Exercise 1

Flow over an inclined plate: Solution

1. The Joukowski transform transforms a circle of radius a in the Z plane into a line segment $-2a \leq x \leq 2a$ of the z plane. It shall be stressed that the transformation $z \to Z$ is unique but the inverse transform has two solutiuons
$$Z = \frac{1}{2}\left(z \pm \sqrt{z^2 - 4a^2}\right) . \tag{1}$$
It is necessary to chose between these two. One solution maps on the inside of the circle (- solution) and the other on the exterior of the circle (+ solution). Only the latter is interesting for us and conserves the physics of our problem.
Setting $Z = a\, e^{\mathrm{i}\theta}$ in the transform on the exercise sheet reveals that the transform has no imaginary part (hence the geometry is a straight line at $y = 0$) and extends from $-2a$ to $2a$.

2. Conform transforms preserve the angles (important for perpendicular streamlines and equipotentials) and conserve the velocity at infinity. One deduces the complex potential in the Z plane (under consideration of the rotation angle α, the angle of attack on the plate):
$$F(Z) = V_\infty \left(Ze^{-\mathrm{i}\alpha} + \frac{a^2}{Z}e^{\mathrm{i}\alpha}\right) - \frac{\mathrm{i}\Gamma}{2\pi}\log Z\, . \tag{2}$$
The circulation is arbitrary and the dipole adapted such as $Z = a\, e^{\mathrm{i}\theta}$ is a stream line (constant imaginary part).

3. At the stagnation points of the circle the streamline has a kink and the velocity becomes zero. The easiest way to find the stagnation points is to identify the points, where the velocity is zero.

   Inserting the $Z = a\, e^{\mathrm{i}\theta}$ in equation (2) holds:
$$F = V_\infty a(e^{\mathrm{i}(\theta-\alpha)} + e^{-\mathrm{i}(\theta-\alpha)}) - \frac{i\,\Gamma}{2\pi}(\ln(a) + \mathrm{i}\theta). \tag{3}$$
   Derivation by θ (we recall that the derivation is independent of the direction for holomorph functions) gives the velocity. Velocity equal zero is satisfied for:
$$\Gamma = 4\pi a V_\infty \sin(\theta - \alpha). \tag{4}$$
   Or
$$\theta = \alpha + \arcsin(\frac{\Gamma}{4\pi a V_\infty}). \tag{5}$$

4. The rear tip is located at $Z = a$, where the velocity W is:
$$\frac{dF}{dZ} = V_\infty\left(e^{-i\alpha} - e^{i(\alpha-2\theta)}\right) - i\frac{\Gamma}{2\pi a}e^{-i\theta}.$$
For the transformation of the velocity in the coordinate system of plate we multiply by:
$$\frac{dZ}{dz} = \left(1 - \frac{a^2}{Z^2}\right)^{-1} = \frac{1}{1-1} \to \infty$$
The rear tip is a singluar point.

5. The Kutta condition imposes that the streamline departs from the trailing edge A_1 of the plate, situated at $z = 2a$. Otherwise $w(2a) = \infty$, which is unphysical. Since we know the position on the Z plane (circle) of the transformed trailing edge of the plate in the z plane, we can impose a circulation Γ that makes $\theta = 0$ the rear stagnation point of the circle.
$$\Gamma = -4\pi a V_\infty \sin(\alpha). \tag{6}$$

6. The front stagnation A_2 point has to verify $\sin(\theta - \alpha) = -\sin(\alpha)$ as well. Because of periodicity $\theta = \pi + 2\alpha$. Transforming this point onto the plate gives $z = -2a\cos(2\alpha)$.

7. The velocity on the plate has to be obtained using several trigonometric identities (this is why we resorted to solve most of the problems on the circle instead). The idea to solve for the velocity however is quite straight forward.
The velocity on the circle will be $W(Z)$ and the velocity on the plate $w(z)$:
$$w(z) = \frac{dF(z)}{dz} = \frac{dF(Z)}{dZ}\frac{dZ}{dz} = W(Z)\left(1 - \frac{a^2}{Z^2}\right)^{-1}, \tag{7}$$
gives
$$w(z) = \frac{V_\infty}{1 - e^{-2i\theta}}(e^{-\mathrm{i}\alpha} - e^{\mathrm{i}(\alpha-2\theta)} + \mathrm{i}2\Gamma e^{\mathrm{i}\theta}) \tag{8}$$
We employ the identity
$$\frac{1}{1 - e^{-2\mathrm{i}\theta}} = \frac{1 - e^{2\mathrm{i}\theta}}{4\sin^2(\theta)}, \tag{9}$$
to obtain
$$w(z) = \frac{V_\infty}{2\sin^2(\theta)}(\cos(\alpha) - \cos(\alpha - 2\theta) + 2\sin(\theta)\sin(\alpha)). \tag{10}$$
Using another identity which is
$$\cos(\alpha) - \cos(\alpha - 2\theta) = 2\sin^2(\theta)\cos(\alpha) - 2\sin(\alpha)\sin(\theta)\cos(\theta), \tag{11}$$
to obtain
$$w(z) = V_\infty(\cos(\alpha) + \frac{\sin(\alpha)}{\sin(\theta)}(1 - \cos(\theta)). \tag{12}$$
The relation between z and θ on the circle is $z = 2a\cos(\theta)$. This leads finally to
$$w(z) = V_\infty\left(\cos(\alpha) \pm \sin(\alpha)\sqrt{\frac{2a - z}{2a + z}}\right) \tag{13}$$
Note that the $\pm$ sign in (13) appears because of the $\sin(\theta)$ term which is expressed as $\sqrt{1 - \cos^2(\theta)}$. Hence to decide the sign to be used in (13) one should understand the sign of $\sin(\theta)$ at the stagnation point. One observes that the velocity is zero at A_2. In contrast the velocity at A_1 is non-zero: $w = V_\infty \cos\alpha$. There is however no contradiction because the derivation of the transform, $\mathrm{d}Z/\mathrm{d}z$, is singular at $Z = \pm a$.

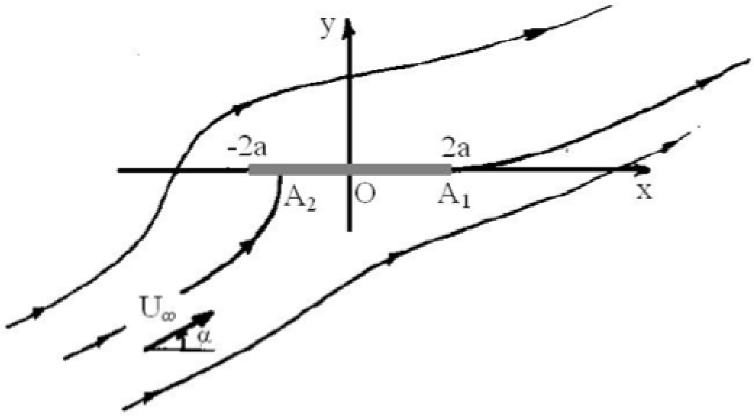

Figure 1

8. The Kutta theorem needs to be projected along the flow direction, which gives:

$$F_y = 4\pi a\rho V_\infty^2 \sin(\alpha)\cos(\alpha).$$

In order to verify the Kutta theorem we can retrieve the force acting on the plate from the pressure, Bernoulli gives $p_\infty + \frac{1}{2}\rho V_\infty^2 = p + \frac{1}{2}\rho u^2$. Using equation (13):

$$p_{\text{top}} - p_{\text{bottom}} = \frac{1}{2}\rho V_\infty^2\left(\left(\cos(\alpha) + \sin(\alpha)\sqrt{\frac{2a-z}{2a+z}}\right)^2 - \left(\cos(\alpha) - \sin(\alpha)\sqrt{\frac{2a-z}{2a+z}}\right)^2\right)$$

$$= 2\rho V_\infty^2\sqrt{\frac{2a-z}{2a+z}}\sin(\alpha)\cos(\alpha).$$

Integrating this expression between $-2a$ and $2a$ gives:

$$F_y = 4\pi a\rho V_\infty^2 \sin(\alpha)\cos(\alpha).$$

As was stated by $F_y = -\rho\, U\,\Gamma$.

9. The couple can be calculated by integration of the pressure forces along the plate with respect to the tip of the plate.

$$\begin{aligned} M =& \int_{-2a}^{2a}(2a+z)(p_{\text{top}} - p_{\text{bottom}})dz \\ =& 2\rho V_\infty^2 \sin(\alpha)\cos(\alpha)\int_{-2a}^{2a}\sqrt{4a^2 - z^2}dz = 4\rho a^2 V_\infty^2 \sin(\alpha)\cos(\alpha) \end{aligned} \qquad (14)$$

In order to find the point of application one looks for the point, in which the force F_y would evoke the same couple,

$$l = \frac{M}{F} = a.$$

The length of the plate is $4a$ the point of application is at distance a from the tip of the plate and therefore located in the first quarter.

Alternative derivation question 7):
Let's first work on each of the terms that we have:

$$\frac{dF(Z)}{dZ} = \left(U_\infty\left[e^{-i\alpha} - e^{-i(2\theta-\alpha)}\right] - \frac{i\Gamma}{2\pi a}e^{-i\theta}\right) \tag{1}$$

$$= U_\infty e^{-i\theta}\left(\underbrace{e^{i(\theta-\alpha)} - e^{-i(\theta-\alpha)}}_{2i\sin(\theta-\alpha)} + 2i\sin(\alpha)\right) \tag{2}$$

$$= 2iU_\infty e^{-i\theta}(\sin(\theta-\alpha)+\sin(\alpha)) \tag{3}$$

$$= 2iU_\infty e^{-i\theta}(\cos(\alpha)\sin(\theta)-\cos(\theta)\sin(\alpha)+\sin(\alpha)) \tag{4}$$

$$= 2iU_\infty e^{-i\theta}(\sin(\theta)\cos(\alpha)+\sin(\alpha)(1-\cos(\theta)) \tag{5}$$

$$\tag{6}$$

$$\left(\frac{dz}{dZ}\right)^{-1} = \frac{1}{1-e^{-2i\theta}} = e^{i\theta}\frac{1}{e^{i\theta}-e^{-i\theta}} = \frac{e^{i\theta}}{2i}\frac{1}{\sin(\theta)} \tag{7}$$

Therefore one can see that the final expression for the velocity is found to be:

$$w(z) = \frac{dF}{dZ}\frac{dZ}{dz} = U_\infty\left(\cos(\alpha) + \frac{\sin(\alpha)}{\sin(\theta)}(1-\cos(\theta)\right) \tag{8}$$