ME-444 Hydrodynamics- exercise22 avril 2024

Flow on a facing plate, boundary layer solution and more.

We consider the potential flow in the $(x-y)$ plane of a liquid (density ρ) on a infinitely thin plate $\mathcal{P}$ located between $x=-2a$ and $x=2a$ on the axis $y=0$. The velocity field is classically denoted by $\mathbf{u}=(u,v)$. A uniform flow impinges on the plate with a far field velocity $\lim_{(x,y)\to\infty}(u,v)=(0,V_\infty)$.

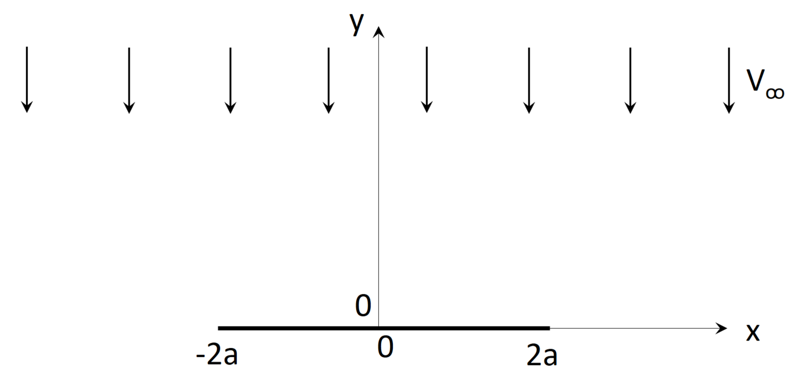

Figure 1: Schematics of the flow on a facing plate.

Part I, boundary layer

We consider the quadrant $x>0, y>0$ of the exterior flow, chosen to be $u_e=Ax, v_e=-Ay$ (remember the exercise of week 1), prevailing in the far field. It is representative of the flow impinging a facing plate of length $4a$ located in $y=0$. The flow is assumed incompressible and to be a viscous Newtonian fluid of kinematic viscosity ν. The Reynolds number is assumed very large $Re=Aa^2/\nu \ll 1$.

1. What characteristic velocity and length scales have been used to build this Reynolds number?
2. Explain why a boundary layer must be introduced in the vicinity of the wall $y=0$. By a scaling analysis, determine its characteristic gauge of y in the boundary layer as well as that of the pressure and fill in the table.

quantity	dimensionless variable	expression
horizontal length	$\bar{x}$	$\frac{x}{a}$
vertical length	$\bar{y}$	
horizontal velocity	$\bar{u}$	$\frac{u}{V_\infty}$
vertical velocity	$\bar{v}$	$\frac{vRe^{1/2}}{V_\infty}$
pressure	$\bar{p}$	

Table 1: Boundary layer scales. The vertical length and pressure gauges have to be determined.

3. [3 pts] Using the aforementioned scales, show that the $\frac{\partial \bar{p}}{\partial \bar{y}} = 0$ in the boundary layer such that $\bar{p}(\bar{y}, \bar{x}) = \bar{p}_e = -\bar{x}^2/2$. Demonstrate that the Prandtl equations write

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0 \tag{1}$$

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = \bar{x} + \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} \tag{2}$$

$$\bar{u}(\bar{x} > 0, 0) = 0 \qquad \bar{v}(\bar{x} > 0, 0) = 0 \tag{3}$$

$$\bar{u}(\bar{x}, \bar{y} \to \infty) = \bar{x} \tag{4}$$

4. Why does the vertical momentum balance seem to have disappeared from this set of boundary layer equations?

5. [3 pts] We now look for self similar solutions, using expansion coefficients denoted by $*$ and rescaled variables denoted by $\tilde{}$, i.e $\bar{u} = u^*\tilde{u}$, $\bar{v} = v^*\tilde{v}$, $\bar{x} = x^*\tilde{x}$, $\bar{y} = y^*\tilde{y}$. Show that the following conditions must hold

$$\frac{u^*}{x^*} = \frac{v^*}{y^*} \tag{5}$$

$$\frac{(u^*)^2}{x^*} = x^* \tag{6}$$

$$u^* = \frac{x^*}{(y^*)^2} \tag{7}$$

and deduce the following self-similar ansatz

$$\bar{u} = \bar{x}g(\bar{y}); \bar{v} = -f(\bar{y}), \tag{8}$$

where a minus sign has been introduced for later convenience.

6. Is this ansatz compatible with the matching condition at $\bar{y} \to \infty$?

7. [2 pts] Show that $g = f'$ and find the missing term [...] in the following ordinary differential equation

$$f''' + [\ldots] + 1 - f'^2 = 0 \tag{9}$$

8. What are the boundary conditions for f? The solution can only be obtained numerically and is reported in figure 2, together with its approximation by a third order polynomial

9. [2 pts] Show that the displacement thickness $\delta_1(\bar{x}) = \int_0^\infty (1 - \bar{u}V_\infty/u_e)d\bar{y}$ does actually not depend on $\bar{x}$. Contrast this result with the boundary layer on a flat plate. Which of these two boundary layers is the thinnest? Does the momentum thickness depend on $\bar{x}$?

10. [2 pts] Assuming that this solution hold on the entire plate and coming back to expressions with dimensions, determine the shear force (per unit length) acting on the semi-plate $x \in [0; 2a]$.

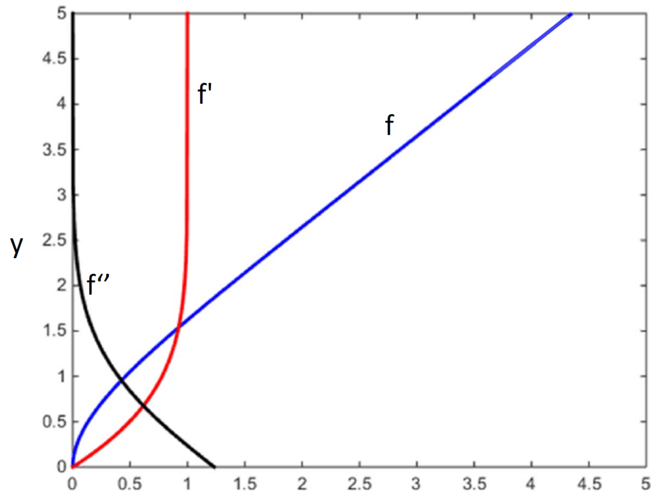

Figure 2: Solution f and its derivatives f' and f'' of the differential equation (9) with suitable boundary conditions

Part II, a solution of the Navier-Stokes equations

11. Remarkably, it turns out the boundary layer approximation obtained before is an exact solution of the Navier-Stokes equations, as surprising as it may sound. Show that the full incompressible Navier-Stokes equations in the $^{-}$ gauge system write

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0 \tag{10}$$

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = -\frac{\partial \bar{p}}{\partial \bar{x}} + \frac{1}{Re}\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} \tag{11}$$

$$\bar{u}\frac{\partial \bar{v}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{v}}{\partial \bar{y}} = -Re\frac{\partial \bar{p}}{\partial \bar{y}} + \frac{1}{Re}\frac{\partial^2 \bar{v}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{v}}{\partial \bar{y}^2} \tag{12}$$

$$\bar{u}(\bar{x}, 0) = \bar{v}(\bar{x}, 0) = 0 \tag{13}$$

12. Suppose now that $\bar{u} = \bar{x}f'(\bar{y})$, $\bar{v} = -f(\bar{y})$ and in addition that $\bar{p} = -\frac{\bar{x}^2}{2} + C(\bar{y})$. Show that the continuity equation is fulfilled as well as the horizontal momentum balance

13. Show that the vertical momentum balance hold if and only if: $C(\bar{y}) = -Re^{-1}f'(\bar{y}) - Re^{-1}f^2(\bar{y})/2 + cst.$.

14. What boundary conditions are fulfilled?

    We have now a rare exact solution of the Navier-Stokes equations!

15. Express the pressure field where $\bar{x} \to \infty$ but $\bar{y}$ remains fixed and finite. Is it compatible with the expression valid in the boundary layer $\bar{p_e} = -\bar{x}^2/2 + cst.$?

16. Express the pressure field in the limit $\bar{x} \to \infty$ and $\bar{y} \to \infty$. Is it consistent with the pressure prevailing in the potential exterior region writing dimensionally $p_e = -A^2(x^2 + y^2)/2 + cst.$?