Exercise 1

Spreading of a viscous mass under its own weight.

A mass of syrup is spreading on a horizontal floor in y=0 under the action of gravity (Figure 1). Syrup is assimilated to a homogeneous, incompressible and Newtonian fluid of density ρ and dynamic viscosity μ. The intensity of the gravity field is noted g. The external pressure is uniform and equal to p_0. Since syrup viscosity is very large, its motion occurs in a slow, quasi-static regime; there-

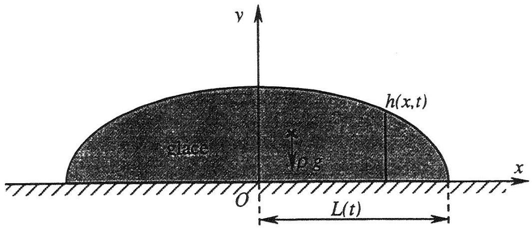

Figure 1

fore the fluid acceleration will be neglected throughout this problem. The dimension of the mass in the Oz-direction is assumed to be large enough for the flow to be considered as two-dimensional, i.e.:

$$\frac{\partial}{\partial z} = 0, \quad \boldsymbol{U} = u(x, y, t)\boldsymbol{e_x} + v(x, y, t)\boldsymbol{e_y}. \tag{1}$$

This mass spreads symmetrically in the Ox-direction, such that at time t it is occupying the volume defined by

$$0 \leq y \leq h(x, t) = h(-x, t) \quad \text{for } |x| \leq L(t), \tag{2}$$

where $h(x, t)$ is the syrup thickness at $\pm x$, and $L(t)$ is the syrup width on the floor; thus, by definition:

$$h(L(t), t) = 0; \qquad h(x, t) = 0 \quad \text{for } |x| \geq L(t). \tag{3}$$

Initial conditions are written as:

$$L(0) = L_0, \quad h(0, 0) = h_0. \tag{4}$$

The aim of the problem is to determine the time evolution of the syrup height $h(x, t)$ and width $L(t)$. The initial height-over-width ratio is assumed to be small,

$$\epsilon = \frac{h_0}{L_0} \ll 1, \tag{5}$$

and the following dimensionless parameters, of order 1, are introduced:

$$\bar{x} = \frac{x}{L_0}, \quad \bar{y} = \frac{y}{h_0}, \quad \bar{u} = \frac{u}{U}, \quad \bar{v} = \frac{v}{V}, \quad \bar{p} = \frac{p}{P}, \tag{6}$$

where $p(x, y, t)$ is the internal pressure, and U, V, P are gauges (characteristic velocities and pressure) to be determined later.

In the first part, simple dimensional and physical arguments allowed to find the asymptotic expression of $L(T)$. In this second part, we will use a more rigorous (and more difficult) method to establish to find the long-time expression of $L(t)$.

A physically motivated order of magnitude approach.

1. Write in *dimensionless* form the equations of fluid motion that apply here.
2. It is assumed that the pressure force in the vertical direction has the same order of magnitude as the gravity force, and that pressure stresses in the horizontal direction have the same order of magnitude as viscous stresses. Comparing the orders of magnitude of the different terms in the equations obtained in question 1. What are gauges that we have to chose?
3. Using these gauges and keeping in mind that ϵ is a small quantity, simplify the governing equation system from the Navier-Stokes equations using the dominant balance approach.
4. Express the characteristic horizontal velocity scale U as a function of the characteristic time T. Express volume conservation in terms of characteristic length scales. Combine these two expressions to obtain a relation for the temporal evolution of the characteristic syrup width, $L \sim T^{\alpha}$.

A rigorous mathematical approach using self-similarity

5. Taking into account that the free syrup/air surface is a material surface, show that:

$$v(x, h(x,t), t) = \frac{\partial h}{\partial t} + u(x, h(x,t), t)\frac{\partial h}{\partial x}. \tag{7}$$

   Then show from the continuity equation that:

$$\frac{\partial h}{\partial t} = -\frac{\partial}{\partial x}\int_0^{h(x,t)} u(x,y,t)\,dy. \tag{8}$$

6. Derive the expression at leading order of each component of the stress tensor $\boldsymbol{\sigma}$ as a function of pressure p and velocity $\boldsymbol{U}$.

   Show also, applying the dynamic boundary condition at the syrup/air surface and the boundary condition on the floor, that the pressure p and the horizontal velocity u must satisfy the following boundary conditions:

$$p(x, h(x,t), t) = p_0, \quad \frac{\partial u}{\partial y}(x, h(x,t), t) = 0, \quad u(x, 0, t) = 0 \quad \text{for } |x| \leq L. \tag{9}$$

7. From the equation system obtained in part 2 and boundary conditions (9), derive the expression of the pressure field $p(x,y,t)$ as a function of $h(x,t)$, and then the expression of the velocity field $u(x,y,t)$ as a function of the gradient $\partial h/\partial x$. Deduce that, at leading order, the syrup height $h(x,t)$ must satisfy the following differential equation:

$$\frac{\partial h}{\partial t} = k\frac{\partial^2 h^4}{\partial x^2}, \quad \text{where } k = \frac{g}{12\nu}, \tag{10}$$

   subject to the boundary condition:

$$\frac{\partial h}{\partial x}(0,t) = 0. \tag{11}$$

8. The motion of the syrup for long times is considered. In these conditions, the initial shape of the surface has little influence, and only the syrup volume $2q$ by unit length in the Oz direction is important. Therefore, volume conservation must be imposed:

$$\int_0^{L(t)} h(x,t)\,dx = q. \tag{12}$$

   Studying the invariants of the differential system (10)-(11)-(12), show that the solution $h(x,t)$ necessarily has the self-similar form:

$$\left(\frac{kt}{q^2}\right)^{\alpha} h = \bar{f}(\eta), \quad \eta = \frac{x}{(kq^3 t)^{\alpha}}, \tag{13}$$

   where α is a constant to be determined.

9. Show that the function $\bar{f}(\eta)$ has the form:

$$\bar{f}(\eta) = \frac{1}{2}\left(\frac{3}{5}\right)^{1/3}(\eta_0^2 - \eta^2)^{1/3}, \tag{14}$$

where η_0 is a constant characterized by the relation:

$$\int_0^{\eta_0} \bar{f}(\eta)\, d\eta = 1. \tag{15}$$

Deduce the expression of $L(t)$ for long times.