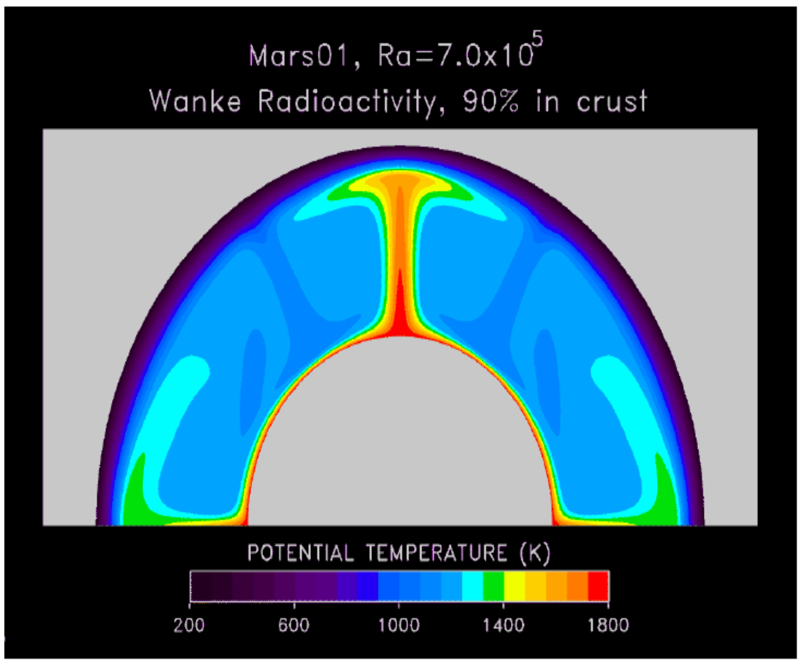
The earth's mantel

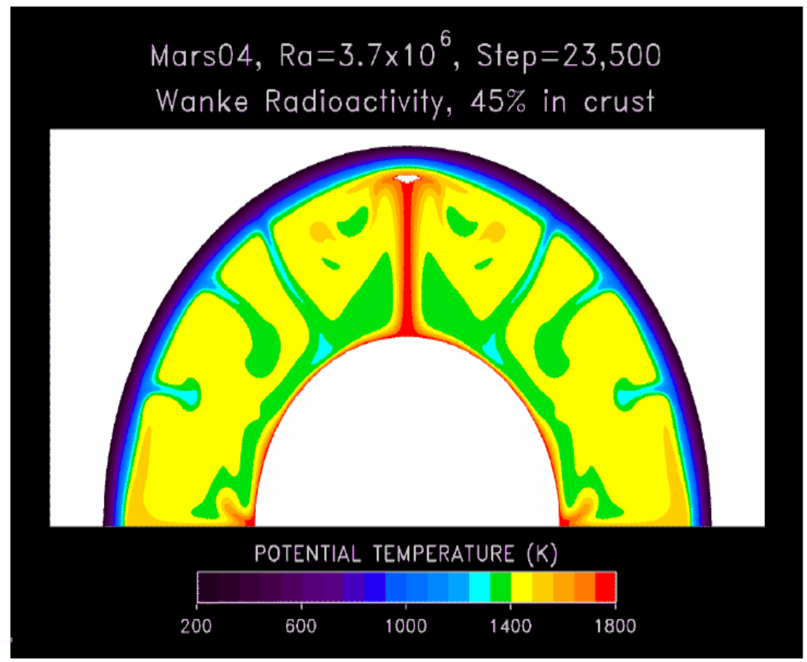
$$\rho = 5\times10^{3}\,\mathrm{kg/m^{3}} \quad c_t = 5\times10^{3}\,\mathrm{m/s} \quad \mu = 10^{22}\,\mathrm{kg/(m\times s)}$$

$$\tau \sim 3000 \text{ ans}$$

$T << 3000$ ans solide (ondes sismiques)

$T >> 3000$ ans liquide (convection)





Kieffer (2001)

Deborah number

$$De = \frac{\tau}{T}$$

$De \leq 1$ **Fluide**

$De \geq 1$ **Solide**

La prophétesse Déborah par Chagall

« Les montagnes ruisselèrent devant le Seigneur »

La prophétesse Déborah dans le livre des Juges