Hydrodynamics
Exam 2019

Waves in shallow pool

We consider a shallow pool filled with a liquid of density ρ, dynamic viscosity μ, surface tension γ, depth H and length L with $H \ll L$. The flow is submitted to gravity $\mathbf{g} = -g\mathbf{e}_y$. The coordinate system is (x, y) as defined in the figure. The kinematic viscosity writes ν.

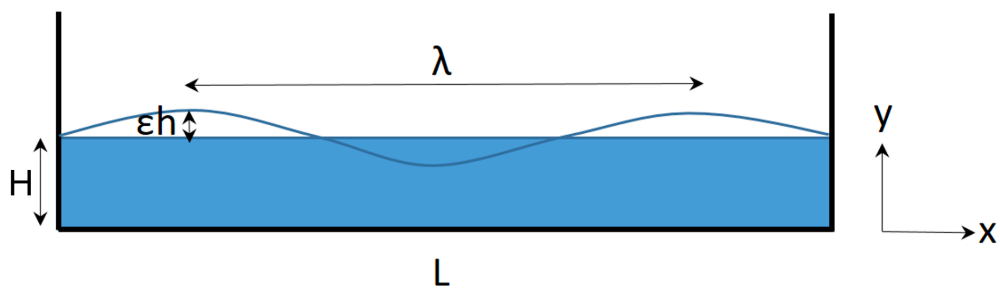

Figure 1: Waves in shallow pool.

We consider waves at the surface of this pool of the form $\exp(i(kx - \omega t))$.

1. What is the dispersion relation found in class (you can use dimensional analysis)

   (a)
   $$\omega = \sqrt{(gk + (\gamma/\rho)\, k^3) \tanh(kH)}$$

   (b)
   $$\omega = \sqrt{gk + \gamma k^3} \tanh(kH)$$

   (c)
   $$\omega = \sqrt{\rho g k + \gamma k^3} \tanh(kH)$$

2. For a bassin of length $L \gg H \gg l_c$, show that
   $$\omega = \sqrt{gH} 2\pi/L \propto \sqrt{gH}/L \tag{1}$$

3. Let us set $L = 1\,m$ and $H = 10\,cm$, among the three following figures, which one depicts a correct time evolution?

4. Is the wavemotion eventually damped? What assumption done to obtain the expression of question 1 is at the origin of this peculiar phenomenon?

5. Since the layer is shallow, we now propose to use a completely different equation, which we have considered in class: the lubrication equation
   $$\frac{\partial h}{\partial t} = \frac{1}{3\mu} \frac{\partial}{\partial x} h^3 \frac{\partial p}{\partial x}, \tag{2}$$

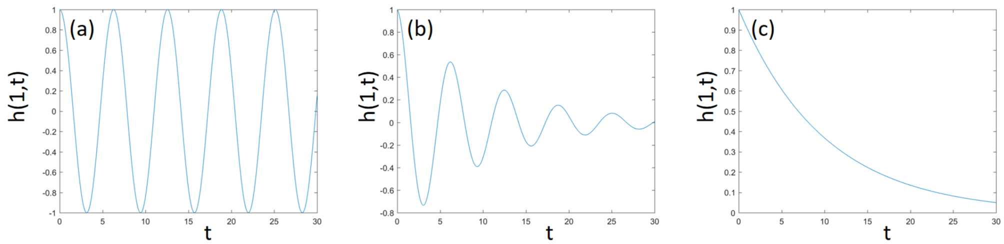

Figure 2: Possible time-evolutions.

where p is the value of the pressure in the liquid in $z = h$ (free surface). Considering small departures from the nominal height H, $h = H + \epsilon h'$ (same for pressure $p = P + \epsilon p'$), show that

$$\frac{\partial h}{\partial t} = \frac{\rho g}{3\mu} H^3 \left(\frac{\partial^2 h}{\partial x^2} - \frac{\gamma}{\rho g} \frac{\partial^4 h}{\partial x^4} \right), \tag{3}$$

6. Using the wave expansion, deduce that

$$\omega = -i \frac{\rho g H^3}{3\mu} (k^2 + l_c^2 k^4) \tag{4}$$

where $l_c = \sqrt{\frac{\gamma}{\rho g}}$ is the so called capillary length. Note the purely imaginary nature of ω.

7. Which of the previous figures plotting a possible time evolution corresponds to this case?

8. Show that the time scale of damping scales like $3\mu L^2/\rho g H^3$ for a typical $L \gg l_c$ (i.e. $k l_c \ll 1$).

9. In order to reconcile the two approaches followed so far, we remember how the lubrication equation was found starting with the x-momentum equation

$$\rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\frac{\partial p}{\partial x} + \mu (\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) \tag{5}$$

We now set the x scale to L, the y scale to H, the pressure gauge to P, the x–velocity scale to U, the y–velocity scale to V and the time scale to τ . From left to right, fill in the two missing scalings:

(a) $\rho U/\tau$
(b)
(c) $\rho U V/H$
(d) P/L
(e)
(f) $\mu U/H^2$

10. Show that if $\tau \sim H^2/\nu$, the first inertial term cannot be neglected with respect to the last viscous term. Using the expressions for τ found in questions 2 and 8, show that this is physically possible.

11. However, we can keep the displacement arbitrary small such that $U/L \ll 1/\tau$ and set the pressure gauge P to end with and equation studied at length in class
$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2} \tag{6}$$

12. How does the y momentum equation simplify under the above choice of gauges?

13. What boundary condition should be imposed at leading order at the free surface $y = H$?

    (a) $\left.\frac{\partial u}{\partial y}\right|_{y=H} = 0$

    (b) $\left.u\right|_{y=H} = 0$

    (c) $\left.v\right|_{y=H} = \frac{dh}{dt}$

    (multiple solutions might be correct).

14. Show that the solution is (noting $K = -\frac{1}{\rho}\frac{\partial p}{\partial x}$)
$$\hat{u}(y) = \frac{iK}{\omega} + A\cosh\left((1-i)\sqrt{\omega/2\nu}\; y\right) + B\sinh\left((1-i)\sqrt{\omega/2\nu}\; y\right) \tag{7}$$

15. Determine A and B using the boundary conditions and show that
$$\hat{u}(y) = \frac{iK}{\omega}\left(1 - \cosh\left((1-i)\sqrt{\omega/2\nu}\; y\right) + \tanh\left((1-i)\sqrt{\omega/2\nu}\; H\right)\sinh\left((1-i)\sqrt{\omega/2\nu}\; y\right)\right) \tag{8}$$

16. Use the continuity equation to show that
$$\hat{v}(H) = -\int_0^H \frac{\partial \hat{u}}{\partial x} dy = -\frac{i\partial K/\partial x}{\omega}\left(H - \frac{\tanh((1-i)\sqrt{\omega/2\nu}H)}{(1-i)\sqrt{\omega/2\nu}}\right) \tag{9}$$
$$\frac{\partial h'}{\partial t} = i\frac{g}{\omega}\left(\frac{\partial^2 h'}{\partial x^2} - \frac{\gamma}{\rho g}\frac{\partial^4 h'}{\partial x^4}\right)\left(H - \frac{\tanh\left[(1-i)\sqrt{\frac{\omega}{2\nu}}H\right]}{(1-i)\sqrt{\frac{\omega}{2\nu}}}\right) \tag{10}$$
and derive the dispersion delation
$$\omega^2 = k\left(gk + \frac{\gamma}{\rho}k^3\right)\left(H - \frac{\tanh\left[(1-i)\sqrt{\frac{\omega}{2\nu}}H\right]}{(1-i)\sqrt{\frac{\omega}{2\nu}}}\right) \tag{11}$$
where the length scale of the viscous boundary layer is $l_{vb} = \sqrt{2\nu/\omega}$.

17. The length scale of the viscous boundary layer is $l_{vb} = \sqrt{2\nu/\omega}$. We can define two limits:

- $H/l_{vb} \ll 1$ (purely damped). Show that one retrieves

$$\omega = -i\frac{H^2}{3\nu}\left(gk + \frac{\gamma}{\rho}k^3\right)kH \tag{12}$$

and comment the associated figure

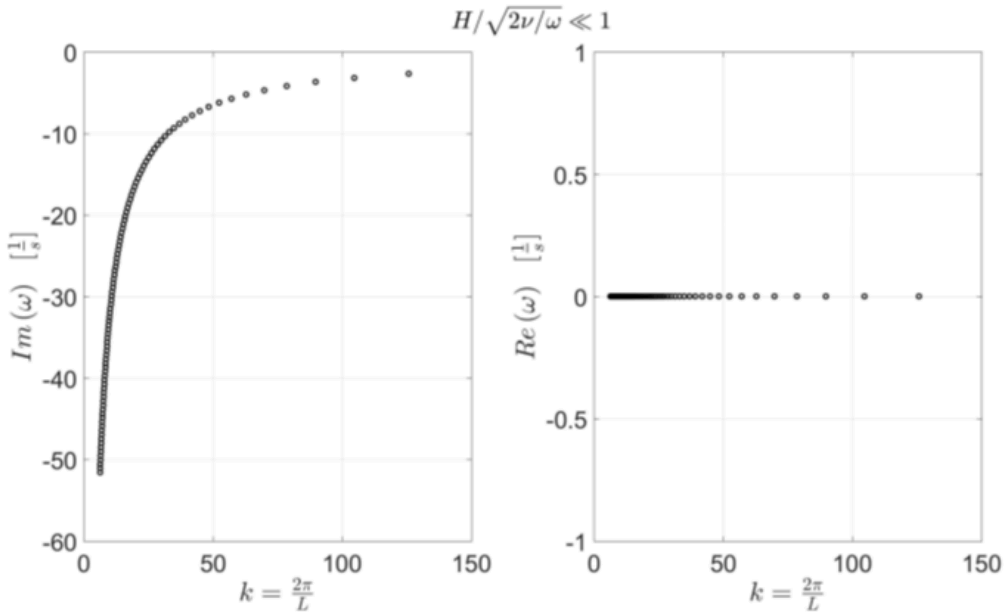

Figure 3: Possible time-evolutions.

- $H/l_{vb} \gg 1$ (marginally stable). Show that one retrieves

$$\omega^2 = \left(gk + \frac{\gamma}{\rho}k^3\right)kH \tag{13}$$

and comment the associated figure

- **Bonus question**: In intermediate cases, the values of ω are found numerically as reported in the following figure. Comment.

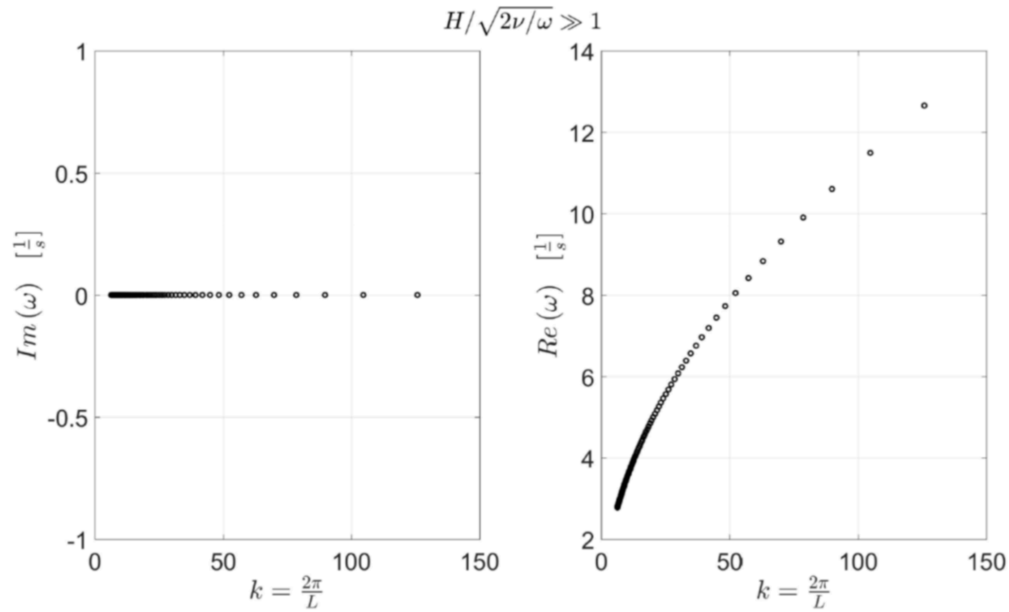

Figure 4: Possible time-evolutions.

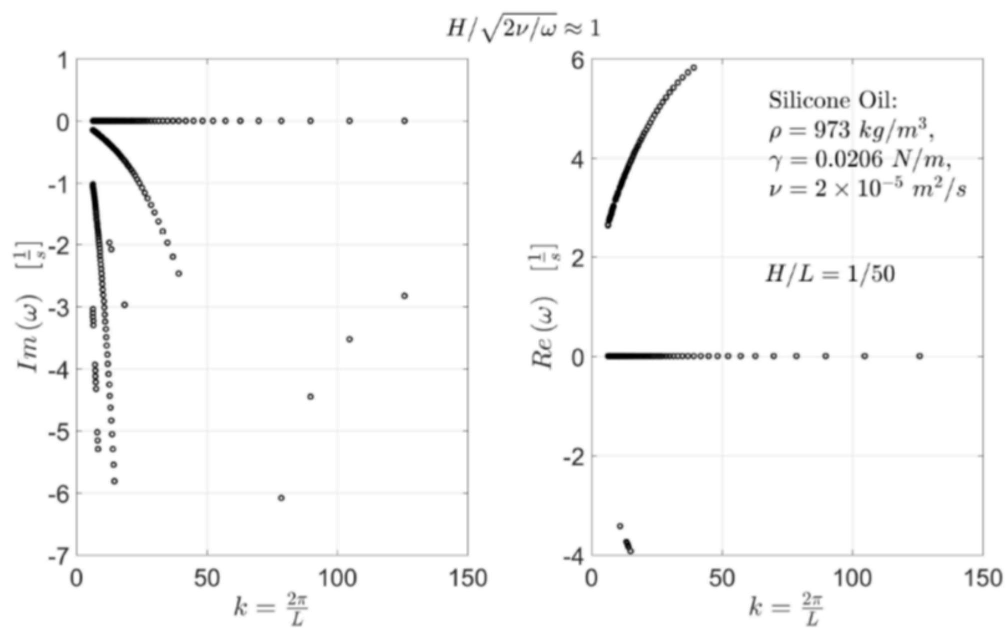

Figure 5: Possible time-evolutions.