Exercise 1

A Newtonian, homogeneous, incompressible liquid of density ρ and dynamic viscosity μ drains along an inclined plane of angle α, as depicted in Fig. 1. We aim here at determining the flow features within the liquid layer, assuming that:

- the thickness of the liquid layer is constant and equal to h,
- the flow is steady and two-dimensional in the Oxy-plane, with a velocity field $\boldsymbol{u} = u\boldsymbol{e_x} + v\boldsymbol{e_y}$,
- the motion induced in the air by the displacement of the liquid is negligible,
- the air pressure above the interface is constant and equal to p_a.

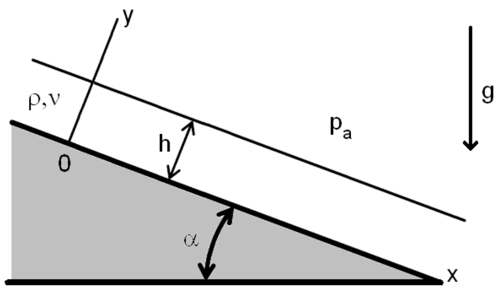

Figure 1

1. Without doing any calculations, discuss which of the velocity profiles shown in Fig. 2 may represent the fluid motion better.

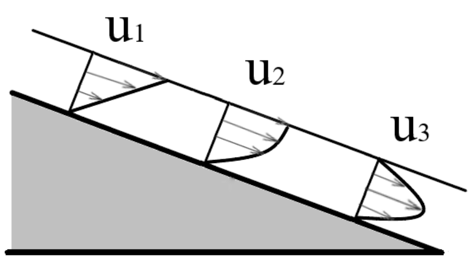

Figure 2

2. Determine all fields u, v and p, assuming all unknown quantities are functions of y only. Plot the corresponding profiles in the liquid layer.
3. Compute the volumetric flow rate Q per unit length in z.

Exercise 2

We consider the flow in the pipe of radius R and length $L \gg R$ shown in Figure 3. In the following, ρ and μ are the density and viscosity of the fluid, $\boldsymbol{V} = U\boldsymbol{e_r} + V\boldsymbol{e_\theta} + W\boldsymbol{e_z}$ is the velocity field and P is the pressure. We neglect buoyancy, and the motion is assumed to be steady and parallel to the (Oz) axis (the flow is fully developed).

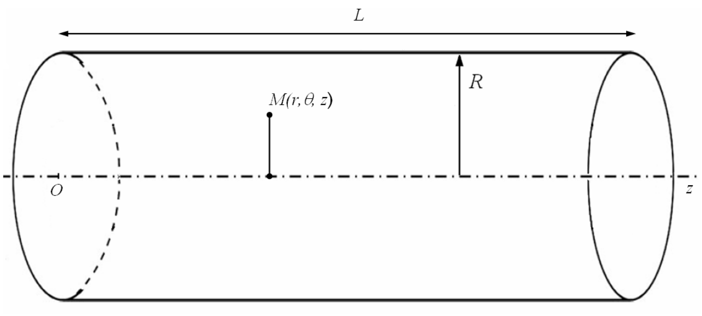

Figure 3

1. Show that the pressure gradient in the z direction is constant. In the following, we will use $\beta = \partial p/\partial z$ to ease the notation.
2. Determine the velocity profiles and the pressure distribution in the pipe. What do you think about the sign of β? What does it mean, physically speaking?
3. Determine the viscous friction force exerted by the fluid on the pipe wall.
4. Is momentum conserved? Propose an explanation by examining all forces at work.
5. Here we investigate the effect of the viscous dissipation on the temperature of the fluid. Determine the temperature profile in a cross-section, assuming the pipe wall is maintained at a constant temperature T_w. How do the temperatures at the center and at the wall compare? Are you surprised and why? ("*No*" is not an option). Propose an explanation.

For hydraulic applications, it is of crucial importance to estimate the pressure drop ΔP that occurs owing to viscous friction at the pipe wall.

6. Determine the pressure drop as a function of the volumetric flow rate Q.
7. Compute the pressure drop for water flowing in a pipe of radius $R = 45mm$, length $L = 100m$, the flow rate being $250L/mn$ (operating conditions for a fire hose).

We assume now that the fluid does not flow through a single pipe, but through a network of n identical pipes of small radius $a \ll R$. Both n and a are chosen for the overall surface of the cross-section to be identical in both cases. (Figure 4).

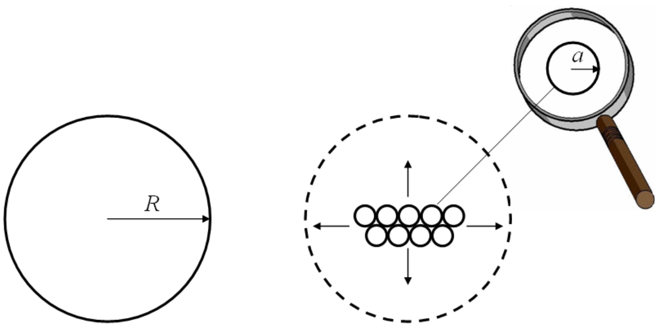

Figure 4

8. Compute the pressure distribution required in each pipe for the total volumetric flow rate to be equal to that achieved using the pipe studied in the preceding questions. How do the corresponding pressure gradients compare? Comment your results.

To avoid significant pressure drops, it may therefore be tempting to use pipes of large diameters. Such a solution is not easily tractable in practice, as one has to pay attention to the weight of the fluid flowing in the pipe (think about the fire hose). Another solution is to reduce the viscous friction, which can be achieved by means of a second fluid of lower viscosity confined close to the pipe wall. Such a configuration is sketched in Figure 5, where μ_1 and $\mu_2 \geq \mu_1$ now stand for the viscosities of the outer and inner fluids, assumed to have the same density, and h stands for the position of the interface between both immiscible fluids. Again, the motion is assumed to be steady and one-dimensional along the $(0z)$ axis, in particular the position of the interface does not vary along z.

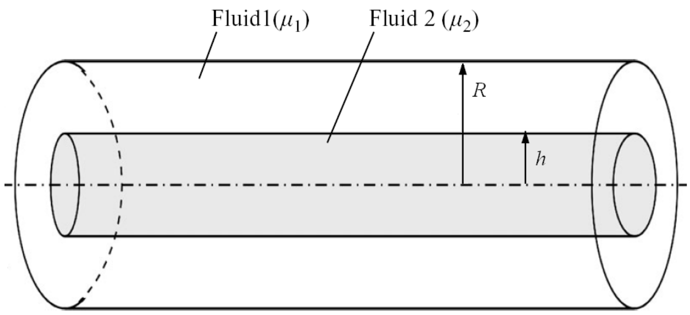

Figure 5

We assume that the flow rate of the inner fluid Q_2 is identical to the one determined in question 6., i.e. that of the fluid 2 alone, flowing through the whole pipe. The flow rate of the outer fluid is chosen as $Q_1 = \alpha Q_2$, with α a free fraction parameter.

9. Determine the pressure and velocity profiles. Show in particular that the position of the interface can be fully determined as a function of the problem parameters (no analytical expression required).

10. Compare the viscous friction force exerted on the pipe wall to that determined in question 3. This is an open question, investigate by yourselves for different parameter values. What happens if α is too high?

Exercise 3

Impinging jet

One wants to know the stream lines of a fluid jet that flows against a plate. As a first approximation use a homogeneous velocity profile: $v = -V_0 y$, as in figure 6(a). Remember, streamlines need incompressibility to be verified.

In a second approach it can be shown that far from a jet outflow the velocity profile v has more likely the form: $v = -V_0 y \cosh^{-2}(x)$, see figure 6(b).

Determine the streamlines for the two types of velocity profiles, (a) and (b).

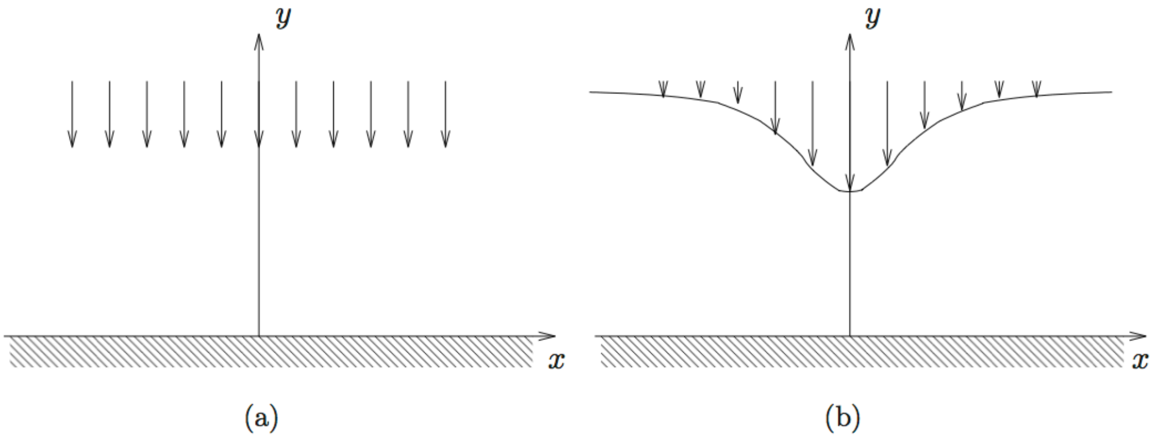

Figure 6. *Jets with a uniform velocity profile as in (a) and with a more realistic profile (b).*

Navier-Stokes equations in Cartesian coordinates

Continuity

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0$$

Momentum

$$\rho\left(\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} + W\frac{\partial U}{\partial z}\right) = -\frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial V}{\partial t} + U\frac{\partial V}{\partial x} + V\frac{\partial V}{\partial y} + W\frac{\partial V}{\partial z}\right) = -\frac{\partial P}{\partial y} + \mu\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial W}{\partial t} + U\frac{\partial W}{\partial x} + V\frac{\partial W}{\partial y} + W\frac{\partial W}{\partial z}\right) = -\frac{\partial P}{\partial z} + \mu\left(\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} + \frac{\partial^2 W}{\partial z^2}\right)$$

Viscous stress tensor in Cartesian coordinates

$$\tau_{xx} = 2\mu\frac{\partial U}{\partial x}\,, \quad \tau_{yy} = 2\mu\frac{\partial V}{\partial y}\,, \quad \tau_{zz} = 2\mu\frac{\partial W}{\partial z}\,,$$

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right)\,,$$

$$\tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial U}{\partial z} + \frac{\partial W}{\partial x}\right)\,,$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial V}{\partial z} + \frac{\partial W}{\partial y}\right)\,.$$

Navier-Stokes equations in cylindrical coordinates

Continuity

$$\frac{1}{r}\frac{\partial}{\partial r}(rU) + \frac{1}{r}\frac{\partial V}{\partial \theta} + \frac{\partial W}{\partial z} = 0$$

Momentum

$$\rho\left(\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial r} + \frac{V}{r}\frac{\partial U}{\partial \theta} + W\frac{\partial U}{\partial z} - \frac{V^2}{r}\right) = -\frac{\partial P}{\partial r} + \mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rU)\right) + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial V}{\partial \theta} + \frac{\partial^2 U}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial V}{\partial t} + U\frac{\partial V}{\partial r} + \frac{V}{r}\frac{\partial V}{\partial \theta} + W\frac{\partial V}{\partial z} + \frac{UV}{r}\right) = -\frac{1}{r}\frac{\partial P}{\partial \theta} + \mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rV)\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial U}{\partial \theta} + \frac{\partial^2 V}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial W}{\partial t} + U\frac{\partial W}{\partial r} + \frac{V}{r}\frac{\partial W}{\partial \theta} + W\frac{\partial W}{\partial z}\right) = -\frac{\partial P}{\partial z} + \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial W}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 W}{\partial \theta^2} + \frac{\partial^2 W}{\partial z^2}\right)$$

Energy

$$\rho c_v\left(\frac{\partial T}{\partial t} + U\frac{\partial T}{\partial r} + \frac{V}{r}\frac{\partial T}{\partial \theta} + W\frac{\partial T}{\partial z}\right) =$$

$$\mu\left(\left(\frac{1}{r}\frac{\partial W}{\partial \theta} + \frac{\partial V}{\partial z}\right)^2 + \left(\frac{\partial W}{\partial r} + \frac{\partial U}{\partial z}\right)^2 + \left(r\frac{\partial}{\partial r}\left(\frac{V}{r}\right) + \frac{1}{r}\frac{\partial U}{\partial \theta}\right)^2\right) + \kappa\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2}\right)$$

Viscous stress tensor in cylindrical coordinates

$$\tau_{rr} = 2\mu\frac{\partial U}{\partial r}\,, \quad \tau_{\theta\theta} = \frac{2\mu}{r}\left(\frac{\partial V}{\partial \theta} + U\right)\,, \quad \tau_{zz} = 2\mu\frac{\partial W}{\partial z}\,,$$

$$\tau_{rz} = \tau_{zr} = \mu\left(\frac{\partial U}{\partial z} + \frac{\partial W}{\partial r}\right)\,,$$

$$\tau_{r\theta} = \tau_{\theta r} = \mu\left(\frac{1}{r}\frac{\partial U}{\partial \theta} + r\frac{\partial}{\partial r}\left(\frac{V}{r}\right)\right)\,,$$

$$\tau_{\theta z} = \tau_{z\theta} = \mu\left(\frac{\partial V}{\partial z} + \frac{1}{r}\frac{\partial W}{\partial \theta}\right)\,.$$