Exercise 1

Tsunami correction

1. The flow is described by the 2D instationary Euler equations because the flow is inviscid. In the (x,z)-plane these are:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0,$$
$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial y} = -g,$$
$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0,$$

2. The fluid is inviscid and the ground is fixed and impermeable. The boundary condition is therefore the impermeability condition (with slip) being $\vec{u} \cdot \vec{n} = 0$ at $z = f(x)$. The normal $\vec{n}$ is collinear to the vector $-f'\vec{e}_x + \vec{e}_z$ and perpendicular to the tangent of the ground $\vec{e}_x + f'(x)\vec{e}_z$. One deduces:

$$w = u\, f' \quad \text{at} \quad z = f(x).$$

3. The atmospheric pressure is p_0 and the atmosphere is supposed to be at rest. The kinetic boundary condition comes from impermeability of the water-air interface is given as $\vec{u} \cdot \vec{n} = \partial h/\partial t$ at $z = h(x)$. In contrast to the ground the the water-air interface is a free surface. One deduces:

$$w = \frac{\partial h}{\partial t} + u\, h' \quad \text{at} \quad z = h(x,t).$$

4. The non-dimensionalized equations (we omit the tilde and write in small letters directly the non-dimensional variables) are:

$$\frac{U}{T}\frac{\partial u}{\partial t} + \frac{U^2}{L}u\frac{\partial u}{\partial x} + \frac{WU}{H}w\frac{\partial u}{\partial z} + \frac{P}{L}\frac{1}{\rho}\frac{\partial p}{\partial x} = 0,$$
$$\frac{W}{T}\frac{\partial w}{\partial t} + \frac{UW}{L}u\frac{\partial w}{\partial x} + \frac{W^2}{H}w\frac{\partial w}{\partial z} + \frac{P}{H}\frac{1}{\rho}\frac{\partial p}{\partial z} = -g,$$
$$\frac{U}{L}\frac{\partial u}{\partial x} + \frac{W}{H}\frac{\partial w}{\partial z} = 0,$$

   One wishes to find relations for W, T and P such that the non-dimensional variables have comparable size. These gauges might then indicate which terms are to be neglected because of $H/L \ll 1$. From logical reason one might think that the horizontal velocity is much larger then the vertical velocity. Looking at the continuity equation reveals that only a scaling of $W = UH/L$ leads to a scaling where the two scaled velocity components are of equal weight (otherwise one might be lead to discard the u component if $W = U$).
   This simply leads to the same continuity equation as before but this time non-dimensional.

Turning our attention the the momentum equation in the x-direction (which is judged to be more important), we replace W. We then chose T and P such that all terms are of equal weight. The derivative $\partial u/\partial z$ drops because u is supposed to be a function of x and t only.
The gauges are all determined now:

$$W = U\frac{H}{L}, \quad T = \frac{L}{U} \quad \text{and} \quad P = \rho U^2.$$

Inserting these gauges into the momentum equation in the z-direction one is left with terms of H^2/L^2 which are much smaller then the rest and are therefore neglected. Finally one rewrites $U/\sqrt{g\,H} = \text{Fr}$ as the Froude number.

5. Rescaling the boundary conditions gives the same expressions as in problem 2 and 3 but this time non-dimensional. It should be mentioned that a different scaling would have lead to a negligible small terms in the boundary conditions.

6. Integration of equation (3) of the exercise sheet gives:

$$p = C_1 - \frac{z}{\text{Fr}^2}.$$

With the dynamic boundary condition at the sea surface

$$p(h) = p_0 = C_1 - \frac{h}{\text{Fr}^2},$$

one fixes C_1 and finally obtains

$$p(h) = \frac{h}{\text{Fr}^2}\left(1 - \frac{z}{h}\right) + p_0,$$

7. Integration of equation (4) of the exercise sheet gives:

$$w + z\frac{\partial u}{\partial x} = C_2.$$

Using the kinematic boundary condition on the sea ground and substituting w:

$$-f\frac{\partial u}{\partial x} + C_2 = u\frac{\partial f}{\partial x},$$

which fixes

$$C_2 = \frac{\partial(u\,f)}{\partial x},$$

finally obtaining

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(uh) = \frac{\partial}{\partial x}(uf) \quad \text{at} \quad z = h(x,t).$$

8. Proceeding with the momentum equation in x-direction, where we replace the pressure p, the z term drops as differentiation of z by x is zero.

9. Linearization of the two equations (6) and (7) of the exercise sheet leaves:

$$\frac{\partial h'}{\partial t} + \frac{\partial u'}{\partial x} = 0,$$

and

$$\frac{\partial u'}{\partial t} + \frac{1}{\mathrm{Fr}^2}\frac{\partial h'}{\partial x} = 0,$$

Differentiating the first equation by t and substituting $\partial u/\partial t$ with the second equation leads to the desired expression with $c_0 = 1/\mathrm{Fr}$

10. Inserting $h'(x,t) = \cos(kx - \omega t)$ solves the equation and gives $\omega = \pm c_0 k$. This is called a dispersion relation and describes the relation between the pulsation ω and the wave number k. c_0 is the celerity, which is the propagation speed of the waves. Since all waves are transported at the same velocity the equation is non-dispersive.

    A tsunami is generated with $H = 4000\mathrm{m}$, $g = 10m/s^2$ and $U = 1m/s$. $c_0 = 1/\mathrm{Fr} = 200m/s$ or $720km/h$! This is the wave speed, not the fluid velocity. The tsunami arrives at the coast in $5h\ 33min$.

    In setting $U = 1m/s$ we simply choose a velocity scale, we could have injected $1\mathrm{km/h}$ as well. At this point however I want to make a comment. It might have seemed odd that we established the dominant balance under question 4 with an arbitrary velocity scale U. An alternative, and in my point of view nicer approach would have been to built a pressure based on $P = \rho g H$ and a velocity on $U = \sqrt{g\,H}$. This way a velocity scale appears naturally with Froude number $Fr = 1$ and inserting $H = 4000\mathrm{m}$ and $g = 10m/s^2$ results in the same wave speed. This approach might seem more coherent to you with respect to the other exercises.

11. In the presence of slope of the sea ground, the wave equation becomes:

    $$\mathrm{Fr}^2\frac{\partial^2 h'}{\partial t^2} - (1 - ax)\frac{\partial^2 h'}{\partial x^2} + a\frac{\partial h}{\partial x} = 0.$$

    Inserting $h' = \exp(i(kx - \omega t))$ with ω real and $k = k_r + ik_i$. First one simplifies $1 - ax \approx 1$, inserting h' and gathering all terms with imaginary i results in $k_i = -\frac{a}{2}$. Solving for the real part and substituting k_i gives $k_r = \pm\sqrt{\mathrm{Fr}\ \omega^2 - \frac{a^2}{4}}$. Hence the sea surface becomes:

    $$h(x,t) = 1 + \epsilon e^{-k_i x} e^{i(k_r x - \omega t} = 1 + \epsilon e^{a/2x}\cos(k_r x - \omega t) \quad \text{(omitting the imaginary part).}$$

    Hence the wave growth exponentially, however this is only for a short period as the relation is only valid in the linear case.