Exercise 1

The Corner Eddies of Keith Moffatt

Corner eddies appear almost everywhere in real fluid motion.

Suppose that a fluid is contained between fixed boundaries $\theta = \pm\alpha$, and that a flow is driven by the application of some stirring mechanism far from the corner. For example, we might rotate a cylinder far from the corner as shown in figure 1. What is the nature of the flow near $r = 0$?

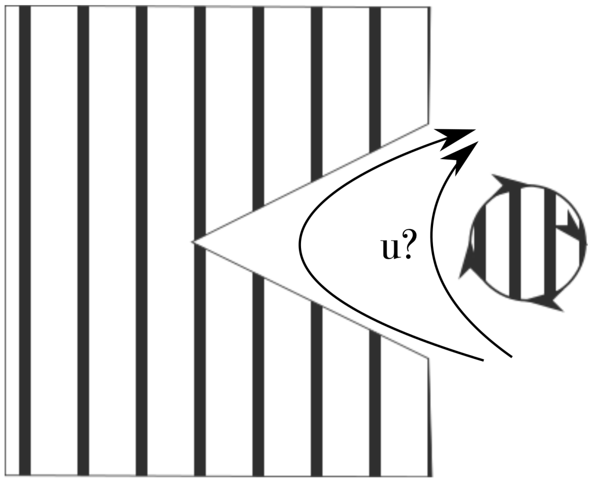

Figure 1. *Corner flow induced by remote stirring.*

It is natural to assume that the streamfunction near $r = 0$ takes the form

$$\psi = r^{\lambda} f(\theta), \tag{1}$$

where, in order to ensure that the velocity vanishes as $r \to 0$, real part of lambda $Re(\lambda) > 1$. If (as indeed turns out to be the case) λ is complex, then the real part of eq. 1 must be understood. We shall assume that the radial velocity $u = r^{-1}\partial\psi/\partial\theta$ is anti-symmetric about $\theta = 0$ (as in figure 1); then $f(\theta)$ is an even function of θ:

$$f(\theta) = f(-\theta). \tag{2}$$

The flow problem is solved if:

- ψ is a solution of the double Laplacian

$$\Delta^2\psi = 0, \tag{3}$$

- ψ is symmetric along the center plane at $\theta = 0$
- and fulfills the no-slip boundary conditions at $\theta = \pm\alpha$,

$$u = \frac{\partial\psi}{r\partial\theta} = 0, \quad \text{and} \quad v = -\frac{\partial\psi}{\partial r} = 0. \tag{4}$$

In order to solve the flow field near a corner use the following questions as a guideline:

1. As an option you could try to solve this problem with your prefered[1] flow solver numerically to get an idea.

2. Apply the Laplacian once to the Ansatz (eq. 1) and reformulate the result into: $r^\gamma F(\theta)$. The Laplacian in cylindrical coordinates is:

$$\Delta\psi = \frac{1}{r}\left(\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right)\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}.$$

3. Apply the Laplacian once more to $r^\gamma F(\theta)$. Solve the differential equation for $F(\theta)$.

4. Using the definition of $F(\theta)$ in terms of $f(\theta)$ and λ solve the differential equation for $f(\theta)$. Determine λ by applying the boundary conditions $u = v = 0$ at $\theta \pm \alpha$.

   Hint: $(\lambda - 2)\sin(a)\cos(b) - \lambda\cos(a)\sin(b) =$
   $(\lambda - 1)(\sin(a)\cos(b) - \cos(a)\sin(b)) - (\sin(a)\cos(b) + \cos(a)\sin(b)) =$
   $(\lambda - 1)\sin(a - b) - \sin(a + b)$,
   using the trigonometric summation identity: $\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$.

5. Plot each of the two terms of the solution in one graph for a fixed α. Can you predict at what angle α corner eddies start to appear?

6. Calculate for some complex solution ($\lambda = p + i\,q$) the vertical velocity component at the center line, $v(r, \theta = 0)$. What can be said about the length of the eddy cells and the decay of the velocity magnitude?

7. Plot a flow field.

[1] If you have none, I believe that Freefem++ can do this quite easily if you modify the cavity example.