Drainage in a cylinder

We consider a thin liquid film coating the interior of a cylindrical cavity of radius R. The liquid has density ρ, dynamic viscosity μ and surface tension γ. The gravity acceleration on earth is denoted by g. Initially, at time $t = 0$, the film has an homogeneous thickness $\delta_0 \ll R$ (fig. 1a). At time progresses, gravity is going to pull the liquid down tangentially (fig. 1b) in a so-called drainage flow to eventually form a liquid pool. Actually, this is only true if the so called Bond number ($Bo = \rho g \delta_0 R \gamma^{-1}$) is sufficiently small (fig. 1c), but this is another story. We introduce the polar angle θ defined from the north pole and call $\delta(\theta, t)$ the film thickness. We want to exploit the aspect ratio $\delta/R = \epsilon \ll 1$ to obtain a lubrication equation. We denote by u and v the tangential (azimuthal) and radial velocities, which both depend on time, as well as polar coordinates r and θ.

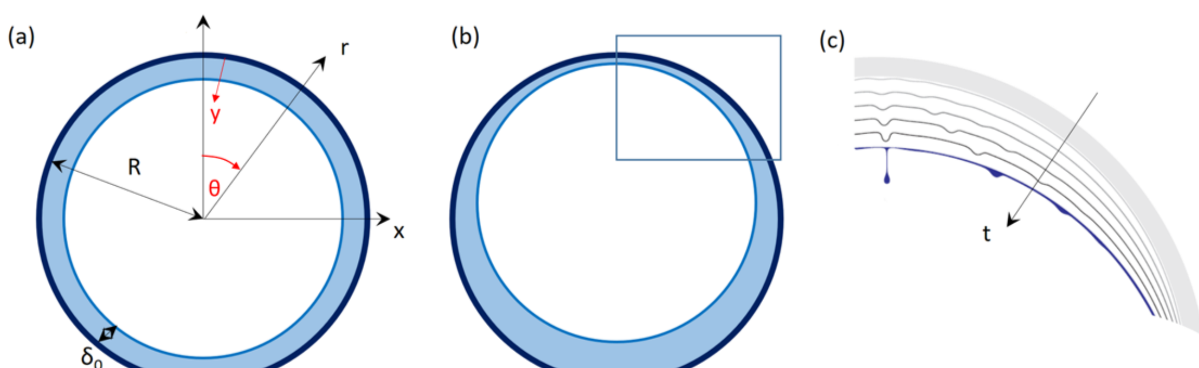

Figure 1: (a) Initial condition for the cylinder coating considered in this exercise and notations. (b) Evolution of the thickness distribution along the azimut. (c) For sufficiently large Bond numbers, a zoom of the upper right quadrant shows a cascade plot of the thickness and the dripping phenomenon.

1. Check that the Bond number is indeed a dimensionless number!

2. Write the continuity equation (use internet or a formulary to see that in radial coordinates there are 3 terms!) to show that the gauge of tangential (and radial) velocities U (and V) must follow the scaling relation

$$V = \frac{\delta_0}{R} U \tag{1}$$

3. What is the principle that was used to obtain this relation?
4. Why could the term $\frac{v}{r}$ be neglected? compare its gauge to that of $\frac{\partial v}{\partial r}$.
5. The azimuthal momentum equation writes

$$-\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial \theta} + \nu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial ru}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right) + g\sin(\theta) = 0 \tag{2}$$

We introduce the variable $y = R - r$, which spans the interval $[0;\delta]$ in the liquid film and has a typical scale δ_0, determine the appropriate pressure gauge P enabling the simplification of this equation into

$$-\frac{1}{\rho}\frac{1}{R}\frac{\partial p}{\partial \theta} + \nu\frac{\partial^2 u}{\partial y^2} + g\sin(\theta) = 0 \tag{3}$$

6. Show that the application of the dominant balance principle yields for the radial momentum equation

$$+\frac{1}{\rho}\frac{\partial p}{\partial y} - g\cos(\theta) = 0 \tag{4}$$

7. Introducing the curvature of the interface κ, one can obtain the following expression for the pressure.

$$p(y) = p_{atm} - \gamma\kappa + \rho g\cos(\theta)(y - \delta) \tag{5}$$

From which interface condition was this relation deduced? Which famous law taking into account surface tension was used?

8. Turning back into the azimuthal momentum equation, show that at leading order

$$0 = \frac{\gamma\kappa_\theta}{\rho R} + \nu\frac{\partial^2 u}{\partial y^2} + g\sin(\theta) + \frac{g\cos(\theta)\delta_\theta}{R} \tag{6}$$

where the index θ designates derivative with respect to θ. Which term has been neglected to arrive to this equation?

9. The velocity field can be integrated into

$$u = \left(\frac{\gamma\kappa_\theta}{\mu R} + \frac{\rho g\sin(\theta)}{\mu} + \frac{\rho g\cos(\theta)\delta_\theta}{\mu R}\right)(y(\delta - y/2)) \tag{7}$$

What are the boundary conditions for the velocity field which enabled to obtain this expression?

10. Using the kinematic boundary condition at the interface, one obtains the following lubrication equation

$$\delta_t = -\frac{1}{3\mu R}\left(\delta^3\left(\frac{\gamma\kappa_\theta}{R} + \rho g\sin(\theta) + \frac{\rho g\cos(\theta)\delta_\theta}{R}\right)\right)_\theta \tag{8}$$

Show that since $\delta \ll R$ and assuming $Bo = \mathcal{O}(1)$ that this equation simplifies into

$$\delta_t = -\frac{\rho g}{3\mu R}\left(\delta^3\sin(\theta)\right)_\theta \tag{9}$$

11. What is the characteristic time-scale τ of drainage?

12. We make the problem dimensionless with $\delta = \delta_0 \tilde{\delta}$ and $t = \tau \tilde{t}$ and now focus on the north pole region $\theta \ll 1$. We look for a solution of separate variables $\tilde{\delta}(\theta, \tilde{t}) = f(\tilde{t})(1 + a\theta^2)$. By remembering the Taylor expansion of $\sin(\theta) \approx \theta - \frac{\theta^3}{6}$ show that $a = \frac{1}{16}$ and write the ordinary differential equation governing $f(\tilde{t})$.

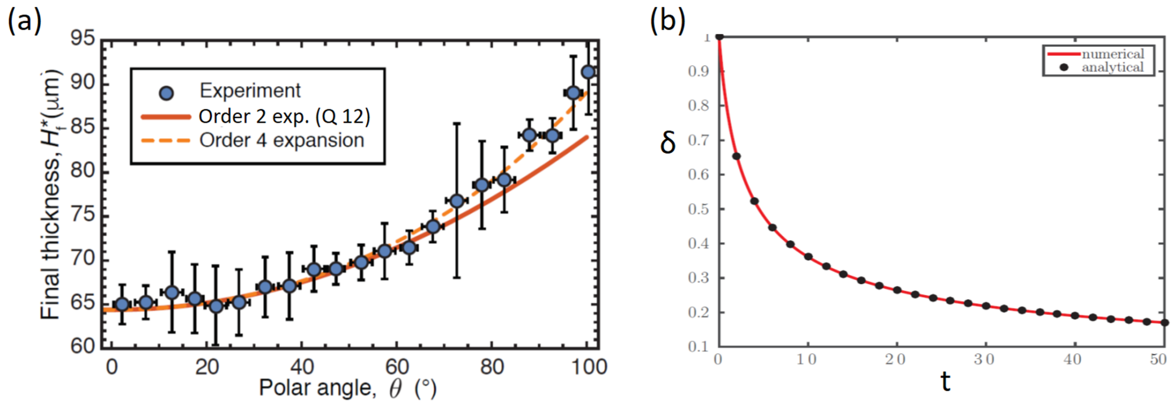

Figure 2: (a) Azimutal thickness distribution at a given time-instant given by experimental measurements, the solution found in question 12 and a refined 4-th order expansion. (b)Approximate non dimensional evolution of the thickness at the pole $f(\tilde{t})$ and its value from the numerical resolution of eq. (8).

13. The solution
$$f(\tilde{t}) = (1 + \frac{2\tilde{t}}{3})^{-\frac{1}{2}} \tag{10}$$
is shown in figure 2b together with the solution at the north pole from the full differential equation (8). What do you think of the comparison?

14. The second order approximation $f(\tilde{t})(1 + a\theta^2)$ is compared at a specific time to experiments and to a refined approximation at fourth order in θ in figure 2a. What do you think about the comparison?