Exercise 1

Pulse flow between two flat plates

One considers the 2D flow in the xy-plane between two parallel plates with a unidirectional flow field. The fluid is newtonian, homogeneous, incompressible and free of gravitational effects. The fluid is moved by imposing an oscillating pressure gradient in the x-direction with a constant amplitude A and a pulsation ω

$$\frac{dp}{dx} = Acos(\omega t). \tag{1}$$

We aim at describing the established flow regime, once the transient phenomena have vanished.

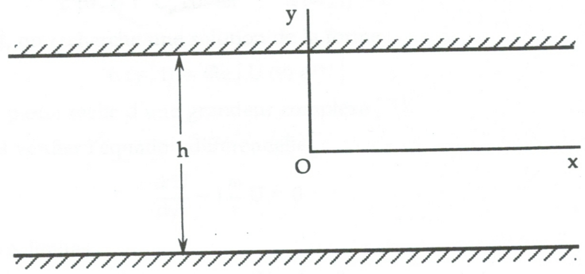

Figure 1. *Pulsed flow between two flat plates*

1. Simplify and non-dimensionalize the Navier-Stokes equations, using h as a gauge for the length and U for the velocity. Show that two dominant balances are possible, leading to a low or high frequency regime.
2. Solve the fully Navier-Stokes equations.
3. Consider the low frequency limit and solve the equations. Do you recognize any popular flow regime?
4. Show that in the opposed limit of high frequencies, the neglected term introduces a singular perturbation near the walls in a region, called the Stokes layer, the thickness of which you will determine.
5. Solve at first the exterior problem and then the problem in the boundary layer using matching conditions.

Exercise 2

A thermal boundary layer

We consider the flow in cylindrical pipe of radius R and axis (Oz). The flow is driven by a constant longitudinal pressure gradient $K = \mathrm{d}p/\mathrm{d}z$ and therefore adopts a classical Poiseuille profile.

The fluid is homogeneous, incompressible of density ρ, viscosity μ, specific heat c_p, and conductivity κ. The tube wall is maintained at constant temperature T_0 for $z < 0$ and T_w for $z > 0$ (figure 2).

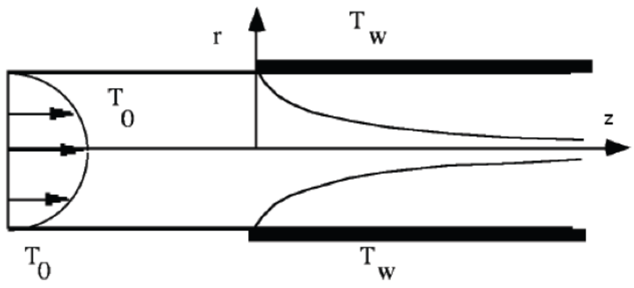

Figure 2

1. Determine the velocity field in the pipe as well as the temperature field when $T_w = T_0$.
2. Let us introduce the non-dimensional quantities $\bar{r}$, $\bar{z}$, $\bar{u}$, $\bar{T}$ defined as

$$r = R\bar{r}\,, \qquad z = R\bar{z}\,, \qquad u = U_0\bar{u}\,, \qquad T = T_0 + (T_w - T_0)\bar{T}\,,$$

   where U_0 is the maximal longitudinal velocity along (Oz). Write the heat equation as well as its boundary conditions in its non-dimensional form, using Reynolds, Prandtl and Eckhert numbers

$$Re = \frac{\rho U_0 R}{\mu}\,, \qquad Pr = \frac{\mu c_p}{\kappa}\,, \qquad E = \frac{U_0^2}{c_p \delta T}\,,$$

   What is the physical meaning of the Prandtl number? Show that the Eckert number quantifies the importance of the dissipated energy by viscous friction.
3. We now assume that the viscous production of heat is negligible, i.e. $E = 0$. Show that the solution of the heat equation obtained in the limit $PrRe \to \infty$ is not admissible, because it does not satisfy all boundary conditions.
4. We therefore consider a thermal boundary layer of unknown thickness ϵ at the pipe wall, and we introduce the interior variable $\tilde{r}$ defined as $\bar{r} = 1 - \epsilon\tilde{r}$. Give the equation at leading order satififsied by the inner solution $\tilde{T}(\tilde{r})$.
5. Use the principle of distinguished limit (principe de moindre dégénérescence), to determine the boundary layer thickness as well as the equation to be satisifed by the temperature in the inner region.
6. Show that this equation has a self-similar solution $\eta = rz^\alpha$, where α is a constant to be determined.
7. Find an integral solution of the resulting self-similar equation.

Navier-Stokes in cylindrical coordinates

Continuity

$$\frac{1}{r}\frac{\partial}{\partial r}(rU) + \frac{1}{r}\frac{\partial V}{\partial \theta} + \frac{\partial W}{\partial z} = 0$$

Momentum

$$\rho\left(\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial r} + \frac{V}{r}\frac{\partial U}{\partial \theta} + W\frac{\partial U}{\partial z} - \frac{V^2}{r}\right) = -\frac{\partial P}{\partial r} + \mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rU)\right) + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial V}{\partial \theta} + \frac{\partial^2 U}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial V}{\partial t} + U\frac{\partial V}{\partial r} + \frac{V}{r}\frac{\partial V}{\partial \theta} + W\frac{\partial V}{\partial z} + \frac{UV}{r}\right) = -\frac{1}{r}\frac{\partial P}{\partial \theta} + \mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rV)\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial U}{\partial \theta} + \frac{\partial^2 V}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial W}{\partial t}+U\frac{\partial W}{\partial r}+\frac{V}{r}\frac{\partial W}{\partial \theta}+W\frac{\partial W}{\partial z}\right)=-\frac{\partial P}{\partial z}+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial W}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 W}{\partial \theta^2}+\frac{\partial^2 W}{\partial z^2}\right)$$

Heat equation

$$\rho c_p\left(\frac{\partial T}{\partial t}+U\frac{\partial T}{\partial r}+\frac{V}{r}\frac{\partial T}{\partial \theta}+W\frac{\partial T}{\partial z}\right)=$$
$$\mu\left(\left(\frac{1}{r}\frac{\partial W}{\partial \theta}+\frac{\partial V}{\partial z}\right)^2+\left(\frac{\partial W}{\partial r}+\frac{\partial U}{\partial z}\right)^2+\left(r\frac{\partial}{\partial r}\left(\frac{V}{r}\right)+\frac{1}{r}\frac{\partial U}{\partial \theta}\right)^2\right)+\kappa\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}+\frac{\partial^2 T}{\partial z^2}\right)$$

Viscous stress tensor

$$\tau_{rr}=2\mu\frac{\partial U}{\partial r}\,,\quad \tau_{\theta\theta}=\frac{2\mu}{r}\left(\frac{\partial V}{\partial \theta}+U\right)\,,\quad \tau_{zz}=2\mu\frac{\partial W}{\partial z}\,,$$

$$\tau_{rz}=\tau_{zr}=\mu\left(\frac{\partial U}{\partial z}+\frac{\partial W}{\partial r}\right)\,,$$

$$\tau_{r\theta}=\tau_{\theta r}=\mu\left(\frac{1}{r}\frac{\partial U}{\partial \theta}+r\frac{\partial}{\partial r}\left(\frac{V}{r}\right)\right)\,,$$

$$\tau_{\theta z}=\tau_{z\theta}=\mu\left(\frac{\partial V}{\partial z}+\frac{1}{r}\frac{\partial W}{\partial \theta}\right)\,.$$