Exercise 1

Waves in shallow pool

1. Under the inviscid approximation, capillary–gravity waves in a horizontally infinite domain (since $L \gg H$, the presence of sidewalls is neglected) are governed by the Laplace equation subjected to the kinematic and dynamic boundary conditions at the free surface and the non–penetration condition at the solid bottom:

$$\Delta\Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0, \qquad \left.\frac{\partial \Phi}{\partial y}\right|_{y=0} = 0, \tag{1}$$

$$\frac{\partial \eta}{\partial t} - \left.\frac{\partial \Phi}{\partial y}\right|_{y=H} = 0, \tag{2}$$

$$\left.\frac{\partial \Phi}{\partial t}\right|_{y=H} + g\eta - \frac{\gamma}{\rho}\frac{\partial^2 \eta}{\partial x^2} = 0, \tag{3}$$

where η is the free surface and Φ is the potential velocity field ($\mathbf{u} = \nabla\Phi$). We consider a solution having the following form:

$$\Phi = \hat{\Phi}(y)\, e^{\mathrm{i}(kx-\omega t)}, \qquad \eta = C e^{\mathrm{i}(kx-\omega t)}, \tag{4}$$

Substituting (4) in (1):

$$\frac{\partial^2 \hat{\Phi}}{\partial y^2} - k^2\hat{\Phi} = 0, \tag{5}$$

whose general solution reads

$$\hat{\Phi}(y) = A\cosh ky + B\sinh ky. \tag{6}$$

Imposing the non–penetration condition at the solid bottom, $y = 0$, and the kinematic boundary condition at the free surface, $y = H$, we get,

$$\left.\frac{\partial \hat{\Phi}}{\partial y}\right|_{y=0} = 0 = kB \quad \longrightarrow B = 0, \tag{7}$$

$$\left.\frac{\partial \hat{\Phi}}{\partial y}\right|_{y=H} = \frac{\partial \eta}{\partial t} = -\mathrm{i}\omega C = kA\sinh kH \qquad \longrightarrow A = -\mathrm{i}C\frac{\omega}{k\sinh kH}. \tag{8}$$

Solving for A and B,

$$\hat{\Phi}(y) = -\mathrm{i}\omega C\frac{\cosh ky}{k\sinh kH}. \tag{9}$$

Using (9) and (4) in (3) evaluated at $y = H$ we end up with the following dispersion relation for capillary–gravity waves:

$$\omega = \sqrt{(gk + (\gamma/\rho)\, k^3) \tanh kH} = \sqrt{\left(1 + \frac{\gamma}{g\rho} k^2\right) gk \tanh kH}. \tag{10}$$

2. The wavenumber k is defined as $2\pi/L$ where L is the reference wavelength. If $L \gg H$, then $kH \ll 1$ and $\tanh kH \approx kH$. If $L \gg l_c = \sqrt{\frac{\gamma}{g\rho}}$, then $\left(1 + \frac{\gamma}{g\rho} k^2\right) \approx 1$. Consequently,

$$\omega \approx \sqrt{gHk^2} = k\sqrt{gH} = 2\pi\sqrt{gH}/L \propto \sqrt{gH}/L. \tag{11}$$

   and the time scale is $\tau \sim 1/\omega \propto L\sqrt{gH}$.

   In the (k, ω)–plane, the dispersion relation above is given by a linear variation of the oscillation frequency ω with the wavenumber k. The slope of the linear law is $\sqrt{gH}$.

3. The eigenvalue ω is a real quantities, meaning that in our conventional normal mode form, $e^{-\mathrm{i}\omega t}$, it represents a purely oscillatory motion (undamped), as depicted in figure 3-(a).

4. The wavemotion associated to the dispersion relation expressed in (11), as well as the classic capillary–gravity wavemotion (10), is neutrally or marginally stable. Indeed, under the fundamental assumption of an inviscid fluid, there is no dissipation (and thus no damping) in the present model.

5. Using the expansions $h = H + \epsilon h'$ and $p = P + \epsilon p'$ and recalling that the base–flow or nominal pressure is hydrostatic, $P = -\rho g y$, for which $P|_{y=H} = -\rho g H$, with $H = constant$ ($\frac{\partial^n P}{\partial x^n} = -\rho g \frac{\partial^n H}{\partial x^n} = 0$), we obtain the following ϵ–order linear problem:

$$\frac{\partial h'}{\partial t} = \frac{H^3}{3\mu} \frac{\partial^2 p'}{\partial x^2}. \tag{12}$$

   From the normal component of the linearized dynamic boundary condition at the free surface (applying the flattening procedure) we have:

$$p' = \rho g h' - \gamma \frac{\partial^2 h'}{\partial x^2} \tag{13}$$

   Note that the term $2\mu \frac{\partial v}{\partial y}$ which usually appears in the normal component of the linearized dynamic boundary condition has been neglected by virtue of the lubrication approximation (such a term is negligible when compared with the other terms during the classic dimensional analysis). Thus we obtain

$$\frac{\partial h'}{\partial t} = \frac{\rho g H^2}{3\mu} \left(\frac{\partial^2 h'}{\partial x^2} - \frac{\gamma}{\rho g} \frac{\partial^4 h'}{\partial x^4} \right). \tag{14}$$

6. Assuming a linear solution having the form $h' \propto e^{\mathrm{i}(kx - \omega t)}$ (classic wave expansion), we end up with the following dispersion relation:

$$\omega = -\mathrm{i} \frac{\rho g H^3}{3\mu} \left(k^2 + l_c^2 k^4\right), \tag{15}$$

   Note that the eigenvalue ω is now purely imaginary and thus it represent a pure damping coefficient.

7. In this case the eigenvalue is an imaginary quantity, which leads to an exponentially decaying wavemotion (with no oscillations), as depicted in figure 3-(c).

8. Denoting the damping by σ, if $L \gg l_c$ ($kl_c \ll 1$), we have $\sigma = \frac{\rho g H^3 4\pi^2}{3\mu L^2}$ ($k = 2\pi/L$). The time scale of the damping is $\tau = 1/\sigma \propto \frac{3\mu L^2}{\rho g H^3}$.

9. The dimensional analysis at the heart of the lubrication approximation is based on the following scaling for the various physical quantities:

$$x \sim L, \quad y \sim H, \quad t \sim \tau, \quad u \sim U, \quad v \sim V, \quad p \sim P, \tag{16}$$

With the scaling above, the x–momentum equation reads:

$$\rho \frac{U}{\tau}\frac{\partial \bar{u}}{\partial \bar{t}} + \rho\left(\frac{U^2}{L}\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{VU}{H}\bar{v}\frac{\partial \bar{u}}{\partial \bar{y}}\right) = -\frac{P}{L}\frac{\partial \bar{p}}{\partial \bar{x}} + \mu U\left(\frac{1}{L^2}\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \frac{1}{H^2}\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}\right) \tag{17}$$

where the bar denotes the dimensionless variable. As we can see from (17), the various terms scale like: $\rho U/\tau$, $\rho U^2/L$, $\rho VU/H$, P/L, $\mu U/L^2$, $\mu U/H^2$.
The continuity equation reads:

$$\frac{U}{L}\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{V}{H}\frac{\partial \bar{v}}{\partial \bar{y}} = 0, \tag{18}$$

From the so–called dominant balance of (18), $\frac{U}{L} \sim \frac{V}{H}$. It follows that $V \sim U\frac{H}{L}$. The fundamental idea of the lubrication approximation is that $H/L = \epsilon \ll 1$ (the fluid layer is thin if compared with the wavelength). With this assumption, the viscous term in (17):

$$\mu U\left(\frac{1}{L^2}\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \frac{1}{H^2}\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}\right) = \frac{\mu U}{L^2}\left(\cancel{\frac{\partial^2 \bar{u}}{\partial \bar{x}^2}} + \underbrace{\frac{L^2}{H^2}}_{1/\epsilon^2 \gg 1}\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}\right) \approx \frac{\mu U}{H^2}\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}. \tag{19}$$

Consequently, from the dominant balance of the right hand side of equation (17), we can set the pressure gauge to:

$$\frac{P}{L} \sim \frac{\mu U}{H^2}, \quad \longrightarrow P \sim \frac{\mu U L}{H^2}. \tag{20}$$

Dividing then each term in equation (17) by $\frac{\mu U}{H^2}$ we obtain:

$$\frac{1}{\tau}\frac{H^2}{\nu}\frac{\partial \bar{u}}{\partial \bar{t}} + \cancel{\underbrace{\frac{\rho U L}{\mu}\frac{H^2}{L^2}}_{Re\ \epsilon^2 \ll 1}\left(\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}}\right)} = -\frac{\partial \bar{p}}{\partial \bar{x}} + \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} \tag{21}$$

where Re is the Reynolds number and if $Re \ll 1/\epsilon^2$, then the inertial term is negligible.

10. The order of magnitude of the first term in the left hand side of equation (21) depends on the time scale τ. In example, if $\tau \sim H^2/\nu$ then $H^2/\tau\nu \sim 1$ and

$$\frac{\partial \bar{u}}{\partial \bar{t}} = -\frac{\partial \bar{p}}{\partial \bar{x}} + \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} \tag{22}$$

11. Dimensional form of equation (22), whose derivation has been demonstrated above.

12. Using previous arguments, $V = U\frac{H}{L}$ and $P = \mu\frac{UL}{H^2}$, $\tau = H^2/\nu$ and dividing by $\frac{\mu}{\rho}\frac{V}{H^2}$:

$$\cancel{\frac{\partial \bar{v}}{\partial \bar{t}}} + \cancel{Re\,\epsilon^2\left(\bar{u}\frac{\partial \bar{v}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{v}}{\partial \bar{y}}\right)} = -\underbrace{\frac{1}{\epsilon^2}}_{\gg 1}\frac{\partial \bar{p}}{\partial \bar{y}} + \cancel{\frac{\partial^2 \bar{v}}{\partial \bar{y}^2}} - \underbrace{\frac{\rho L H g}{\mu U}}_{not\ negligible} \tag{23}$$

In dimensional terms we have:

$$\frac{1}{\rho}\frac{\partial p}{\partial y} + g = 0. \tag{24}$$

13. At leading order the free surface obeys the kinematic boundary conditions (no flux through the surface), $\frac{dh}{dt} = v|_{y=H}$. The other boundary conditions comes from the dimensional analysis of the x–component of the dynamic boundary condition:

$$2\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\Bigg|_{y=H} = 0, \quad \longrightarrow \frac{U}{H}\frac{\partial \bar{u}}{\partial \bar{y}}\Bigg|_{y=1} + \frac{V}{L}\frac{\partial \bar{v}}{\partial \bar{x}}\Bigg|_{y=1} = 0 \tag{25}$$

$$\frac{U}{H}\frac{\partial \bar{u}}{\partial \bar{y}}\Bigg|_{y=1} + \frac{UH}{L^2}\frac{\partial \bar{v}}{\partial \bar{x}}\Bigg|_{y=1} = 0, \quad \longrightarrow \frac{\partial \bar{u}}{\partial \bar{y}}\Bigg|_{y=1} + \cancel{\underbrace{\frac{H^2}{L^2}}_{\epsilon^2 \ll 1}\frac{\partial \bar{v}}{\partial \bar{x}}\Bigg|_{y=1}} = 0. \tag{26}$$

The second free surface boundary conditions is thus $\frac{\partial u}{\partial y}\Big|_{y=H} = 0$ (dimensional b.c.).

14. The solution of equation (28)

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}, \quad \longrightarrow \frac{\partial u}{\partial t} = K + \nu\frac{\partial^2 u}{\partial y^2}, \tag{27}$$

is based on the following expansion: $u = \epsilon u'$ with $u' = \hat{u}(y)\,e^{\mathrm{i}(kx-\omega t)}$, for which

$$-\,i\omega\hat{u} = K + \nu\frac{\partial^2 \hat{u}}{\partial y^2}. \tag{28}$$

It can be verified, substituting (34) in (36), that (34) solution.

15. Recalling the expansion for the free surface, $h = H + \epsilon h'$ with $h' = Ce^{\mathrm{i}(kx-\omega t)}$, equation (36) is subjected to the no–slip b.c. at the bottom $u|_{y=0} = 0$ and $\frac{\partial u}{\partial y}\Big|_{y=H} = 0$ at the free surface.

$$\hat{u}(0) = 0 = \frac{iK}{\omega} + A, \quad \longrightarrow A = -\frac{iK}{\omega} \tag{29}$$

$$\frac{\partial \hat{u}}{\partial y}(H) = 0 = (1-i)\sqrt{\frac{\omega}{2\nu}}\left[A\sinh\left((1-i)\sqrt{\frac{\omega}{2\nu}}\,H\right) + B\cosh\left((1-i)\sqrt{\frac{\omega}{2\nu}}\,H\right)\right] \tag{30}$$

From which,

$$B = \frac{iK}{\omega}\tanh\left((1-i)\sqrt{\frac{\omega}{2\nu}}\,H\right) \tag{31}$$

Hence the solution reads,

$$\hat{u}(y) = \frac{iK}{\omega}\left(1 - \cosh\left((1-i)\sqrt{\omega/2\nu}\, y\right) + \tanh\left((1-i)\sqrt{\omega/2\nu}\, H\right)\sinh\left((1-i)\sqrt{\omega/2\nu}\, y\right)\right) \tag{32}$$

16. Integrating the continuity equation from $y = 0$ to $y = H$ (unperturbed or nominal height) one gets:

$$\int_0^H \frac{\partial \hat{u}}{\partial x}\, dy + \int_0^H \frac{\partial \hat{v}}{\partial y}\, dy = 0, \quad \longrightarrow \hat{v}(H) - \cancel{\hat{v}(0)} = -\int_0^H \frac{\partial \hat{u}}{\partial x}\, dy, \tag{33}$$

where $v(0) = 0$ because of the non–penetration b.c. Deriving with respect to x ($K = -\frac{1}{\rho}\frac{\partial p}{\partial x}$) and then integrating in y solution (32), the expression for $\hat{v}(H)$ is obtained.
From the ϵ–order kinematic boundary condition at the free surface, $y = H$,

$$\frac{\partial h'}{\partial t} = \hat{v}(H) \tag{34}$$

Using the normal component of the linearized dynamic boundary condition to express the ϵ–order pressure, we have $\frac{\partial K}{\partial x} = -\frac{1}{\rho}\frac{\partial^2 p'}{\partial x^2} = -g\left(\frac{\partial^2 h'}{\partial x^2} - \frac{\gamma}{\rho g}\frac{\partial^4 h'}{\partial x^4}\right)$, which substituted in the expression for $\hat{v}(H)$ leads to the proposed expression of $\frac{\partial h'}{\partial t}$.
Using then the normal form (wave expansion)

$$h' = Ce^{\mathrm{i}(kx - \omega t)} \tag{35}$$

the dispersion relation is easily found.

17. Derivation of the following limits:

- $H/l_{vb} \ll 1$ (purely damped)

  If $H/l_{vb} \ll 1$, then $\tanh((1-i)H_{vb}) \approx (1-i)H/l_{vb}$. Hence,

$$\left(H - \frac{\tanh(1-i)H/l_{vb}}{(1-i)/l_v b}\right) \approx 0, \quad \longrightarrow \omega^2 \approx 0. \tag{36}$$

  Indeed, one needs to consider the second term of the Taylor expansion, $\tanh x \approx x - \frac{x^3}{3} + \ldots$

$$\left(H - \frac{\tanh(1-i)H/l_{vb}}{(1-i)/l_v b}\right) \approx \left(H - \frac{(1-i)H/l_{vb} - (1-i)^3 H^3/3l_{vb}^3}{(1-i)/l_{vb}}\right) = -\frac{i}{3}\frac{H^3}{\nu}\omega \tag{37}$$

  When $H/l_{vb} \ll 1$, the thickness of the viscous boundary layer, l_{vb}, is much larger than the actual fluid depth, H. It derives that in the present limit, the model lies in the regime described by the lubrication approximation, for which the wavemotion is purely damped, as previously commented.

- $H/l_{vb} \gg 1$ (marginally stable)

  If $H/l_{vb} \gg 1$, $\tanh\left((1-i)\,H/l_{vb}\right) \approx 1$, consequently

$$\left(H - \frac{\tanh\left(1-i\right)H/l_{vb}}{\left(1-i\right)/l_{v}b}\right) \approx \left(H - \frac{l_{vb}}{(1-i)}\right) = H\left(1 - \underbrace{\frac{l_{vb}}{H}}_{\ll 1}\frac{1}{(1-i)}\right) \approx H \quad (38)$$

  When $H/l_{vb} \gg 1$, the thickness of the viscous boundary layer, l_{vb}, is negligible when compared with the fluid depth, H. As a consequence, the viscous dissipation is negligible and a pure oscillatory wevemotion, in analogy with the shallow water approximation of inviscid capillary–gravity waves, is retrieved.

In the intermediate case, when H and l_{vb} are comparable, the wavemotion appears as an oscillatory motion damped by viscous effects. The strong nonlinear nature of the complex dispersion relation leads to the existence of many different branches in the (k, ω)–plane.