Exercise 1

Part 1

1. In the absence of volumic forces, the equations of motion of a perfect, incompressible fluid are the Euler equations. It is not a potential flow since the flow field is not rotation free.

$$\partial\vec{u}/\partial t + (\vec{u}\cdot\nabla)\vec{u} + \nabla p/\rho = 0 \quad \text{and} \quad \nabla\cdot\vec{u} = 0.$$

2. The flow is stationary, the acceleration of the fluid is $(\vec{u}\cdot\nabla)\vec{u}$, being:

$$(\vec{u}\cdot\nabla)\vec{u} = \text{rot}(\vec{u})\times\vec{u} + \nabla(\frac{1}{2}||\vec{u}||^2) = \omega\vec{e}_z\times\vec{u} + \nabla(\frac{1}{2}||\vec{u}||^2).$$

Or, the flow field being two dimensional and incompressible, there exists a function $\psi(x,y)$ such as: $\vec{u} = (\partial\psi/\partial y)\vec{e}_x - (\partial\psi/\partial x)\vec{e}_y$. Consequently:

$$\vec{e}_z\times\vec{u} = (\partial\psi/\partial x)\vec{e}_x + (\partial\psi/\partial y)\vec{e}_y = \nabla\psi.$$

Then assuming ω being constant, one deduces that:

$$\text{rot}\vec{u}\times\vec{u} = \omega\nabla\psi = \nabla(\omega\psi),$$

and the acceleration becomes the derivative of a potential:

$$(\vec{u}\cdot\nabla)\vec{u} = \nabla(\omega\psi + \frac{1}{2}||\vec{u}||^2).$$

Since the pressure appears also as a gradient, we transformed the Euler equations into the gradient of a scalar potential.

3. Since the Euler equation appears as a scalar potential, we have independence of the integration path and the integral for any scalar potential becomes:

$$\int_P \nabla\phi\cdot dr = \phi(B) - \phi(A),$$

for an arbitrary integration path P between position A and B. In our case this gives:

$$\rho\omega\psi + \frac{1}{2}\rho||\vec{u}||^2 + p = C,$$

where C is a constant.

4. The force exerted by a fluid on an obstacle is: $\vec{F} = -\int\int_\Sigma p\vec{n}dS$. Or, using preceding results,

$$\vec{F} = \rho\omega\int\int_\Sigma \psi\vec{n}dS + \frac{\rho}{2}\int\int_\Sigma ||\vec{u}||^2\vec{n}dS - C\int\int_\Sigma \vec{n}dS.$$

The integral of the normal around a closed interface is zero. Since the streamline ψ is constant on the obstacle, not only the last integral but also the first are zero.

Part 2

1. $\mathrm{rot}\,\vec{U} = -S\vec{e}_z$.

2. $\mathrm{rot}\,\vec{u} = \mathrm{rot}\,(\vec{U} + \nabla\phi) = -S\vec{e}_z$ because $\mathrm{rot}\,\nabla\phi = 0$. Applying the incompressibility condition, we find $\Delta\phi = 0$.

3. On Σ, the slip condition $\vec{u}|_\Sigma \cdot \vec{n} = 0$ becomes:

$$0 = (\vec{U} + \nabla\phi)|_\Sigma \cdot \vec{e}_r = (V + SR\sin(\theta))\cos(\theta) + (\partial\phi/\partial r)|_\Sigma.$$

4. The Laplacian in polar coordinates is given as:

$$\Delta\phi = \frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial\phi}{\partial r}\Big) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2},$$

the two functions $\phi = \Gamma\theta/2\pi$ and $\phi = A\ln(r)$ are harmonics, they verify $\Delta\phi = 0$.

5. Since $\Delta\ln(r) = 0$, also $\nabla\Delta\ln(r) = 0 = \Delta\nabla\ln(r)$, by linear combination $\Delta(\vec{B}\cdot\nabla\ln(r)) = 0$ for all vectors $\vec{B}$ being constant. By iteration this is the same for $\bar{\bar{C}}$.

6. In matching all the boundary conditions of Part 2.3 and identifying coefficients, one obtains:

$$A = 0, \quad B_1 = -VR^2, \quad B_2 = 0, \quad C_{11} = C_{22}, \quad C_{12} = \frac{1}{8}SR^4.$$

7. The velocity components are:

$$u_r = V\Big(1 - \frac{R^2}{r^2}\Big)\cos(\theta) + \frac{Sr}{2}\Big(1 - \frac{R^4}{r^4}\Big)\sin(2\theta)$$

$$u_\theta = \frac{\Gamma}{2\pi r} - V\Big(1 + \frac{R^2}{r^2}\Big)\sin(\theta) - \frac{Sr}{2}\Big(1 - \Big(1 + \frac{R^4}{r^4}\Big)\cos(2\theta)\Big).$$

Without shear (S=0), one recovers the potential flow with circulation Γ around a cylinder of radius R.

Part 3

3. At $r = R$ one verifies $u_r = 0$, because the fluid streams around the cylinder. For u_θ, still on the boundary Σ, one obtains:

$$(u_\theta^2)|_\Sigma = C^2 - 4KV\sin(\theta) + 4(V^2 - CSR)\sin^2(\theta) + 8SVR\sin^3(\theta) + 4S^2R^2\sin^4(\theta),$$

where $C = \Gamma/(2\pi R) + SR/2$.

2. From part 1.4, the force is expressed as

$$\vec{F} = \frac{\rho}{2}\int\int_\Sigma ||\vec{u}||^2\vec{n}dS = \frac{\rho}{2}\int_{z=0}^{L}\int_{\theta=0}^{2\pi} u_\theta^2\vec{e}_r R d\theta dz,$$

projected on the Y and X axis:

$$F_x = \frac{\rho}{2}LR\int_0^{2\pi} u_\theta^2\cos(\theta)d\theta, \quad F_y = \frac{\rho}{2}LR\int_0^{2\pi} u_\theta^2\sin(\theta)d\theta.$$

The calculation gives $F_x = 0$, there is no drag.

3. After the calculation the exerted force on the cylinder is:

$$\vec{F} = \rho V L (2\pi S R^2 - \Gamma)\vec{e}_y.$$

The force is directed perpendicular to the flow (lift). The first term comes from shear and the second from Joukowski.