Exercise 1

Shear flow around a cylinder.

In this problem you will study the flow around a fixed cylinder, whose axis is perpendicular to a shear flow (see picture below). The objective will be to determine the forces exerted by the fluid on the cylinder.

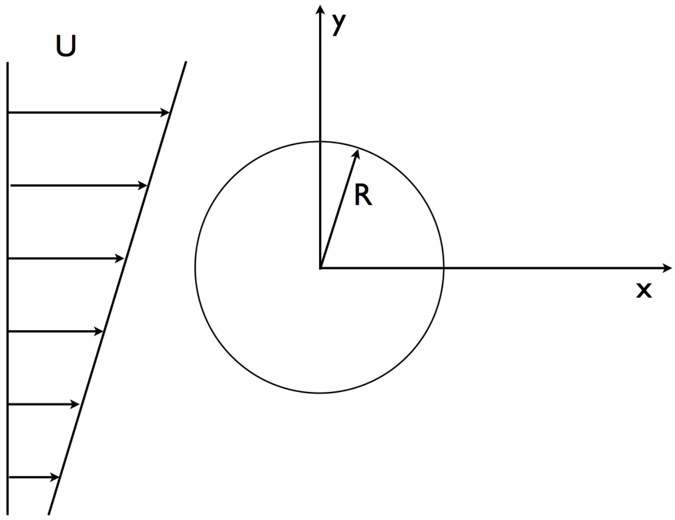

Figure 1

One supposes that the fluid is perfect, incompressible and stationary. It will also be planar in the x,y-plane. Gravitational force shall be neglected.

By definition $(\vec{e}_x, \vec{e}_y)$ are a pair of cartesian base vectors and $(\vec{e}_r, \vec{e}_\theta)$ polar base vectors. The position of a point in the plane shall be defined by $\vec{x} = x\vec{e}_x + y\vec{e}_y = r\vec{e}_r$. One denotes

$$\vec{u}(\vec{x}) = u_x(x,y)\vec{e}_x + u_y(x,y)\vec{e}_y = u_r(r,\theta)\vec{e}_r + u_\theta(r,\theta)\vec{e}_\theta,$$

the velocity vector, $p(\vec{x})$ the pressure and ρ the density of the fluid.

Part 1. General considerations

1. Recall the equations of fluid motion, that apply here.
2. It is supposed that the vorticity $\mathrm{rot}\ \vec{u} = \omega\vec{e}_z$ is constant everywhere in the flow. Then $\psi(x,y)$ being a stream function, defined in cartesian coordinates by the relations:

$$u_x = \partial\psi/\partial y, \quad u_y = -\partial\psi/\partial x.$$

Show that the acceleration term can be written as the derivative of a potential (the potential here is not the potential ϕ in potential flow theory). One shall recall that:

$$(\vec{u}\cdot\nabla)\vec{u} = (\mathrm{rot}\ \vec{u}) \times \vec{u} + \nabla(\frac{1}{2}||\vec{u}||^2).$$

3. Deduce that $\rho\omega\psi + \frac{1}{2}\rho||\vec{u}||^2 + p$ is constant in the flow.
4. Supposing that an obstacle described by its surface Σ and normal $\vec{n}$ is in the flow described above. Show that the force on the object is given by:

$$\vec{F} = \frac{\rho}{2} \int\int_{\Sigma} ||\vec{u}||^2 \vec{n}\, dS.$$

Part 2. Description of the flow

1. The velocity field is given by

$$\vec{U}(y) = (V + S\, y)\vec{e}_x,$$

   With V and S constant. Calculate its vorticity.

2. One places in this flow a cylinder with its axis oriented along the z-coordinate, the length L and the radius R. One is looking for the modification due to the presence of the cylinder. The velocity is supposed to take the form:

$$\vec{u} = \vec{U} + \nabla\phi,$$

   where $\phi(\vec{x})$ is the potential of the velocity field and such that $||\nabla\phi|| \to 0$ when $||\vec{x}|| \to \infty$. It is a priori not given that a potential solves this problem.
   Calculate the vorticity and show that $\Delta\phi = 0$.

3. Show that the boundary conditions on the surface Σ of the cylinder are written as:

$$\partial\phi/\partial r|_{\Sigma} = -V\cos(\theta) - \frac{1}{2} S\, R\, \sin(2\theta).$$

4. Calculate the laplacian of $\phi = \Gamma\theta/2\pi$ and $\phi A\ln(r)$, where Γ and A are constants.
5. Show without calculation that the following function satisfies $\Delta\phi = 0$:

$$\phi = \frac{\Gamma\theta}{2\pi} + A\ln(r) + B_i \frac{\partial}{\partial x_i}\ln(r) + C_{ij}\frac{\partial^2}{\partial x_i \partial x_j}\ln(r).$$

   Here, B_i (i=1,2) are constants of a vector $\vec{B}$ in $(\vec{e}_x, \vec{e}_y)$ and C_{ij} (i,j =1,2) those of a constant tensor $\bar{\bar{C}}$.

6. In polar coordinates one can show that the function ϕ obtained in the previous section is expressed as:

$$\phi = \frac{\Gamma\theta}{2\pi} + A\ln(r) + \frac{1}{r}(B_1\cos(\theta) + B_2\sin(\theta)) - \frac{1}{r^2}((C_{11} - C_{22})\cos(2\theta) + 2C_{12}\sin(2\theta)).$$

   Show that $C_{11} = C_{22}$ and determine A, B_1, B_2 and C_{12}.

7. Determine the components u_r and u_θ of the complete velocity field around the cylinder (one recalls that $\vec{u} = \vec{U} + \nabla\phi$).

Part 3. Force exerted on the cylinder

1. Show, and give explicitly the constants K_n, that:

$$(u_\theta^2)|_{\Sigma} = \sum_{n=0}^{4} K_n \sin^n(\theta).$$

2. By use of the result of 1.4, show that the component F_x of the force exerted by the fluid on the cylinder is zero.

3. Determine the component F_y. Discuss this result. One shall recall that:

$$\int_0^{2\pi} \sin^n(\theta) d\theta = 0 \text{ (n odd)}, \quad \int_0^{2\pi} \sin^2(\theta) d\theta = \pi, \quad \int_0^{2\pi} \sin^4(\theta) d\theta = \frac{3\pi}{4}.$$