Exercise 1

The Lamb-Oseen vortex

1. We consider a planar (2D) flow, steady, incompressible and irrotational with circular streamlines centered around the origin. Show that such a flow can exist and determine the velocity vector at each point.

   Show that the circulation Γ of the flow field is constant no matter along which circle you integrate. Express the angular velocity v_ϑ as a function of the circulation Γ. What do you obtain if you try to obtain the circulation by domain integration?

   Hint:

$$\text{rot}(F) = \begin{pmatrix} \frac{1}{r}\frac{\partial F_z}{\partial \vartheta} - \frac{\partial F_\vartheta}{\partial z} \\ \\ \frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r} \\ \\ \frac{1}{r}\frac{\partial (r\,F_\vartheta)}{\partial r} - \frac{1}{r}\frac{\partial F_r}{\partial \vartheta} \end{pmatrix}.$$

$$\Gamma = \int_\Omega \text{rot}(\vec{v})dA = \oint_{\partial\Omega} \vec{v}ds.$$

2. Show that the obtained flow field, except at the origin, is an exact solution of the Navier-Stokes equations for a viscous and incompressible fluid. What can be said about the origin?

3. Consider now the flow field defined above as an initial condition of an unsteady flow. Calculate the unsteady solution for a flow where the angular velocity at the origin is zero, under the hypothesis that the streamlines are still represented by circles.

   - Start with simplification of the equation system using the hypothesis and non-dimensionalise.
   - Find the dilatation groups that lead to a self-similar solution. Introduce stretching groups $r = r^*\tilde{r}$, etc. Don't forget the initial condition. Transform the variables and the equation using the self-similar transform, where $v_\vartheta = r^\alpha F(r^\beta/t)$ (α and β need to be found).
   - Solve the resulting Ordinary Differential Equation.

4. Express the unsteady flow field in r and t. Is the flow field still irrotational? Express the rotation Γ as a function of the radius at different time steps. Comment.

Exercise 2

The Rankine vortex

In practice, for a slowly decaying vortex, the Lamb-Oseen vortex can be approximated by the Rankine vortex.

Let us consider the velocity field $\mathbf{u}$ of a fluid of constant density ρ.

$$\mathbf{u} = r\Omega\mathbf{e}_\theta, \quad r < a, \quad (1)$$

$$\mathbf{u} = a^2\Omega/r\mathbf{e}_\theta, \quad r \geq a. \quad (2)$$

where (r, θ, z) are the cylindrical coordinates.

1. Plot the flow field and the streamlines
2. Show that the flow is potential in the exterior of the cylinder of radius a.
3. Determine the pressure field in each region. At infinity, $P = P_0$.
4. What can be said about the quantity $u_\theta^2/2 + P/\rho$ in each domain? What can you deduce from it?
5. Determine the central depression of a cyclone with winds of maximal speed 180km/h.
6. Imagine that the vortex is in the ocean-water, deduce from the preceding results the shape of the water surface.

There exist other vortex solutions like the Taylor-Green vortex, which is an infinite array of vortices. These solutions are interesting to study fundamental mechanisms of vortex dynamics or stability. They provide also good test case for numerical algorithms.

Figure 1. *Two counter rotating vortices, induced by wing tips.*