Von Karman disk

Let consider a plane disk of large radius R which is made to rotate in its own plane with a steady angular velocity Ω immerse in a fluid which is initially at rest everywhere. The relative motion of the disk and the fluid sets up viscous stresses, which tend to drag the fluid round the disk. After the transient a steady motion around the disk is achieved. In the limit of infinite disk ($R \to \infty$), Von Karman derived a self-similar solution for the resulting steady motion, which you will derive here step-by-step.

Let define u, v and w the velocity components parallel to the (r,θ,z) coordinates in a cylindrical system with z corresponding to the axis of the disk and r and θ standing in the disk plane.

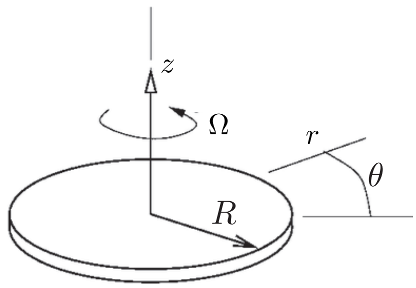

FIGURE 1 –

1. [1pt.] Write the boundary conditions to be imposed at the disk surface.

We now look for a solution such that u/r, v/r and w are function of z alone.

2. [2pt.] With this assumption, show that from the steady z-momentum the following equation for the pressure p is obtained :
$$\frac{p}{\rho} = \nu \frac{\mathrm{d}w}{\mathrm{d}z} - \frac{1}{2} w^2 + F(r) \tag{1}$$
where F is a function of r alone.
Since there is no rotation of the fluid and radial motion far form the disk, p must be independent of r when z is large, hence what you can conclude about $F(r)$? Consequently, what can you say about the pressure distribution in space ?
3. [2pt.] Show that the continuity equation reads :
$$\frac{2u}{r} + \frac{\mathrm{d}w}{\mathrm{d}z} = 0 \tag{2}$$
4. [2pt.] Show that the r and θ momentum reduce to :
$$\begin{aligned} \left(\frac{u}{r}\right)^2 + w \frac{\mathrm{d}u/r}{\mathrm{d}z} - \left(\frac{v}{r}\right)^2 &= \nu \frac{\mathrm{d}^2 u/r}{\mathrm{d}z^2} \\ \frac{2uv}{r^2} + w \frac{\mathrm{d}v/r}{\mathrm{d}z} &= \nu \frac{\mathrm{d}^2 v/r}{\mathrm{d}z^2} \end{aligned} \tag{3}$$

5. [4pt.] Consider now the system of equations given by continuity (2), r and θ momentum (3), and the boundary conditions at the disk surface you derived at question 1. Let introduce in the system the following dilatations :

$$u = Uu', \; v = Vv', \; w = Ww', r = Rr', \; z = Zz', \; \nu = N\nu', \; \Omega = O\Omega', \tag{4}$$

and determine the conditions among dilation groups which has to be satisfied to obtain a self-similar solution.

6. [4pt.] Show that the solution for u, v, w has the self-similar form :

$$v/r = \Omega g(\zeta), \quad w = \sqrt{\Omega\nu} h(\zeta), \quad u/r = -\frac{1}{2}\Omega h' \tag{5}$$

where the self-similar variable ζ is equal to $z/\sqrt{\nu/\Omega}$. Note as the quantity $\sqrt{z/\Omega}$ is the so-called Ekman layer and is a measure of the vorticity layer thickness.
Determine the corresponding self-similar system of equations.

7. [3pt.] The obtained self-similar system can be integrated numerically and the solution applies strictly to an infinite disk. However in the case of a finite disk, provided the Ekman layer is small compared with the disk radius, $\sqrt{\nu/\Omega}/R \ll 1$, it is reasonable to suppose the effect of the edge of the disk is small. Therefore, determine the expression of the torque exerted by the fluid on both sides of the disk.

8. [2pt.] The quantities g, h and h' are reported in figure 2 as function of the self-similar variable ζ. Is the flow at rest far from the disk ? Explain why.

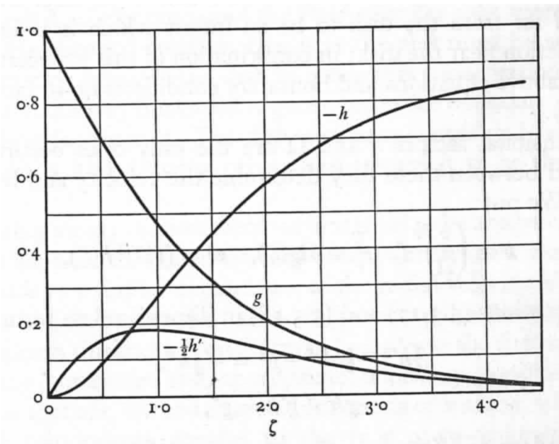

FIGURE 2 –

Stokes disk

Let consider a plane disk of large radius R surrounded by a fluid of viscosity μ and density ρ. The disk executes rotary oscillations about its axis with angle of rotation $\phi(t) = \phi_0 \cos(\omega t)$. We consider here the case of small amplitude oscillations $\phi_0 \ll 1$.
Let define u, v and w the velocity components parallel to the (r,θ,z) coordinates in a cylindrical system with z corresponding to the axis of the disk and r and θ standing in the disk plane.

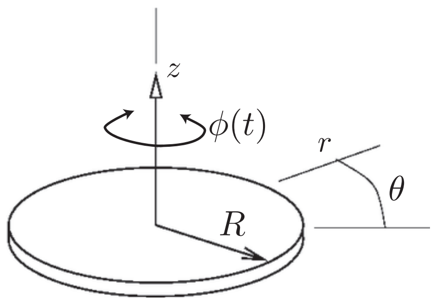

FIGURE 3 –

1. [1pt.] Write the boundary conditions to be imposed at the disk surface.
2. [1pt.] Show that in the limit of small oscillations $\phi_0 \ll 1$ the nonlinear convective term in the Navier-Stokes equations is negligible, whatever the frequency ω.
3. [2pt.] We now seek for an axisymmetric solution of the type $u = w = 0$ and $v = r\Omega(z,t)$. Show that the governing equation for $\Omega(z,t)$ is :

$$\frac{\partial \Omega}{\partial t} = \nu \frac{\partial^2 \Omega}{\partial z^2}, \tag{6}$$

   where ν is the fluid's kinematic viscosity $\nu = \mu/\rho$. What can you conclude about the pressure field ?
4. [4pt.] Let consider the governing equation (6) with the appropriate boundary conditions at $z = 0$ and $z = \infty$ and solve for $\Omega(z,t)$. (Hint : use the modal expansion $\Omega = \hat{\Omega}(z)e^{i\omega t} + c.c.$, where $c.c.$ is the complex conjugate)
5. [2pt.] The quantity $\delta = \sqrt{\nu/\omega}$ is called the Stokes layer. By looking at the expression you derived for $\Omega(z,t)$, what is the physical meaning of δ ?
6. [2pt.] Determine the instantaneous torque $M_z(t)$ exerted by the fluid on both sides of the disk.
7. [1pt.] Draw the instantaneous angle of rotation $\phi(t)$ and the torque $M_z(t)$ as a function of time. Are they synchronized ?
8. [2pt.] Determine the average of the energy dissipation over one period of oscillation.

Navier-Stokes equations in cylindrical coordinates

Continuity

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0$$

Momentum

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \theta} + w\frac{\partial u}{\partial z} - \frac{v^2}{r}\right) = -\frac{\partial P}{\partial r} + \mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(ru)\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial u}{\partial \theta} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \theta} + w\frac{\partial v}{\partial z} + \frac{uv}{r}\right) = -\frac{1}{r}\frac{\partial P}{\partial \theta} + \mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv)\right) + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial v}{\partial \theta} + \frac{\partial^2 v}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial \theta} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial P}{\partial z} + \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

Viscous stress tensor in cylindrical coordinates

$$\tau_{rr} = 2\mu\frac{\partial u}{\partial r}\,, \quad \tau_{\theta\theta} = \frac{2\mu}{r}\left(\frac{\partial v}{\partial \theta} + u\right)\,, \quad \tau_{zz} = 2\mu\frac{\partial w}{\partial z}\,,$$

$$\tau_{rz} = \tau_{zr} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)\,,$$

$$\tau_{r\theta} = \tau_{\theta r} = \mu\left(\frac{1}{r}\frac{\partial u}{\partial \theta} + r\frac{\partial}{\partial r}\left(\frac{v}{r}\right)\right)\,, \quad \tau_{\theta z} = \tau_{z\theta} = \mu\left(\frac{\partial v}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial \theta}\right)\,.$$