Exercise 1

1. The only admissible combination of flow potentials is the constant flow and the dipole. On needs to impose that the normal velocity on a circle of radius a is zero, or equivalently that the streamfunction along the interface is zero. This solution corresponds to the flow around a cylinder, but considering the top half-plane it represents the problem.

   Then we can express the streamfunction as:

$$\psi(r=a) = \psi_{uniformflow} + \psi_{doublet} = U_\infty r sin\theta - \frac{K sin\theta}{r}$$

   Imposing the boundary condition at r=a, we have

$$\psi(r=a) = U_\infty a sin\theta - \frac{K sin\theta}{a} = 0 \to K = U_\infty a^2.$$

   Thus,

$$\psi(r,\theta) = U_\infty sin\theta\Big(r - \frac{a^2}{r}\Big)$$

2. The pressure on the exterior surface of the tent is calculated with Bernoulli's principle:

$$p_\infty + \frac{\rho}{2}U_\infty^2 = p(\theta_a) + \frac{\rho}{2}u_\theta^2(\theta_a).$$

   Instead of the angular velocity one could also use the sum of the square of each velocity component. Using the angular velocity is advantageous because it is the only non zero velocity component on the tent. Thus, $u = \sqrt{{u_\theta}^2 + {u_r}^2} = u_\theta$ as $u_r = 0$

$$u_\theta = \frac{\partial\psi}{\partial r} = \frac{\partial}{\partial r}\left(U_\infty sin\theta\left(r + \frac{a^2}{r}\right)\right) = 2\sin\theta U_\infty.$$

   So the pressure on the tent's surface is:

$$p = p_\infty + \frac{\rho}{2}U_\infty^2(1 - 4\sin^2\theta).$$

3. The pressure inside the tent is assumed constant and equal to the pressure at the hole:

$$p_i(\theta) = p(r=a, \theta=\theta_a) = p_\infty + \frac{1}{2}\rho U_\infty^2(1 - 4\sin^2\theta_a).$$

   The force acting on the tent can be calculated as the integral of the pressure difference $\Delta p(\theta) = p_e(\theta) - p_i(\theta) = 2\rho U_\infty^2(\sin^2\theta_a - \sin^2\theta)$ on the cylinder:

$$\begin{aligned}\boldsymbol{F} = -\int_C \Delta p \boldsymbol{n} dl &= -2\rho U_\infty^2 \int_0^\pi (\sin^2\theta_a - \sin^2\theta)(\cos\theta \boldsymbol{e}_x + \sin\theta \boldsymbol{e}_y) a d\theta \\ &= 4a\rho U_\infty^2\left(\frac{2}{3} - \sin^2\theta_a\right)\boldsymbol{e}_y.\end{aligned}$$

   This force is zero if $\sin\theta_a = \sqrt{2/3}$, or $\theta \simeq \pm 55$.

Exercise 2

Potential flow around a slightly distorted circle

(a) See Figure 1.

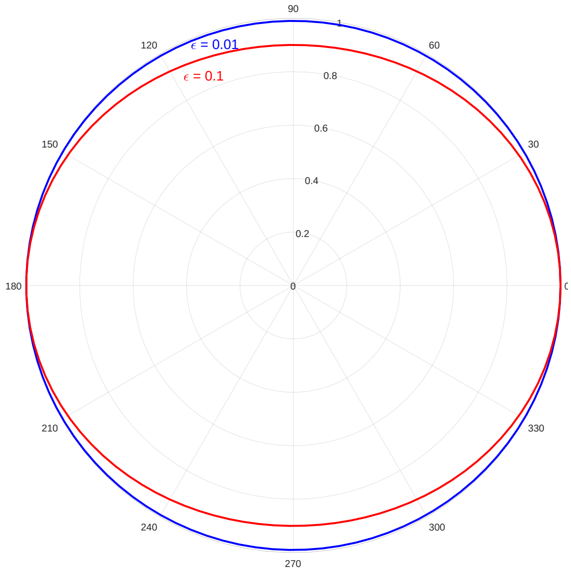

Figure 1. *Distorted circle,* $a = 1$ $\epsilon = 0.01$ *and* $\epsilon = 0.1$.

(b) The acyclic stream function associated to the potential flow around a circular cylinder of radius a with uniform flow $U_\infty \mathbf{e_x}$ at infinity is obtained by superposition of the uniform flow and a doublet which reads

$$\psi = U_\infty \sin\theta \left(r - \frac{a^2}{r} \right).$$

(c) ψ_0 is the one represented above satisfying the boundary conditions $\psi_0(\infty) = U_\infty r \sin\theta$ and $\psi_0 (a, \theta) = 0$.

(d) Knowing that ψ_0 is the solution of potential flow and thanks to the linearity of the Laplace operator we can write

$$\Delta(\psi_0 + \epsilon\psi_1) = 0 \longrightarrow \Delta\psi_1 = 0.$$

Recalling the Taylor's expansion of solution around (a, θ)

$$\psi\left(a\left(1 - \epsilon\sin^2(\theta)\right), \theta\right) = \psi(a, \theta) - a\epsilon\sin^2(\theta)\frac{\partial\psi}{\partial r}(a, \theta) + O(\epsilon^2).$$

The body of distorted cylinder represents the stream line with the associated stream function $\psi\left(a\left(1 - \epsilon\sin^2(\theta)\right), \theta\right) = 0$. The farfield stream function should be the same as the uniform flow, hence $\psi(\infty) = U_\infty r \sin\theta$. Rewriting the stream function in terms of ψ_0 and ψ_1 and keeping the linear order, we get the boundary conditions on ψ_1

$$\psi_1(\infty) = 0; \psi_1(a,\theta) = a\sin^2(\theta)\frac{\partial \psi_0}{\partial r}(a,\theta).$$

(e) Differentiating ψ_0 and replacing in the boundary condition one finds

$$\psi_1(a,\theta) = 2aU_\infty \sin^3(\theta),$$

which can be re-expressed as

$$\psi_1(a,\theta) = \frac{aU_\infty}{2}\left(3\sin\theta - \sin(3\theta)\right).$$

(f) The scalar Laplacian in polar coordinates reads

$$\Delta\psi_1 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi_1}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \psi_1}{\partial \theta^2} = 0.$$

Using the separation of variables we seek a solution of form $\psi_1 = R(r)T(\theta)$ with boundary conditions

$$R(\infty) = 0; R(a)T(\theta) = \frac{aU_\infty}{2}\left(3\sin\theta - \sin(3\theta)\right).$$

Rewriting and manipulating the Laplacian we find

$$\frac{r(rR')'}{R} + \frac{T''}{T} = 0.$$

As R and T are functions of independent variables one deduces

$$\frac{r(rR')'}{R} = -\frac{T''}{T} = \lambda$$

which λ is the eigenvalue coupling two separated ODEs. We start with $T'' = -\lambda T$ and looking at the boundary conditions we find the general solution to be of form $T(\theta) = c_i \sin\left(\sqrt{\lambda_i}\theta\right)$ with $\lambda_1 = 1$ and $\lambda_2 = 9$.

Solving the other ODE $\frac{r(rR')'}{R} = \lambda$ one finds the general solution of $R(r) = \left(C_{1,i}r^{\sqrt{\lambda_i}} + C_{2,i}r^{-\sqrt{\lambda_i}}\right)$.

Applying the two boundary conditions to determine the constants, the solution reads

$$\psi_1 = \frac{3a^2U_\infty}{2r}\sin(\theta) - \frac{a^4U_\infty}{2r^3}\sin(3\theta).$$