Exercise 1

We consider here two coaxial cylinders of respective radii $R_1 < R_2$ and of infinite length along their revolution axis $(0z)$, as depicted in Figure 1. The region enclosed between both cylinders is filled with an incompressible, homogeneous fluid of viscosity μ and density ρ. Buoyancy is neglected in the following. Both cylinders rotate with respective angular velocities Ω_1 and Ω_2. The flow motion is assumed to be steady and two-dimensional, with revolution symmetry. Using standard cylindrical coordinates, the velocity and pressure fields are thus noted $\boldsymbol{U}(r,z) = U\boldsymbol{e_r} + V\boldsymbol{e_\theta}$ and $P(r,z)$.

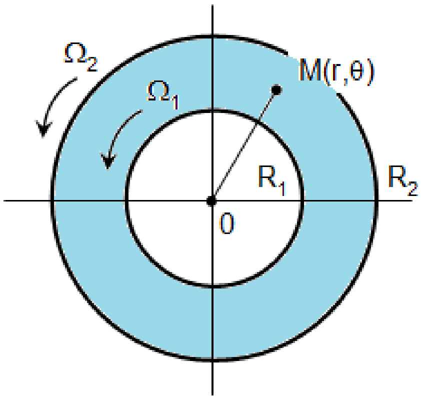


Figure 1

1. Show that the velocity components and the pressure field are functions of r only.
2. Determine the analytical expressions of the velocity and pressure fields.

We assume now that the outer cylinder remains fixed.

3. Determine the corresponding velocity field.
4. Determine the force per unit length exerted by the fluid on the internal wall of the outer cylinder.
5. Determine the corresponding moment with respect to the cylinder revolution axis.
6. Using the preceding results, propose a solution to determine the fluid viscosity, assuming that one knows the angular velocity of the inner cylinder and the resistive couple allowing to cancel the angular velocity of the outer cylinder.

We assume now that the radius of the fixed outer cylinder is arbitrarily large, i.e. $R_2 \to \infty$.

7. Determine the corresponding velocity field.
8. Show that the flow is irrotational.

We consider another limit case for which the radius of the inner cylinder vanishes, i.e. $R_1 \to 0$.

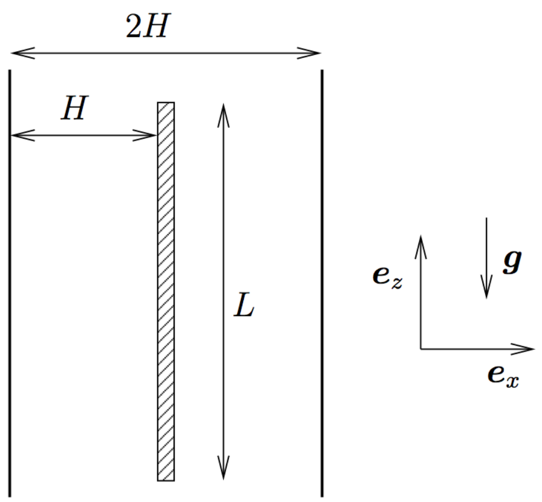
9. Determine the corresponding velocity field and show that the flow is in solid rotation.

10. Is the flow rotational or irrotational? Propose a physically motivated definition for rotational and irrotational flows.

Exercise 2

A Plate falling in a channel

A plate of negligible thickness, mass m and length L, falls at a terminal velocity U_0 at the center of a canal of width $2H$. We set $2H \ll L$, so that a fully developed flow can be considered in the streamwise direction. The canal is filled with a newtonian fluid of constant density and viscosity. We solve the problem in 2D.



1. Determine the velocity field.
2. Express the pressure gradient $\frac{\partial p}{\partial z}$ as a function of the variables and reexpress the velocity field.
3. Determine the force exerted on the plate.
4. Determine the velocity U_0 as a function of m, g, H, μ and L.
5. When now the thickness of the plate cannot be assumed as negligible, what are the limitations of the previous reasoning.

Navier-Stokes equations in Cartesian coordinates

Continuity

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0$$

Momentum

$$\rho\left(\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} + W\frac{\partial U}{\partial z}\right) = -\frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial V}{\partial t} + U\frac{\partial V}{\partial x} + V\frac{\partial V}{\partial y} + W\frac{\partial V}{\partial z}\right) = -\frac{\partial P}{\partial y} + \mu\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial W}{\partial t} + U\frac{\partial W}{\partial x} + V\frac{\partial W}{\partial y} + W\frac{\partial W}{\partial z}\right) = -\frac{\partial P}{\partial z} + \mu\left(\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} + \frac{\partial^2 W}{\partial z^2}\right)$$

Viscous stress tensor in Cartesian coordinates

$$\tau_{xx} = 2\mu\frac{\partial U}{\partial x}\,, \quad \tau_{yy} = 2\mu\frac{\partial V}{\partial y}\,, \quad \tau_{zz} = 2\mu\frac{\partial W}{\partial z}\,,$$

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right)\,,$$

$$\tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial U}{\partial z} + \frac{\partial W}{\partial x}\right)\,,$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial V}{\partial z} + \frac{\partial W}{\partial y}\right).$$

Navier-Stokes equations in cylindrical coordinates

Continuity

$$\frac{1}{r}\frac{\partial}{\partial r}(rU) + \frac{1}{r}\frac{\partial V}{\partial \theta} + \frac{\partial W}{\partial z} = 0$$

Momentum

$$\rho\left(\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial r} + \frac{V}{r}\frac{\partial U}{\partial \theta} + W\frac{\partial U}{\partial z} - \frac{V^2}{r}\right) = -\frac{\partial P}{\partial r} + \mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rU)\right) + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial U}{\partial \theta} + \frac{\partial^2 U}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial V}{\partial t} + U\frac{\partial V}{\partial r} + \frac{V}{r}\frac{\partial V}{\partial \theta} + W\frac{\partial V}{\partial z} + \frac{UV}{r}\right) = -\frac{1}{r}\frac{\partial P}{\partial \theta} + \mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rV)\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial V}{\partial \theta} + \frac{\partial^2 V}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial W}{\partial t} + U\frac{\partial W}{\partial r} + \frac{V}{r}\frac{\partial W}{\partial \theta} + W\frac{\partial W}{\partial z}\right) = -\frac{\partial P}{\partial z} + \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial W}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 W}{\partial \theta^2} + \frac{\partial^2 W}{\partial z^2}\right)$$

Viscous stress tensor in cylindrical coordinates

$$\tau_{rr} = 2\mu\frac{\partial U}{\partial r}\,, \quad \tau_{\theta\theta} = \frac{2\mu}{r}\left(\frac{\partial V}{\partial \theta} + U\right)\,, \quad \tau_{zz} = 2\mu\frac{\partial W}{\partial z}\,,$$

$$\tau_{rz} = \tau_{zr} = \mu\left(\frac{\partial U}{\partial z} + \frac{\partial W}{\partial r}\right)\,,$$

$$\tau_{r\theta} = \tau_{\theta r} = \mu\left(\frac{1}{r}\frac{\partial U}{\partial \theta} + r\frac{\partial}{\partial r}\left(\frac{V}{r}\right)\right)\,, \quad \tau_{\theta z} = \tau_{z\theta} = \mu\left(\frac{\partial V}{\partial z} + \frac{1}{r}\frac{\partial W}{\partial \theta}\right)\,.$$