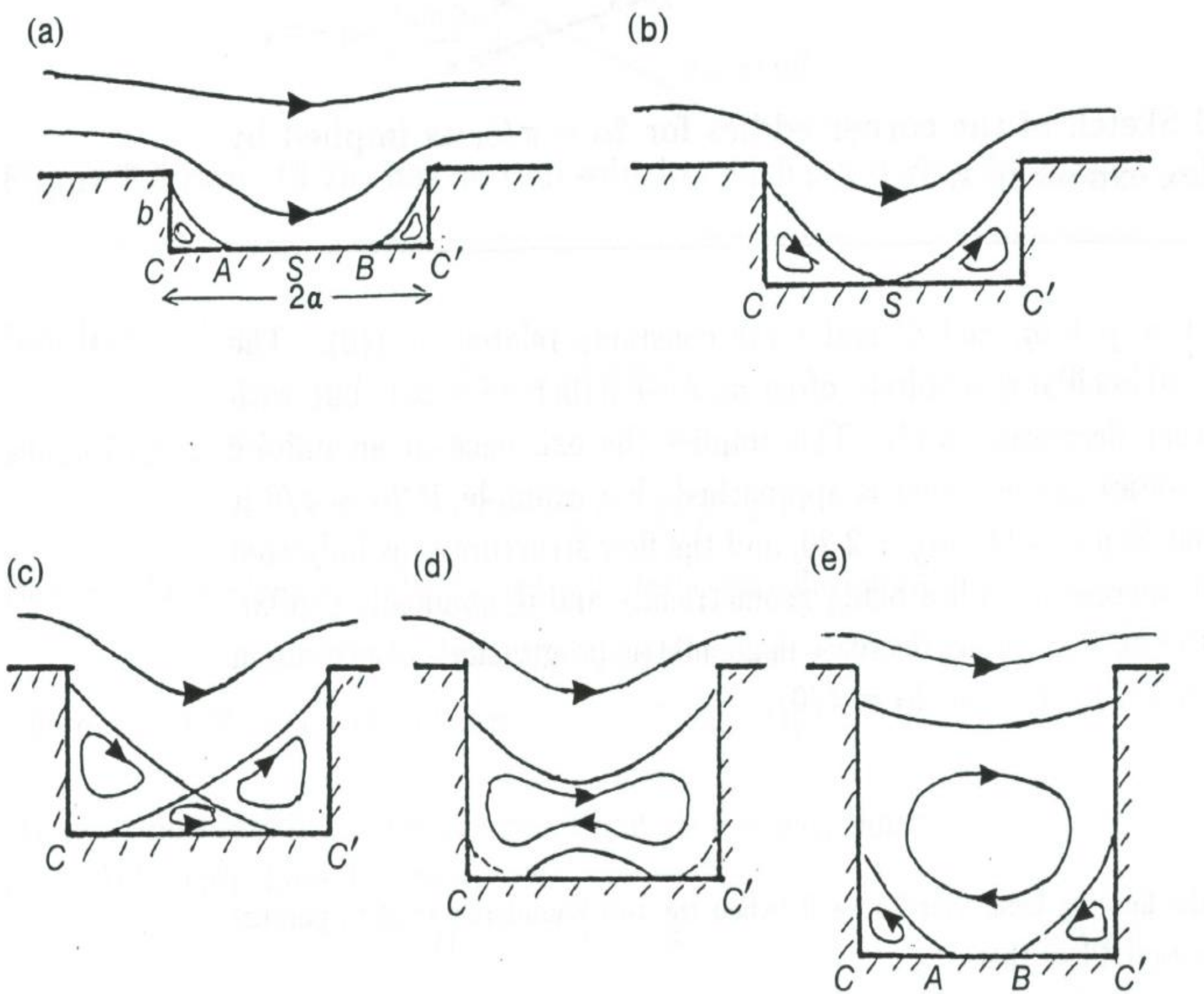
Stokes flow/ Creeping flow



Newtonian incompressible fluid

$$\boxed{\rho \frac{D\boldsymbol{v}}{Dt} = -\boldsymbol{\nabla} p + \mu \, \Delta \boldsymbol{v} + \rho \boldsymbol{f}}$$

$$\boxed{\operatorname{div} \boldsymbol{v} = 0}$$

Stokes flow/ Creeping flow ⇔ Re=0

$$\eta \Delta \mathbf{u} = \nabla p$$

$$\mathrm{div}\mathbf{u} = 0$$

Linear equations

Stokes flow/ Creeping flow

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$$\mathrm{div}\,\mathbf{u} = 0$$

Linear equations

Unicity of solution $\neq$ Navier-Stokes

Stokes flow/ Creeping flow

$$\eta \Delta \mathbf{u} = \nabla p$$

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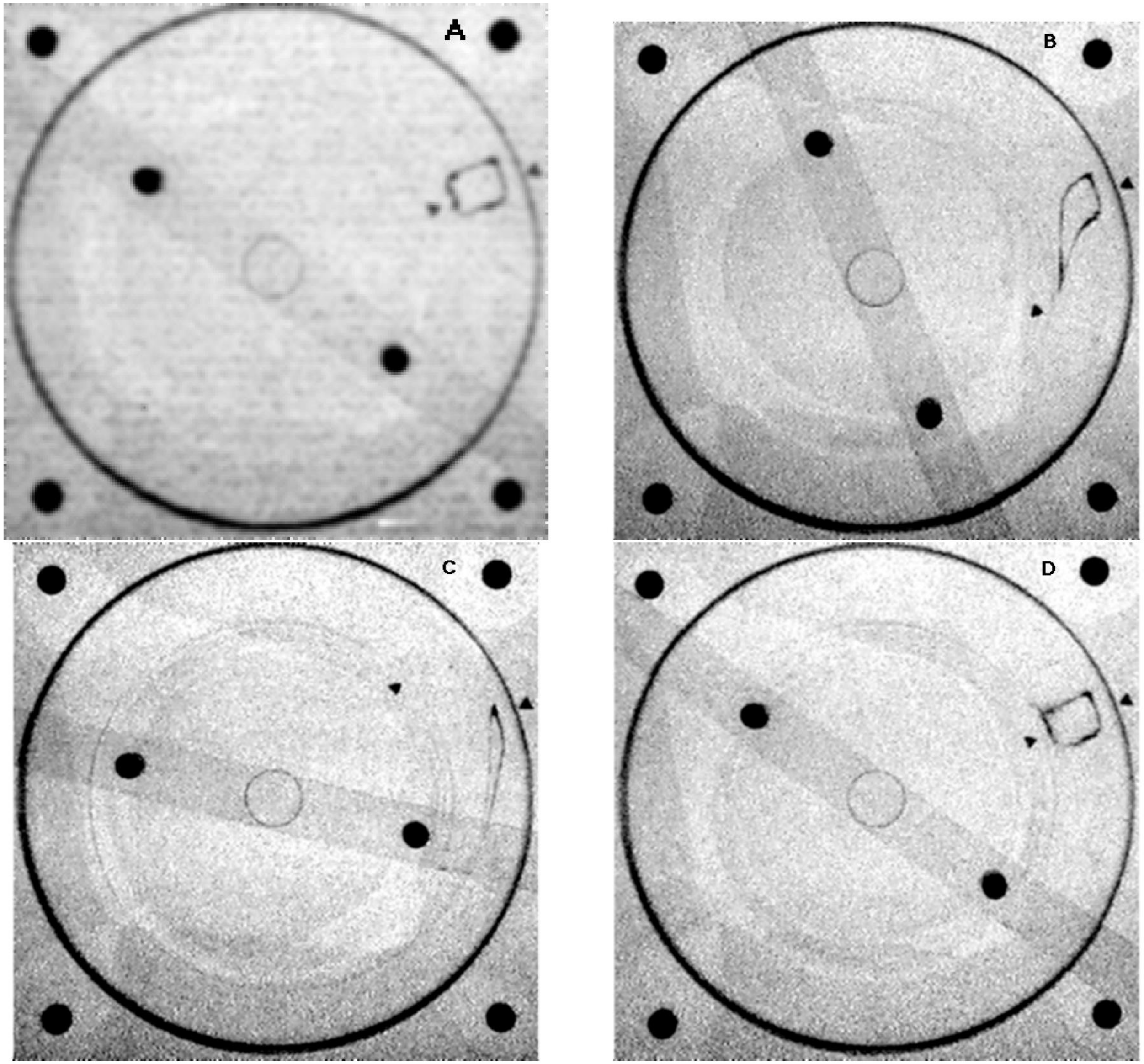
Linear equations

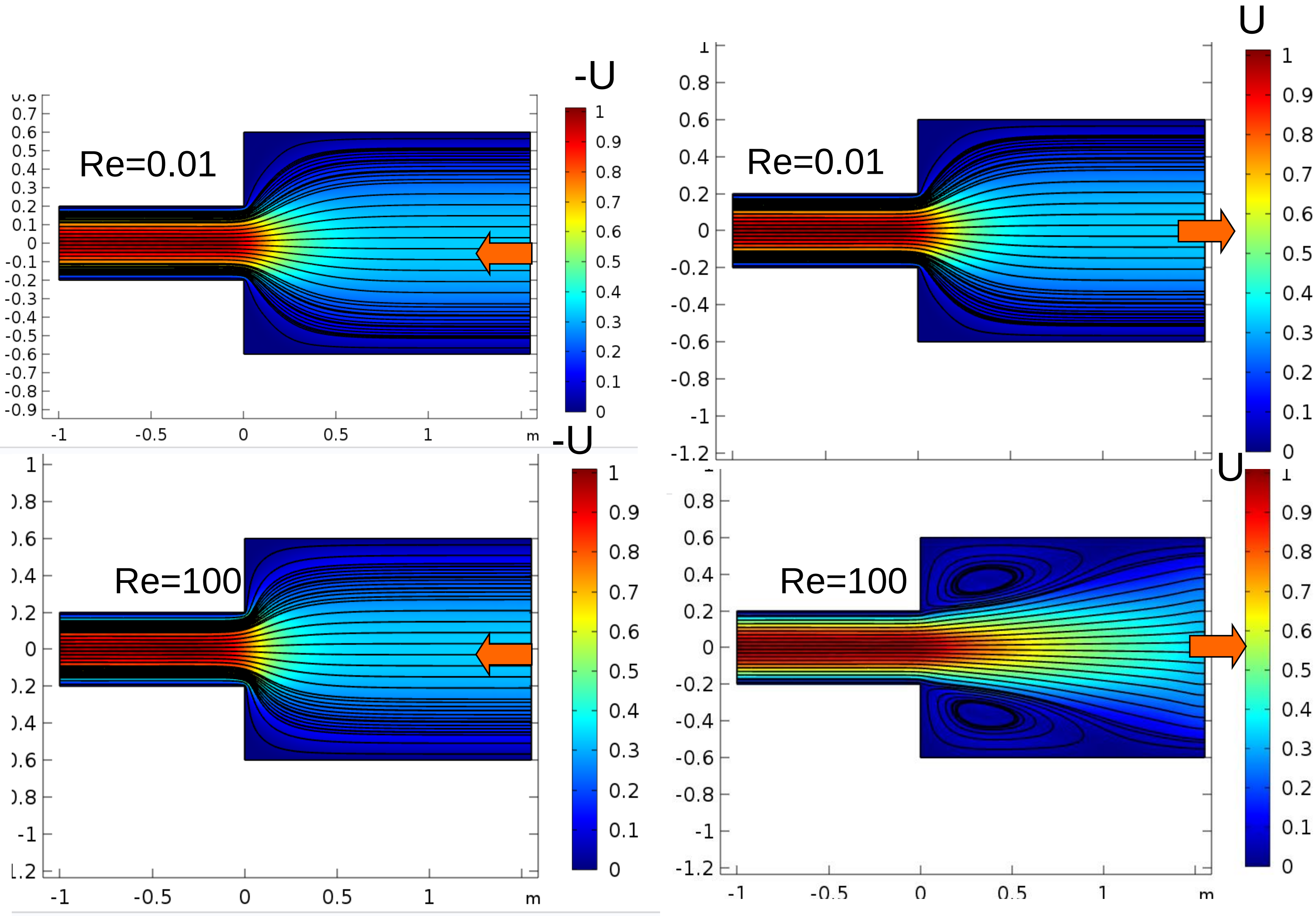
Unicity of solution $\neq$ Navier-Stokes

Superposition principle

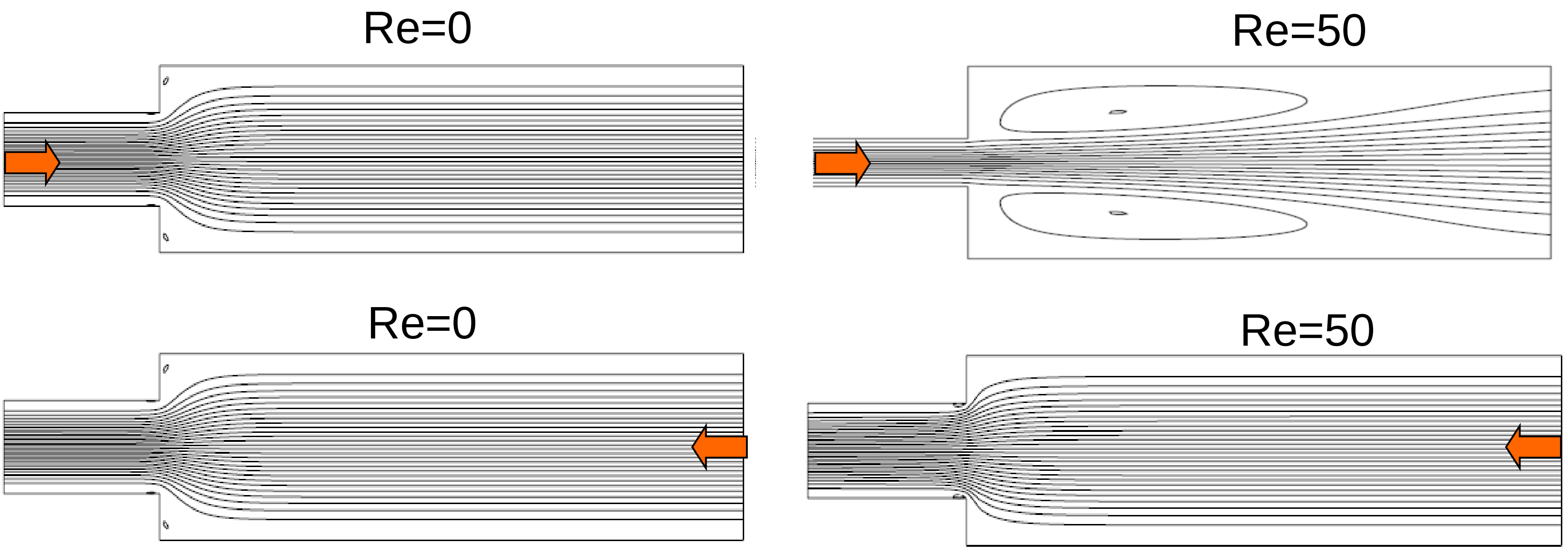
Reversibility



Birthday candle paradox



Reversibility of Stokes equations



Simulations Marc Fermigier

Different formulations

$$\boxed{\frac{\partial}{\partial x_j}\sigma_{ij} = 0, \quad \sigma_{ij} = -p\delta_{ij} + 2\mu e_{ij},}$$

$$\boxed{\Delta p = 0}$$

Different formulations

$$\boxed{\frac{\partial}{\partial x_j}\sigma_{ij} = 0, \quad \sigma_{ij} = -p\delta_{ij} + 2\mu e_{ij},}$$

$$\boxed{\Delta p = 0}$$

$$\boxed{\Delta \omega = 0}$$

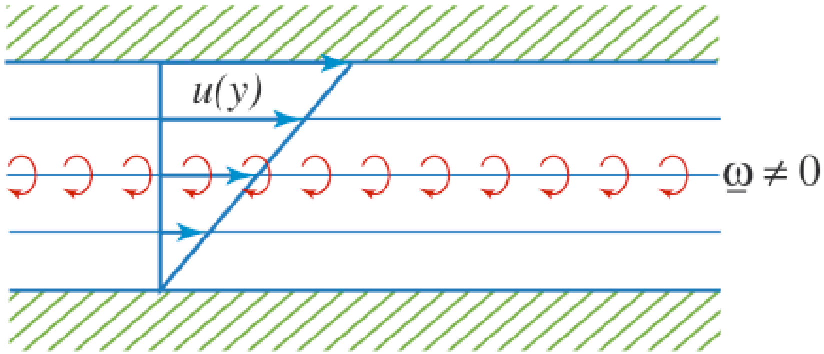
vorticity

Vorticity

$$\boldsymbol{\omega} = \mathbf{rot}(\mathbf{u})$$

Don’t mix up curved streamlines and vorticity!

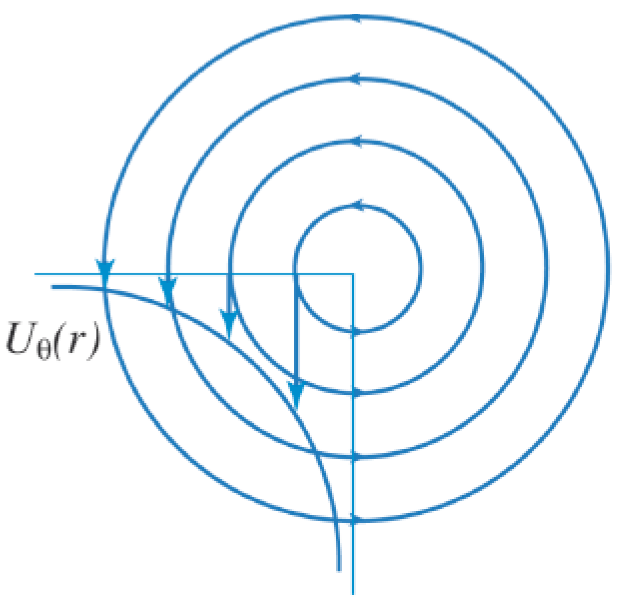
Plane Couette flow



Don’t mix up curved streamlines and vorticity!

Point vortex

$$U_\theta(r) = \frac{\Gamma}{2\pi r}$$

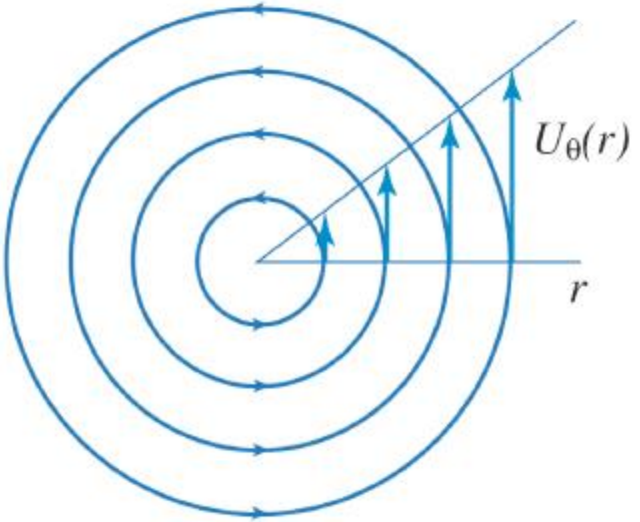


$\underline{\omega} = 0$

Solid body rotation

$$U_\theta(r) = 2\omega r$$

$\underline{\omega} \neq 0$



Vorticity equation on plane

1) $$\frac{\partial}{\partial y}: \quad \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right]$$

2) $$\frac{\partial}{\partial x}: \quad \frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left[\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right]$$

2)-1)= $$\frac{\partial}{\partial t}(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y})+\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+u\frac{\partial^2 v}{\partial x^2}+v\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}$$

$$-u\frac{\partial u}{\partial x\partial y}-\frac{\partial u}{\partial y}\frac{\partial u}{\partial x}-v\frac{\partial^2 u}{\partial y^2}-\frac{\partial v}{\partial y}\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x\partial y}+\frac{1}{\rho}\frac{\partial p}{\partial x\partial y}+$$

$$\nu\left[\frac{\partial^2}{\partial x^2}(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y})+\frac{\partial^2}{\partial y^2}(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y})\right]$$

Vorticity equation on plane

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+u\frac{\partial^2 v}{\partial x^2}-u\frac{\partial^2 u}{\partial x\partial y}+v\frac{\partial^2 v}{\partial x\partial y}-v\frac{\partial^2 u}{\partial y^2}$$

$$=\nu\left[\frac{\partial^2}{\partial x^2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+\frac{\partial^2}{\partial y^2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\right]$$

$$\boxed{\frac{\partial}{\partial t}\omega+u\frac{\partial\,\omega}{\partial x}+v\frac{\partial\,\omega}{\partial y}=\nu\left[\frac{\partial^2\,\omega}{\partial x^2}+\frac{\partial^2\,\omega}{\partial y^2}\right]}$$

Advection-diffusion equation

Vorticity equation: axisymmetric case

1) $$\frac{\partial}{\partial r}: \quad \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial z}+v\frac{\partial u}{\partial r}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\left[\frac{\partial^2 u}{\partial z^2}+\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right]$$

2) $$\frac{\partial}{\partial z}: \quad \frac{\partial v}{\partial t}+u\frac{\partial v}{\partial z}+v\frac{\partial v}{\partial r}=-\frac{1}{\rho}\frac{\partial p}{\partial r}+\nu\left[\frac{\partial^2 v}{\partial z^2}+\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}-\frac{v}{r^2}\right]$$

2)-1)= $$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial z}-\frac{\partial u}{\partial r}\right)+\frac{\partial u}{\partial z}\frac{\partial v}{\partial z}+u\frac{\partial^2 v}{\partial z^2}+v\frac{\partial^2 v}{\partial z\partial r}+\frac{\partial v}{\partial z}\frac{\partial v}{\partial r}+\frac{v}{r}-\frac{v}{r}$$

$$-u\frac{\partial u}{\partial z\partial r}-\frac{\partial u}{\partial r}\frac{\partial u}{\partial z}-v\frac{\partial^2 u}{\partial r^2}-\frac{\partial v}{\partial r}\frac{\partial u}{\partial r}=-\frac{1}{\rho}\frac{\partial p}{\partial z\partial r}+\frac{1}{\rho}\frac{\partial p}{\partial z\partial r}$$

$$\nu\left[\frac{\partial^2}{\partial z^2}\left(\frac{\partial v}{\partial z}-\frac{\partial u}{\partial r}\right)+\frac{\partial^2}{\partial r^2}\left(\frac{\partial v}{\partial z}-\frac{\partial u}{\partial r}\right)+\frac{1}{r}\left(\frac{\partial v}{\partial z}-\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial u}{\partial r}-\frac{1}{r^2}\frac{\partial v}{\partial z}\right]$$

Vorticity equation: axisymmetric case

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial z}-\frac{\partial u}{\partial r}\right)+u\frac{\partial^2 v}{\partial z^2}-u\frac{\partial u}{\partial z\partial r}+v\frac{\partial^2 v}{\partial z\partial r}-v\frac{\partial^2 u}{\partial r^2}$$
$$=\nu\left[\frac{\partial^2}{\partial z^2}\left(\frac{\partial v}{\partial z}-\frac{\partial u}{\partial r}\right)+\frac{\partial^2}{\partial r^2}\left(\frac{\partial v}{\partial z}-\frac{\partial u}{\partial r}\right)+\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial v}{\partial z}-\frac{\partial u}{\partial r}\right)-\frac{1}{r^2}\left(\frac{\partial v}{\partial z}-\frac{\partial u}{\partial r}\right)\right]$$

$$\boxed{\frac{\partial}{\partial t}\omega+u\frac{\partial\omega}{\partial z}+v\frac{\partial\omega}{\partial r}=\nu\left[\frac{\partial^2\omega}{\partial z^2}+\frac{\partial^2\omega}{\partial r^2}+\frac{1}{r}\frac{\partial\omega}{\partial r}-\frac{\omega}{r^2}\right]}$$

Advection-diffusion equation

Different formulations

$$\boxed{\frac{\partial}{\partial x_j}\sigma_{ij} = 0, \quad \sigma_{ij} = -p\delta_{ij} + 2\mu e_{ij},}$$

$$\boxed{\Delta p = 0}$$

$$\boxed{\Delta \omega = 0}$$

vorticity

Incompressible flow
and streamfunction

Streamfunction

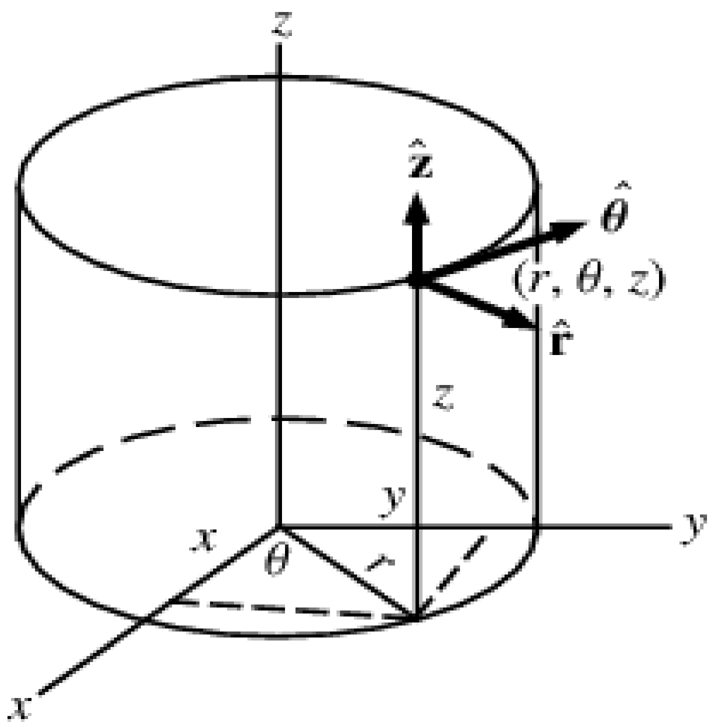
$$u = \frac{\partial \Psi}{\partial y}, v = -\frac{\partial \Psi}{\partial x}$$

Continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial \Psi}{\partial x \partial y} - \frac{\partial \Psi}{\partial x \partial y} = 0$$

Cylindrical coordinate system



In cylindrical coordinates (r , q ,z) with $\partial / \partial\theta = 0$

-axisymmetric case

Streamfunction

$$u = \frac{1}{r}\frac{\partial \Psi}{\partial r}, v = -\frac{1}{r}\frac{\partial \Psi}{\partial z}$$

Continuity equation

$$\frac{\partial (ru)}{\partial z} + \frac{\partial (rv)}{\partial r} = 0$$

$$\frac{\partial (ru)}{\partial z} + \frac{\partial (rv)}{\partial r} = r\frac{\partial u}{\partial z} + r\frac{\partial v}{\partial r} + v =$$

$$r\frac{1}{r}\frac{\partial \Psi}{\partial z \partial r} - r\frac{1}{r}\frac{\partial \Psi}{\partial z \partial r} - \frac{1}{r}\frac{\partial \Psi}{\partial z} + \frac{1}{r}\frac{\partial \Psi}{\partial z} = 0$$

Vorticity

$$\boxed{\frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial x^2} = -\,\omega}$$

Streamfunction formulation

$$\Delta \omega = 0$$

$$\omega = -\nabla^2 \psi \, \mathbf{k}$$

$$\nabla^4 \psi \equiv \nabla^2 (\nabla^2 \psi) = 0$$

Biharmonic equation

Minimal dissipation

The unique solution of the Stokes equations (with suitable boundary conditions) is the divergence free flow field that minimizes the viscous dissipation

$$\Phi = 2\mu \int_V e_{ij} e_{ij} \, \mathrm{d}V,$$

Flow along a sphere

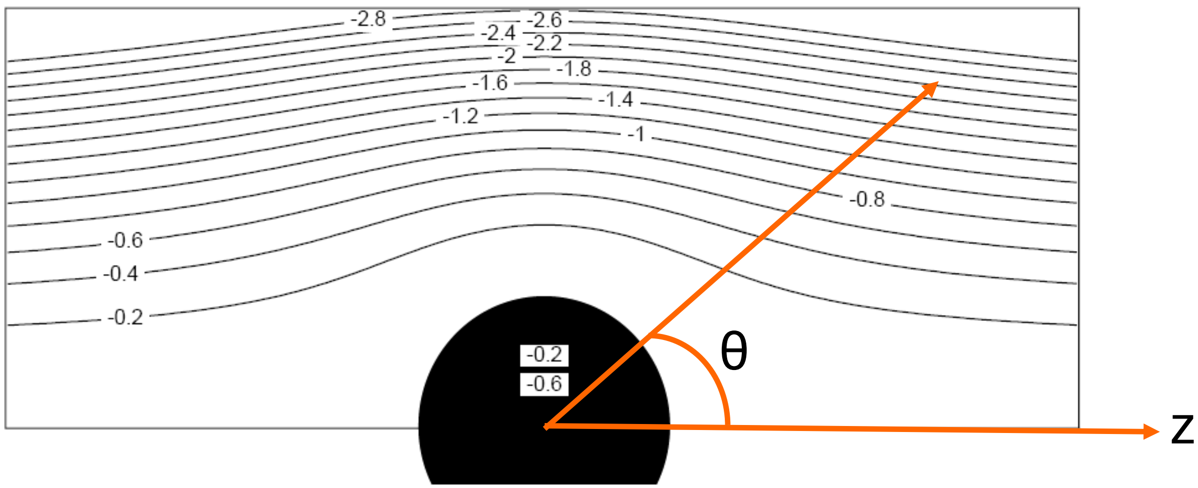
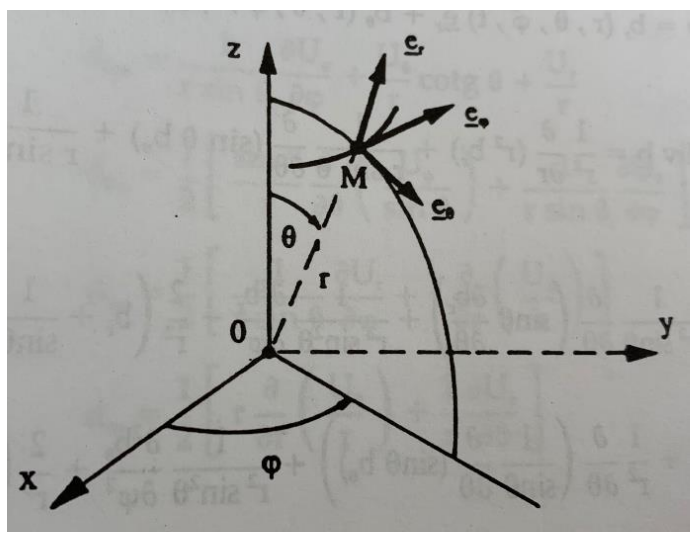


FIG. 6.9 – écoulement à petit Re autour d'une sphère. Lignes de courant dans le repère où la sphère est immobile.



Flow along a sphere
Stokes Formula

Stokes streamfunction in spherical coordinates

$$U_r(r, \theta) = \frac{1}{r^2 \sin\theta} \frac{\partial \psi}{\partial \theta} \quad , \quad U_\theta(r, \theta) = -\frac{1}{r \sin\theta} \frac{\partial \psi}{\partial r} \quad .$$

What should we check?

Flow along a sphere
Stokes Formula

Stokes streamfunction in spherical coordinates

$$\boxed{U_r(r,\theta) = \frac{1}{r^2 \sin\theta}\frac{\partial\psi}{\partial\theta} \quad , \quad U_\theta(r,\theta) = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r} \quad .}$$

What should we check?

$$\operatorname{div} \underline{b} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 b_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, b_\theta) + \frac{1}{r\sin\theta}\frac{\partial b_\varphi}{\partial\varphi}$$

Flow along a sphere
Stokes Formula

Rotational in spherical coordinates yields

$$\underline{\omega} = \underline{\mathrm{rot}}\ \underline{U} = \frac{1}{r}\left(\frac{\partial}{\partial r}(r\, U_\theta) - \frac{\partial U_r}{\partial \theta}\right)\underline{e}_\varphi$$

Therefore

$$\underline{\omega} = \omega_\varphi\, \underline{e}_\varphi = -\frac{1}{r \sin\theta}\left[\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} - \frac{\mathrm{cotg}\,\theta}{r^2}\frac{\partial \psi}{\partial \theta}\right]\underline{e}_\varphi$$

Flow along a sphere
Stokes Formula

$$\underline{\omega} = \omega_\varphi \, \underline{e}_\varphi = -\frac{1}{r \sin\theta}\left[\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} - \frac{\operatorname{cotg}\theta}{r^2}\frac{\partial\psi}{\partial\theta}\right]\underline{e}_\varphi$$

Applying the Laplacian of a vector

$$\boxed{\Delta\,\underline{\omega} = 0 \quad .}$$

One is left with

$$\boxed{\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} - \frac{\operatorname{cotg}\theta}{r^2}\frac{\partial}{\partial\theta}\right)^2 \psi = 0 \quad .}$$

This looks like a Bilaplacian, but be careful…

$$\boxed{\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}-\frac{\operatorname{cotg}\theta}{r^2}\frac{\partial}{\partial\theta}\right)^2\psi=0 \quad .}$$

$$\Delta b = \operatorname{div}\,(\underline{\operatorname{grad}}\ b) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial b}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial b}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 b}{\partial\varphi^2}$$

Boundary conditions (in the frame of sphere)

$$\boxed{u_r(a,\theta)=u_\theta(a,\theta)=0}$$

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$$\boxed{u_r(a,\theta)=u_\theta(a,\theta)=0}$$

$$\boxed{\psi(a,\theta)=\frac{\partial\psi}{\partial r}(a,\theta)=0\,,\quad 0\leqslant\theta\leqslant\pi\quad.}$$

Boundary conditions (in the frame of sphere)

$$\boxed{u_r(a,\theta)=u_\theta(a,\theta)=0}$$

$$\boxed{\psi(a,\theta) = \frac{\partial \psi}{\partial r}(a,\theta) = 0 , \quad 0 \leqslant \theta \leqslant \pi \quad .}$$

$$\boxed{U_r \sim U_\infty \cos\theta \quad , \quad U_\theta \sim -U_\infty \sin\theta \quad , \quad r \to \infty}$$

Boundary conditions (in the frame of sphere)

$$\boxed{u_r(a,\theta)=u_\theta(a,\theta)=0}$$

$$\boxed{\psi(a,\theta) = \frac{\partial\psi}{\partial r}(a,\theta) = 0\ ,\quad 0 \leqslant \theta \leqslant \pi\quad .}$$

$$\boxed{U_r \sim U_\infty \cos\theta\quad ,\quad U_\theta \sim -U_\infty \sin\theta\quad ,\quad r \to \infty}$$

$$\boxed{\psi(r,\theta) \sim \frac{1}{2} U_\infty r^2 \sin^2\theta\ ,\ 0 \leqslant \theta \leqslant \pi\ ,\ r \to \infty\quad .}$$

Separation of variables

$$\psi(r, \theta) = f(r) \sin^2 \theta$$

....

$$\left(\frac{d^2}{dr^2} - \frac{2}{r^2}\right)^2 f(r) = 0$$

....

$$f(r) = A r^4 + B r^2 + C r + \frac{D}{r}$$

Separation of variables

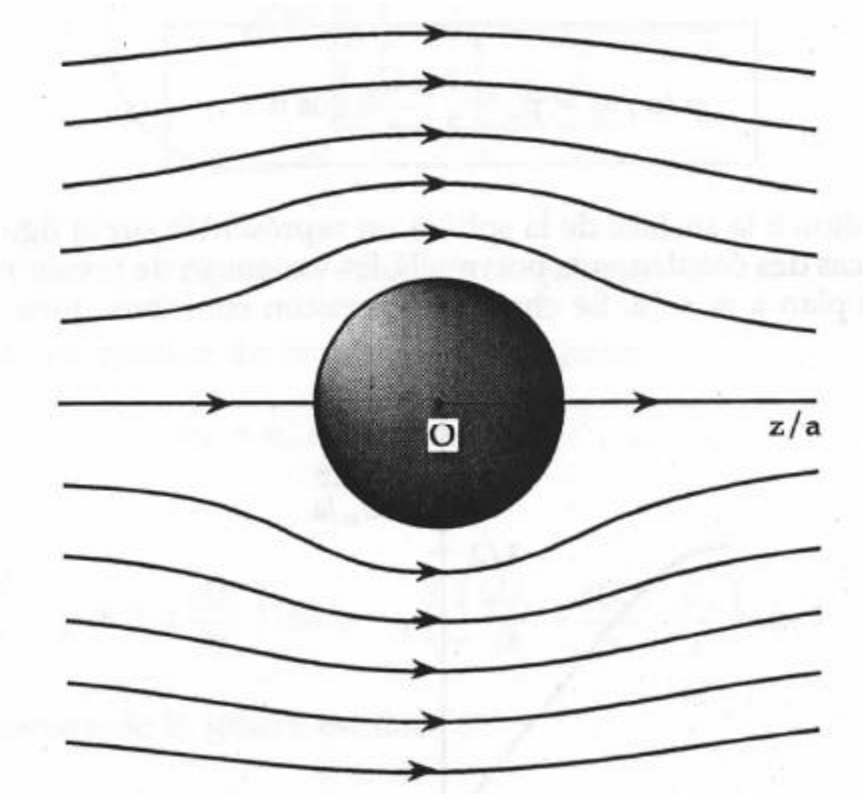
$$f(r) = A r^4 + B r^2 + C r + \frac{D}{r}$$

$$\boxed{\psi(r, \theta) = U_\infty \left(\frac{1}{2} r^2 - \frac{3}{4} a r + \frac{1}{4} \frac{a^3}{r} \right) \sin^2 \theta \quad .}$$

$$U_r(r, \theta) = U_\infty \cos\theta \left(1 - \frac{3}{2}\frac{a}{r} + \frac{1}{2}\frac{a^3}{r^3} \right) \quad ,$$

$$U_\theta(r, \theta) = U_\infty \sin\theta \left(-1 + \frac{3}{4}\frac{a}{r} + \frac{1}{4}\frac{a^3}{r^3} \right) \quad .$$

Streamfunction field



Retrieve pressure…

$$-\frac{\partial p}{\partial r}+\mu\left(\frac{\partial^2 U_r}{\partial r^2}+\frac{2}{r}\frac{\partial U_r}{\partial r}+\frac{1}{r^2}\frac{\partial^2 U_r}{\partial \theta^2}+\frac{1}{r^2}\operatorname{cotg}\theta\frac{\partial U_r}{\partial \theta}-\frac{2}{r^2}U_r-\frac{2}{r^2\sin\theta}\frac{\partial}{\partial\theta}(U_\theta\sin\theta)\right)=0\ ,$$

$$-\frac{1}{r}\frac{\partial p}{\partial \theta}+\mu\left(\frac{\partial^2 U_\theta}{\partial r^2}+\frac{2}{r}\frac{\partial U_\theta}{\partial r}+\frac{1}{r^2}\frac{\partial^2 U_\theta}{\partial \theta^2}+\frac{1}{r^2}\operatorname{cotg}\theta\frac{\partial U_\theta}{\partial \theta}+\frac{2}{r^2}\frac{\partial U_r}{\partial \theta}-\frac{U_\theta}{r^2\sin^2\theta}\right)=0\ ,$$

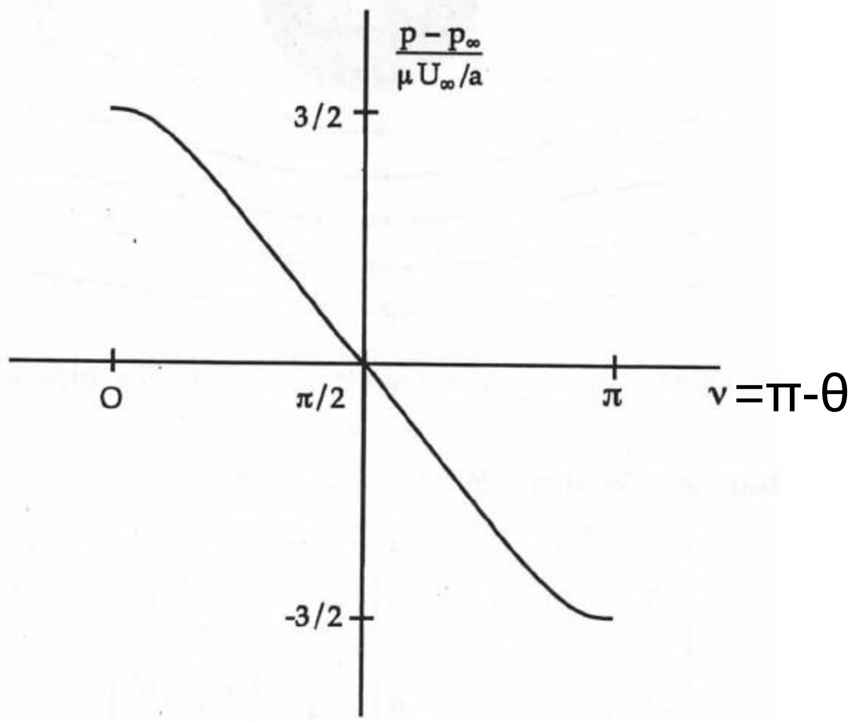
$$p(r,\theta)\sim p_\infty\ ,\quad 0\leqslant\theta\leqslant 2\pi\ ,\quad r\to\infty$$

….

$$\boxed{p(a,\theta)=p_\infty-\frac{3}{2}\frac{\mu U_\infty}{a}\cos\theta\quad .}$$

Retrieve pressure…

$$p(a,\theta) = p_\infty - \frac{3}{2}\frac{\mu U_\infty}{a}\cos\theta \quad .$$



Retrieve stress

$$\underline{\underline{\tau}} = 2\,\mu\,\underline{\underline{d}}$$

Retrieve stress

$$\underline{\underline{\tau}} = 2\,\mu\,\underline{\underline{d}}$$

$$d_{rr} = \frac{\partial U_r}{\partial r} \quad , \quad d_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial U_r}{\partial \theta} + \frac{\partial U_\theta}{\partial r} - \frac{U_\theta}{r}\right)$$

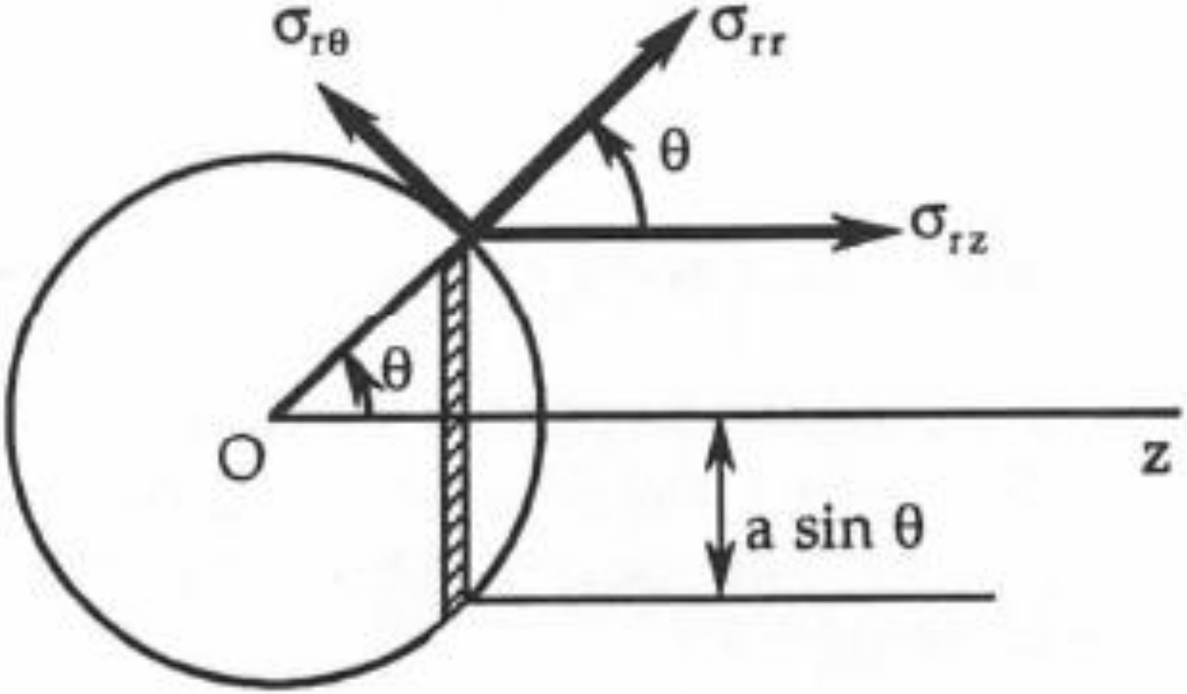
Retrieve stress

$$\underline{\underline{\tau}} = 2\,\mu\,\underline{\underline{d}}$$

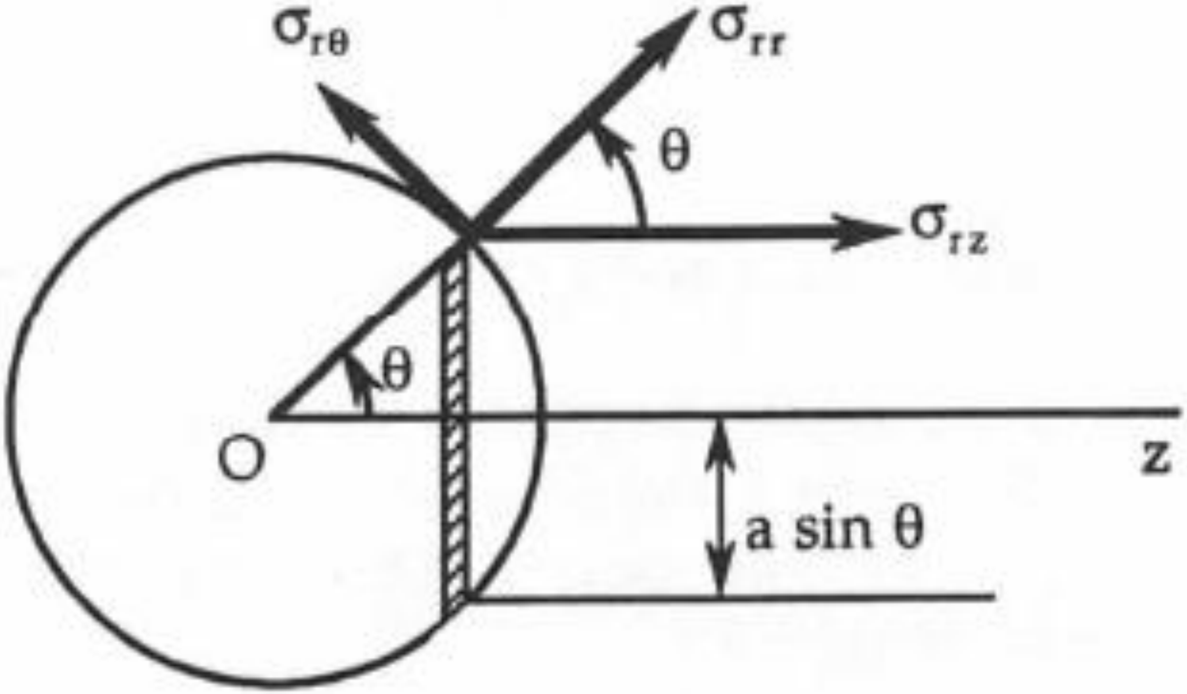
$$d_{rr} = \frac{\partial U_r}{\partial r} \quad , \quad d_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial U_r}{\partial \theta} + \frac{\partial U_\theta}{\partial r} - \frac{U_\theta}{r}\right)$$

$$\boxed{\tau_{rr} = 2\,\mu\,\frac{\partial U_r}{\partial r} \quad , \quad \tau_{r\theta} = \mu\left(\frac{1}{r}\frac{\partial U_r}{\partial \theta} + \frac{\partial U_\theta}{\partial r} - \frac{U_\theta}{r}\right) \quad .}$$

Retrieve stress

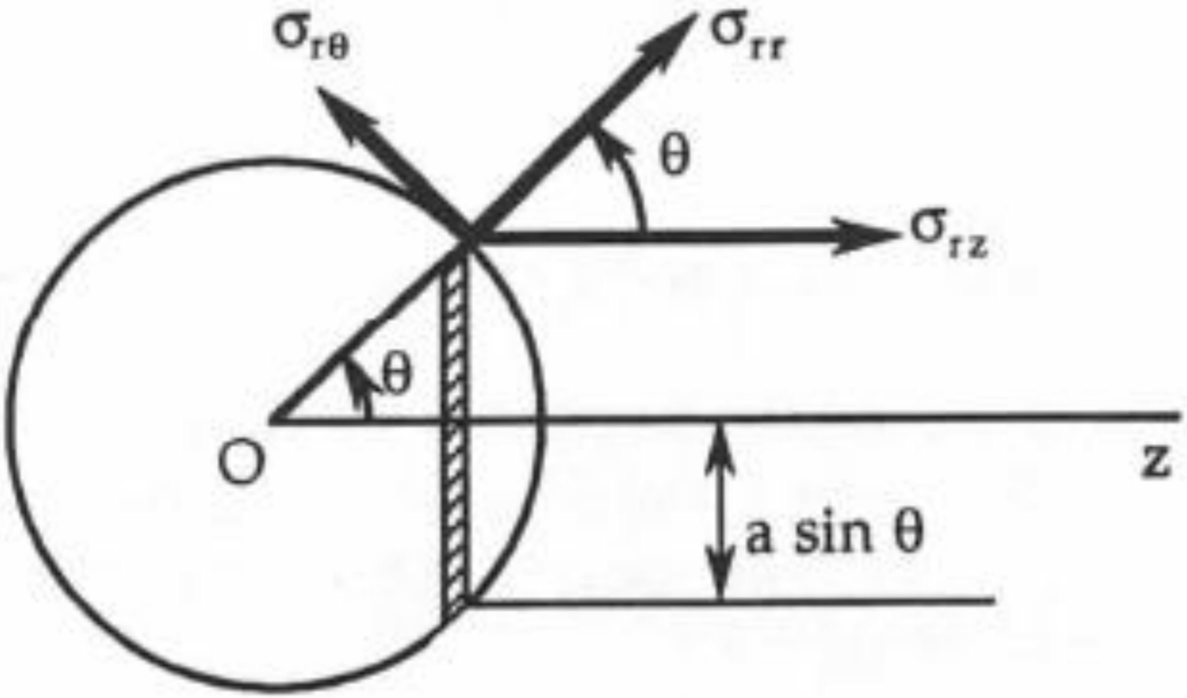


Retrieve stress



$$\sigma_{rz} = \sigma_{rr} \cos\theta - \sigma_{r\theta} \sin\theta$$

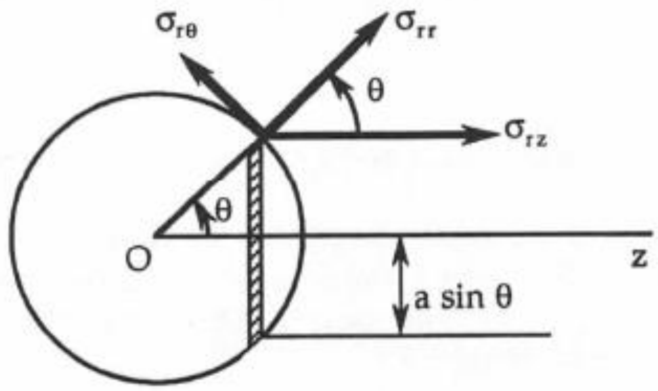
Retrieve stress



$$\sigma_{rz} = \sigma_{rr}\cos\theta - \sigma_{r\theta}\sin\theta$$

$$\sigma_{rz} = \left(-p + 2\mu\frac{\partial U_r}{\partial r}\right)\cos\theta - \mu\left(\frac{1}{r}\frac{\partial U_r}{\partial \theta} + \frac{\partial U_\theta}{\partial r} - \frac{U_\theta}{r}\right)\sin\theta$$

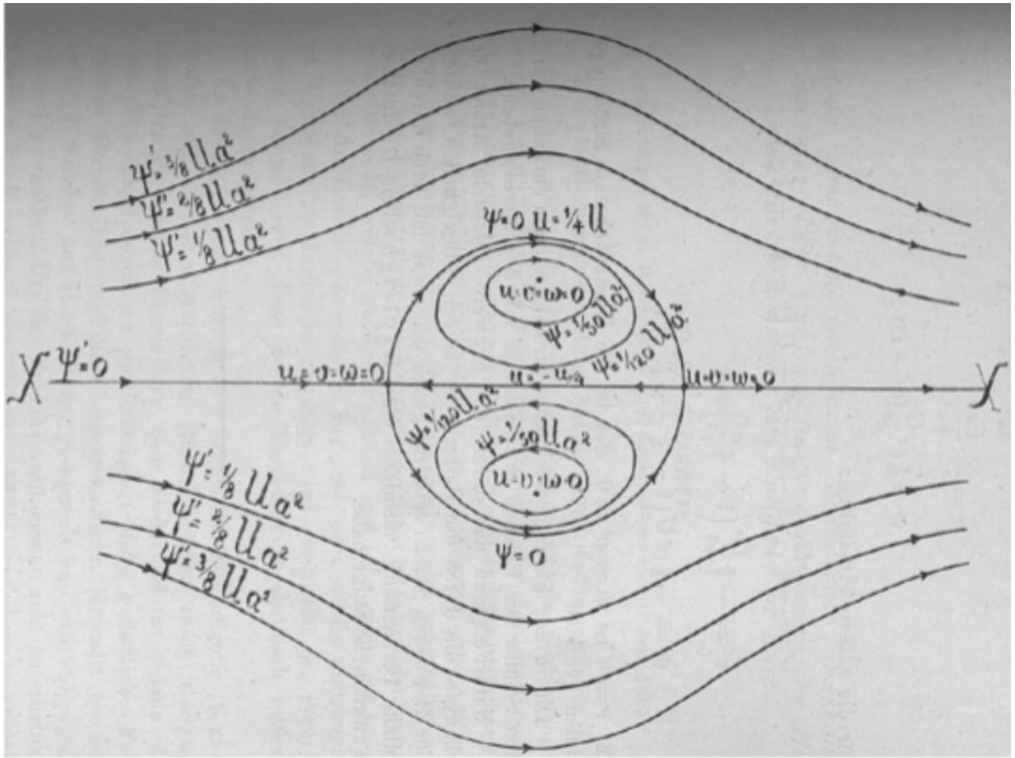
Retrieve stress



$$F_x = \int_0^{\pi} \sigma_{rz}(a,\theta)\, 2\pi a^2 \sin\theta \, d\theta$$

$$\boxed{F_x = 6\,\pi\,\mu\,U_\infty\,a \quad .}$$

Drag of drop



Rybzinski 1911

Drag of drop

$$\boxed{D = 4\pi\mu V a \frac{3\lambda + 2}{2(\lambda + 1)}}$$

$\lambda = \mu_{int} / \mu_{ext}$ viscosity ratio

Hadamard 1911, Rybzinski 1911

$$\boxed{D = 6\pi\mu a V}$$

for a very viscous drop (almost solid)

$$\boxed{D = 4\pi\mu a V}$$

for an inviscid drop

Drag does not depend a lot on exact shape of object
when Re<<1

Flow along sphere

$$u_r = U\cos\phi\left(\frac{3a}{2r} - \frac{a^3}{2r^3}\right)$$

$$u_\phi = U\sin\phi\left(\frac{3a}{4r} + \frac{a^3}{4r^3}\right)$$

$$\mathbf{F} = -6\pi\mu a\mathbf{U}$$

Flow along a sphere

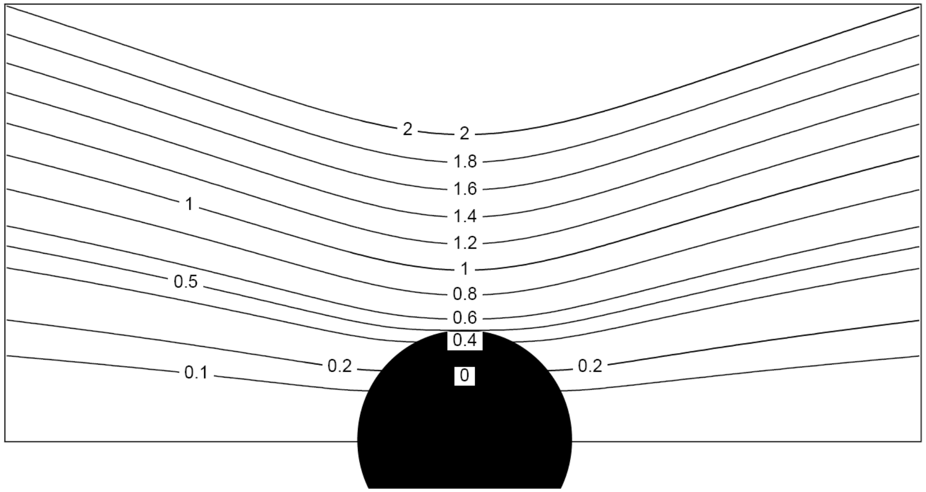
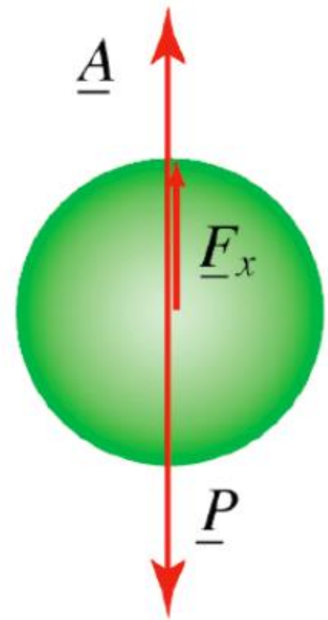


FIG. 6.8 – écoulement à petit Re autour d'une sphère. Lignes de courant dans le repère où le fluide est immobile à l'infini. Les valeurs de ψ sont normalisées par Ua^2.

CHUTE DE BILLES



$$\mathrm{Re} \ll 1$$

$$\text{const.}\,\mu\,U\,a = (\rho_b - \rho)\frac{4}{3}\pi\,a^3 g$$

$$U = \text{const.}\frac{\rho_b - \rho}{\mu}\frac{4}{3}\pi\,a^2 g$$

$$\boxed{U \;\alpha\; a^2}$$

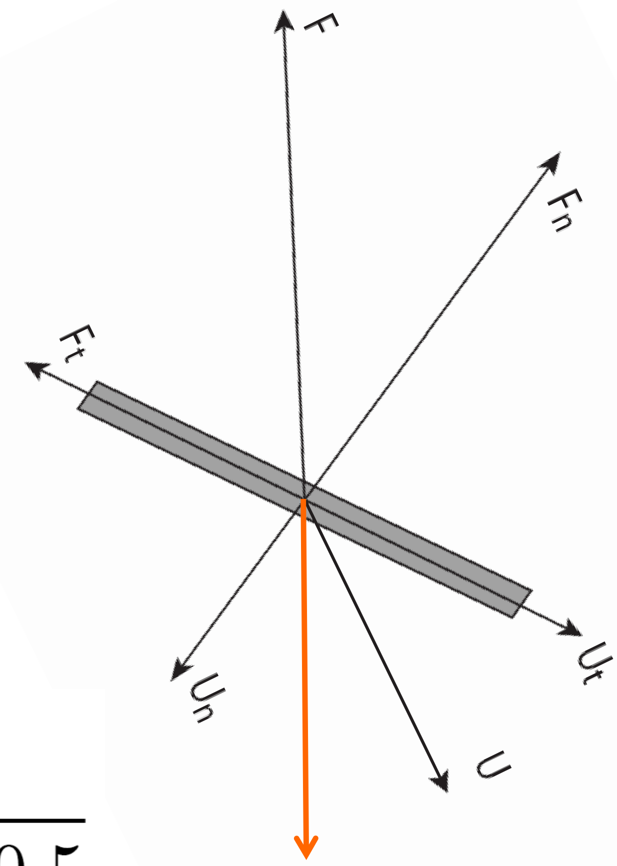
Flow along sphere

$$C_D = \frac{|\mathbf{F}|}{\pi \rho U^2 a^2} \sim \frac{6}{R}\left(1 + \frac{3}{8}R + \frac{9}{40}R^2 \log R + 0(R^2)\right)$$

Developement asymptotique en Reynolds

$$R=\rho Ua/\mu$$

Flow along a stab

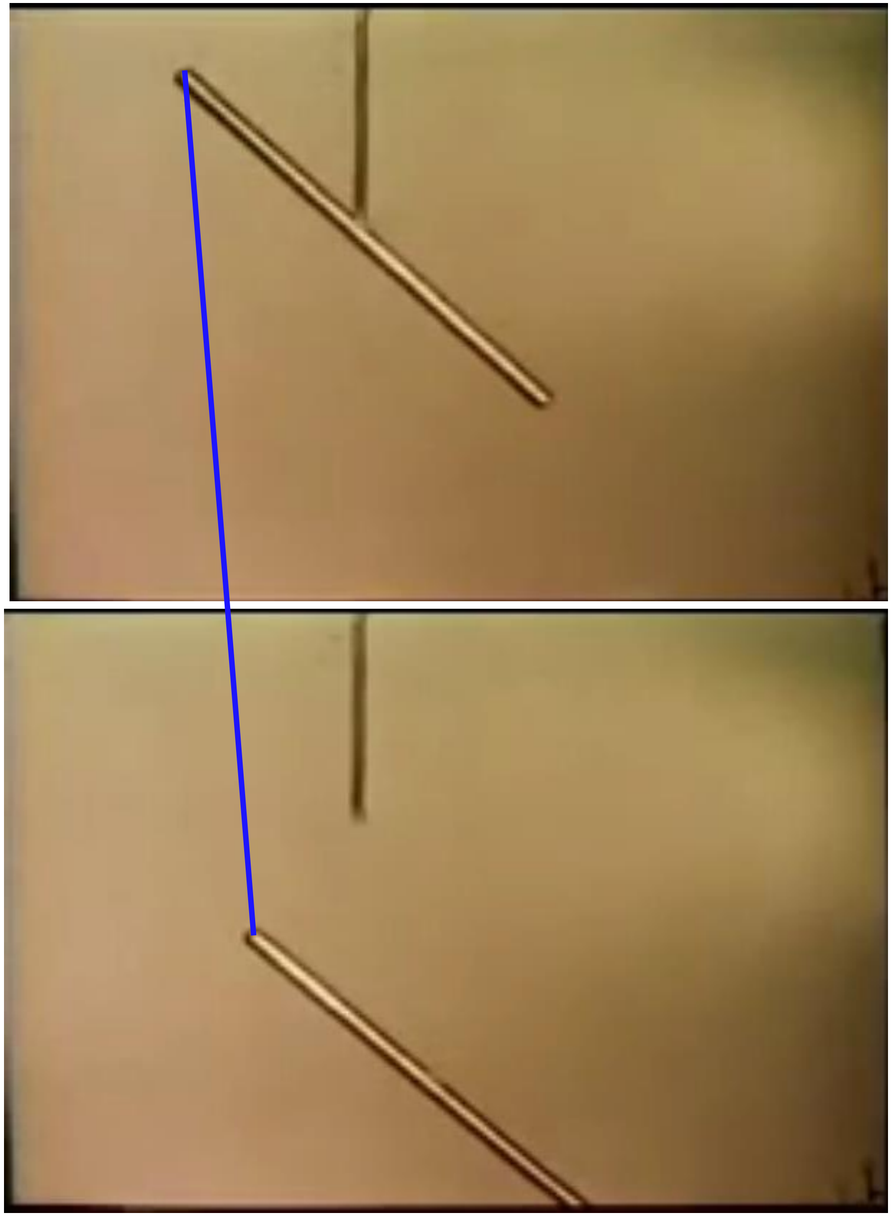


$$F_n = \frac{4\pi\eta L U_n}{\ln(L/a) + 0.5}$$

$$\xrightarrow[\ln(L/a) \gg 1]{} \quad F_n = 2\,F_t$$

$$F_t = \frac{2\pi\eta L U_t}{\ln(L/a) - 0.5}$$

Flow along a stab



Flow along a disk

No solution of the 2D Stokes equations satisfying the far field conditions and the boundary conditions on the disk

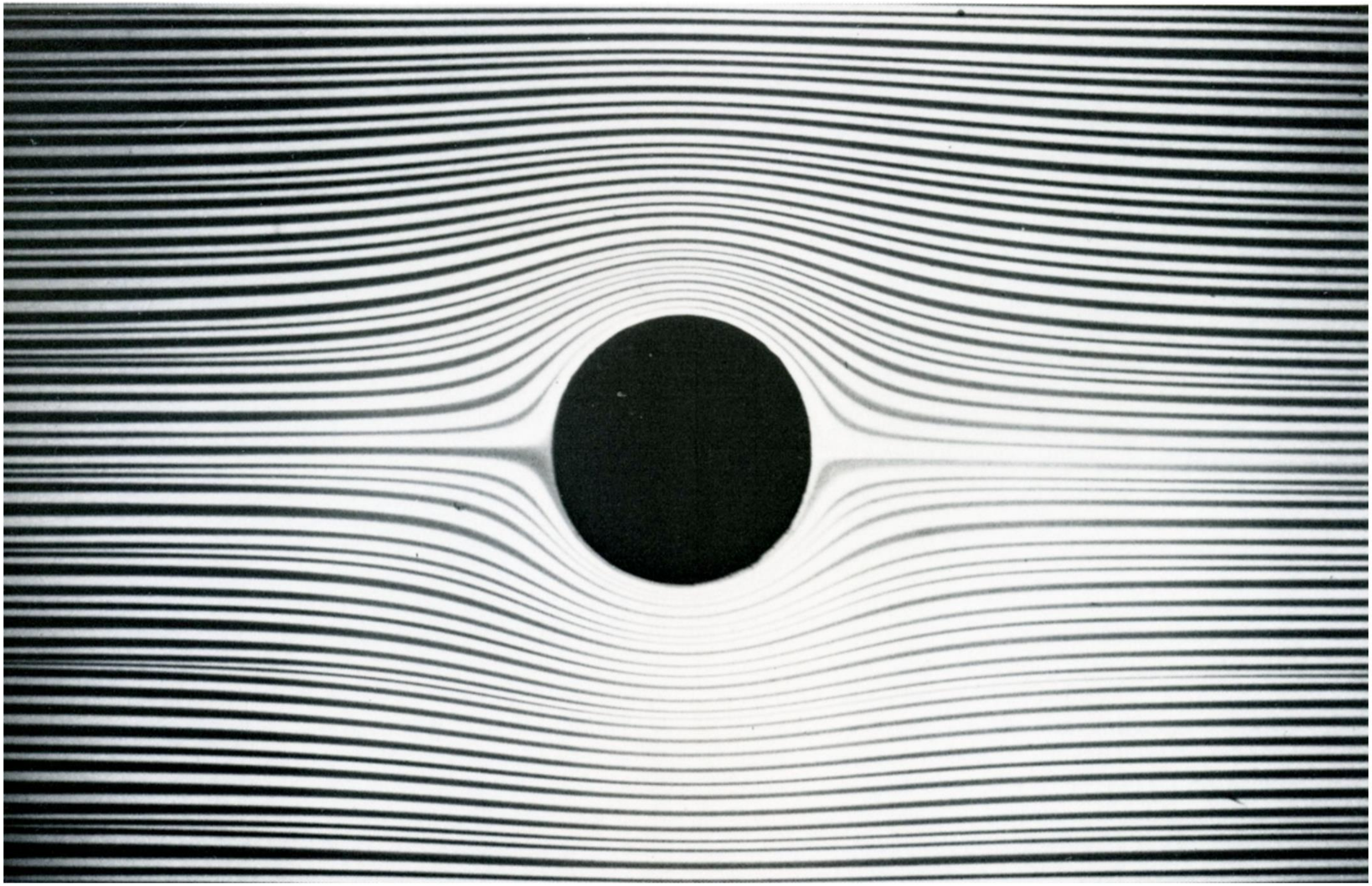
⇨Oseen's paradox! Inertial effects have to be taken into account

$$C_D = \frac{F}{2\rho U^2 a} \sim \frac{16\pi}{R}\epsilon\,(1 + a_2\epsilon^2 + 0(\epsilon^3)),$$

where

$$\epsilon = \frac{1}{\log 8/R + \frac{1}{2} - \gamma}, \quad \gamma \doteqdot 0{\cdot}58 \text{ (Euler's constant)}, \quad a_2 \doteqdot -0{\cdot}87.$$

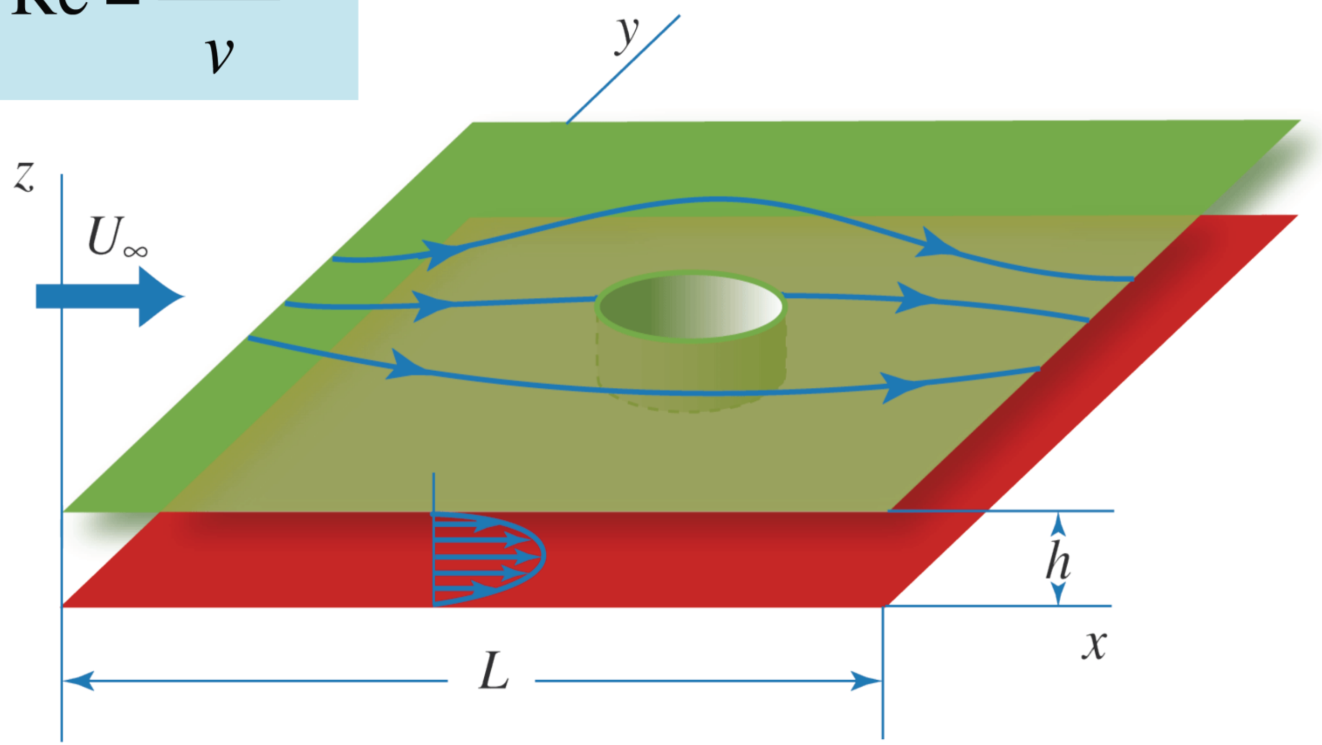
A strange potential limit of the Stokes equations



Peregrine (1982)

ÉCOULEMENT DANS UNE CELLULE DE HELE-SHAW

$$\mathrm{Re} = \frac{U_\infty h}{\nu}$$



$$\varepsilon = h/L << 1$$

(Film A. Garcia)

ÉCOULEMENT DANS UNE CELLULE DE HELE-SHAW

$$(\bar{x},\bar{y}) = \frac{(x,y)}{L} \qquad \bar{z} = \frac{z}{h}$$

$$(\bar{u},\bar{v}) = \frac{(u,v)}{U_\infty} \qquad \bar{w} = \frac{w}{W} \qquad \bar{p} = \frac{p}{P}$$

Continuité:

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} + \frac{W}{U_\infty}\frac{L}{h}\frac{\partial \bar{w}}{\partial z} = 0$$

Navier-Stokes:

$$\mathrm{Re}\,\varepsilon\left(\bar{u}\frac{\partial}{\partial \bar{x}} + \bar{v}\frac{\partial}{\partial \bar{y}} + \frac{W}{U_\infty}\frac{L}{h}\bar{w}\frac{\partial}{\partial \bar{z}}\right)\bar{u}$$

$$= -\frac{Ph^2}{\mu U_\infty L}\frac{\partial \bar{p}}{\partial \bar{x}} + \left(\frac{\partial^2}{\partial \bar{z}^2} + \varepsilon^2\frac{\partial^2}{\partial \bar{x}^2} + \varepsilon^2\frac{\partial^2}{\partial \bar{y}^2}\right)\bar{u}$$

• • •

ÉCOULEMENT DANS UNE CELLULE DE HELE-SHAW
Équations fondamentales

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

$$\mu\frac{\partial^2 u}{\partial z^2}=\frac{\partial p}{\partial x}$$

$$\mu\frac{\partial^2 v}{\partial z^2}=\frac{\partial p}{\partial y}$$

$$\frac{\partial p}{\partial z}=0$$

$\varepsilon << 1$ $\qquad$ $\mathrm{Re}\,\varepsilon << 1$

ÉCOULEMENT DANS UNE CELLULE DE HELE-SHAW

$$u(x,y,z) = -\frac{1}{2\mu}\frac{\partial p}{\partial x} z(h-z)$$

$$v(x,y,z) = -\frac{1}{2\mu}\frac{\partial p}{\partial y} z(h-z)$$

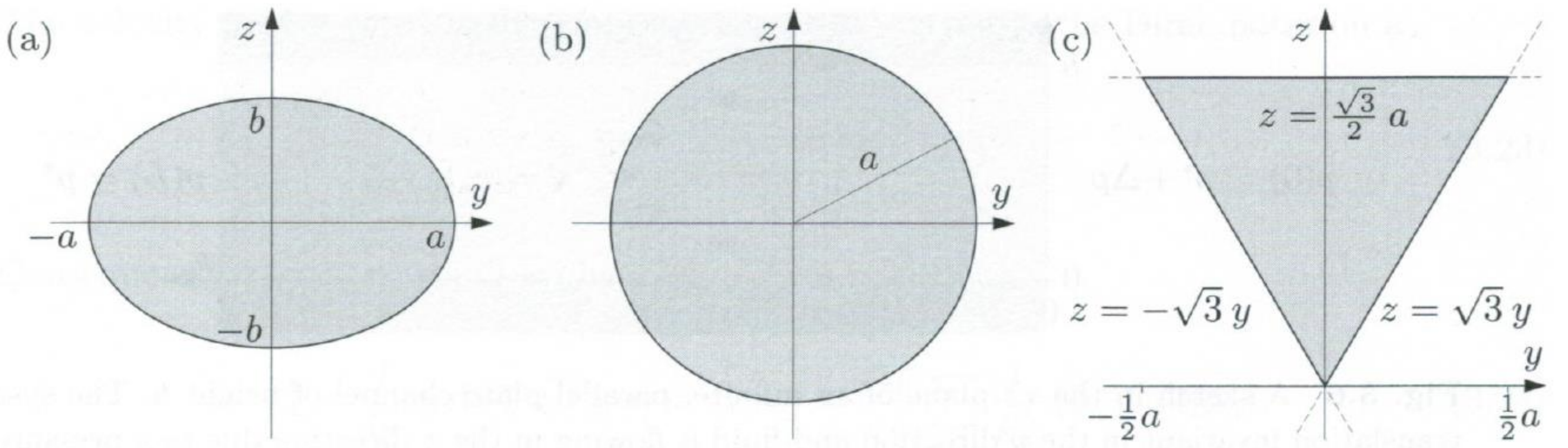
$$\boxed{w(x,y,z) = 0 \ !}$$

$$\boxed{\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \ !}$$

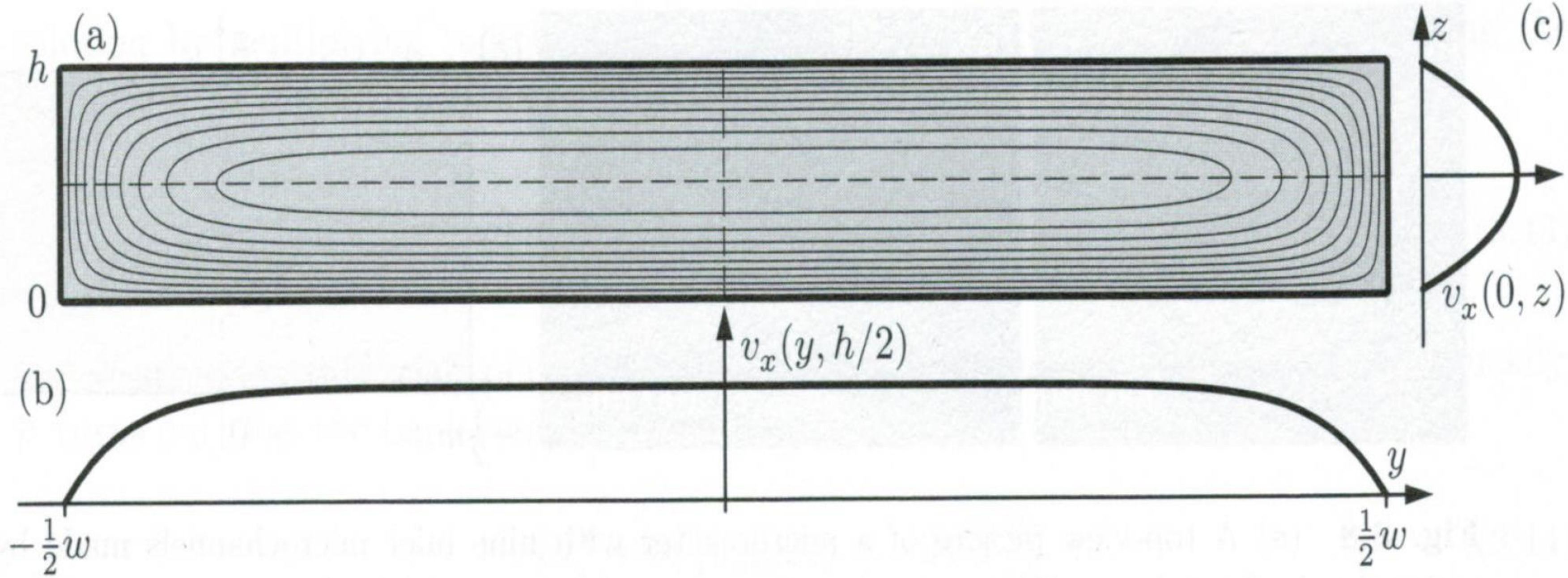
Écoulement potentiel dans le plan *x-y* !

Towards Navier-Stokes equations

In strictly parallel flows, the inertial terms disappear as the consequence of geometry



Flow in a rectangular duct



$$v_x(y,z) = \frac{4h^2\Delta p}{\pi^3\eta L}\sum_{n,\mathrm{odd}}^{\infty}\frac{1}{n^3}\left[1-\frac{\cosh\left(n\pi\frac{y}{h}\right)}{\cosh\left(n\pi\frac{w}{2h}\right)}\right]\sin\left(n\pi\frac{z}{h}\right).$$

Flow in a rectangular duct

$$\begin{aligned} Q &= 2\int_0^{\frac{1}{2}w} \mathrm{d}y \int_0^h \mathrm{d}z\, v_x(y,z) \\ &= \frac{4h^2\Delta p}{\pi^3\eta L} \sum_{n,\mathrm{odd}}^{\infty} \frac{1}{n^3}\,\frac{2h}{n\pi}\left[w - \frac{2h}{n\pi}\tanh\left(n\pi\frac{w}{2h}\right)\right] \\ &= \frac{8h^3 w\Delta p}{\pi^4\eta L} \sum_{n,\mathrm{odd}}^{\infty} \left[\frac{1}{n^4} - \frac{2h}{\pi w}\,\frac{1}{n^5}\tanh\left(n\pi\frac{w}{2h}\right)\right] \\ &= \frac{h^3 w\Delta p}{12\eta L}\left[1 - \sum_{n,\mathrm{odd}}^{\infty} \frac{1}{n^5}\,\frac{192}{\pi^5}\,\frac{h}{w}\tanh\left(n\pi\frac{w}{2h}\right)\right], \end{aligned}$$

$$\sum_{n,\mathrm{odd}}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{96}.$$

Flow in a rectangular duct

$$\approx \frac{h^3 w \Delta p}{12\eta L}\left[1 - 0.630\,\frac{h}{w}\right], \quad \text{for } h < w.$$

13% error for square, 0.2% for aspect ratio 2

ÉCOULEMENT DANS UNE CELLULE DE HELE-SHAW

$$u(x,y,z) = -\frac{1}{2\mu}\frac{\partial p}{\partial x} z(h-z)$$

$$v(x,y,z) = -\frac{1}{2\mu}\frac{\partial p}{\partial y} z(h-z)$$

$$w(x,y,z) = 0 \;!$$

$$\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0\,!$$

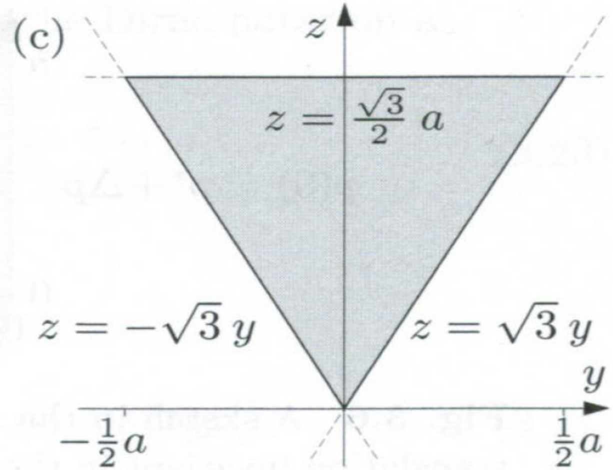
Écoulement potentiel dans le plan *x-y* !

$$Q = \int_0^h \int_{-w/2}^{w/2} -\frac{1}{2\eta}\frac{dp}{dx} z(h-z)dzdy$$

$$Q = \int_0^h \int_{-w/2}^{w/2} \frac{1}{2\eta}\frac{\delta P}{L} z(h-z)dzdy$$

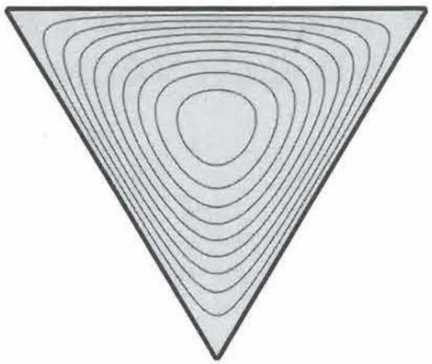
$$Q = \frac{wh^3}{12\mu}\frac{\Delta P}{L}$$

Flow in a triangular duct

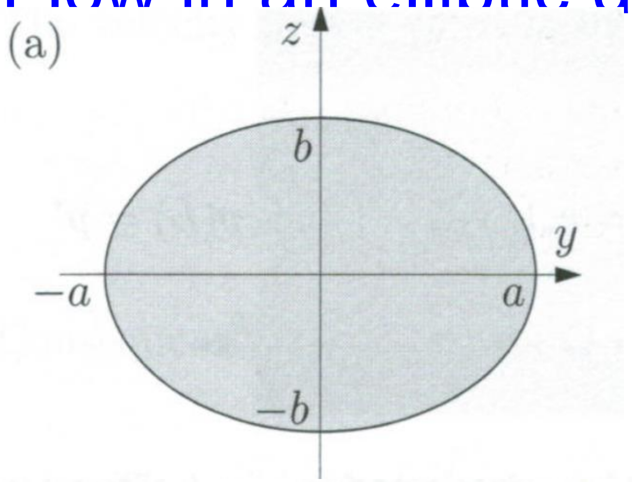


$$v_x(y,z) = \frac{v_0}{a^3}\left(\frac{\sqrt{3}}{2}a - z\right)\left(z - \sqrt{3}y\right)\left(z + \sqrt{3}y\right) = \frac{v_0}{a^3}\left(\frac{\sqrt{3}}{2}a - z\right)\left(z^2 - 3y^2\right).$$

$$v_0 = \frac{1}{2\sqrt{3}}\,\frac{\Delta p}{\eta L}\,a^2$$



Flow in an elliptic duct



$$v_x(y,z) = v_0\left(1 - \frac{y^2}{a^2} - \frac{z^2}{b^2}\right)$$

$$v_0 = \frac{\Delta p}{2\eta L}\,\frac{a^2 b^2}{a^2 + b^2}$$