Exercise 1

1. We assume the flow to be

- unidirectional
- Newtonian, the viscosity is constant in space and time
- incompressible
- not influenced by gravity

than the Navier Stokes equations write simply

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} - \frac{1}{\rho}\frac{dp}{dx} \tag{1}$$

with boundary conditions $u(-h/2) = u(h/2) = 0$. In order to introduce non dimensional variables we select the following gauges

- $u = \tilde{u}U$
- $x = \tilde{x}h$
- $y = \tilde{y}h$
- $t = \tilde{t}T = \frac{\tilde{t}}{\omega}$

inserting them in the Navier-Stokes equation we obtain

$$\omega U \frac{\partial \tilde{u}}{\partial \tilde{t}} = -\frac{A}{\rho}\frac{\partial \tilde{p}}{\partial \tilde{x}} + \frac{\nu U}{h^2}\frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2} \tag{2}$$

Depending on the ratio between the characteristic time of fluctuation of the pressure gradient $1/\omega$ and the viscous time scale h^2/ν some term can be neglected;

- high frequency regime $\omega \gg \frac{\nu}{h^2}$, than the natural velocity scale to choose is $U \sim \frac{A}{\rho\omega}$, omitting the tildes equation becomes

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \tag{3}$$

- low frequency regime $\omega \ll \frac{\nu}{h^2}$, than the natural velocity scale to choose is $U \sim \frac{Ah^2}{\rho\nu}$, omitting the tildes equation becomes

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}. \tag{4}$$

2. For a pulsating pressure gradient, it is natural to look for an oscillating solution, we write this in dimensional form as

$$u(x,t) = \frac{1}{2}\left[u(y)\exp i\omega t + c.c.\right] = \Re\{u(y)\exp(i\omega t)\} \tag{5}$$

so that

$$i\omega u = \nu\frac{\partial^2 u}{\partial y^2} - \frac{A}{\rho}. \tag{6}$$

or in dimensional form using $U = \frac{A}{\rho\omega}$ (note that the choice of the gauge in now arbitrary because we are not doing any assumption)

$$i\hat{u} = \frac{\nu}{h^2\omega}\frac{\partial^2 \hat{u}}{\partial \hat{y}^2} - 1, \tag{7}$$

whose solution is:

$$u(y) = \frac{iA}{\rho\omega}\left(1 - \frac{e^{(1+i)\sqrt{\frac{\omega}{\nu}}y} + e^{-(1+i)\sqrt{\frac{\omega}{\nu}}y}}{e^{(1+i)\sqrt{\frac{\omega}{\nu}}\frac{h}{2}} + e^{-(1+i)\sqrt{\frac{\omega}{\nu}}\frac{h}{2}}}\right), \tag{8}$$

$$u(y,t) = \Re\{u(y)\exp(i\omega t)\}, \tag{9}$$

$$\hat{u}(\hat{y}) = i\left(1 - \frac{e^{(1+i)\hat{y}} + e^{-(1+i)\hat{y}}}{e^{(1+i)\frac{1}{2}} + e^{-(1+i)\frac{1}{2}}}\right). \tag{10}$$

This solution is valid in for every frequency because we didn't simplify any term.

3. Now we solve in the assumption of low frequency $\omega \ll \frac{\nu}{h^2} \rightarrow \sqrt{\frac{\nu}{\omega}} \gg h$, the dominate term is the one in middle, this select the velocity scale as $U = \frac{Ah^2}{\rho\nu}$, thus the non dimensional equation writes

$$i\frac{\omega h^2}{\nu}\hat{u} = \frac{\partial^2 \hat{u}}{\partial \hat{y}^2} - 1 \rightarrow 0 = \frac{\partial^2 \hat{u}}{\partial \hat{y}^2} - 1, \tag{11}$$

which is a regular asymptotic expansion since the higher order derivatives remains on the simplified equation. Applying the no-slip boundary condition we obtain the following solution

$$u(y,t) = -\frac{A}{2\rho\nu}\left(\frac{h^2}{4} - y^2\right)\cos(\omega t). \tag{12}$$

This is a quasi-static poiseuille flow which oscillates in opposite phase with the pressure gradient, figure 1.

4. Now we solve in the assumption of high frequency $\omega \gg \frac{\nu}{h^2} \rightarrow \sqrt{\frac{\nu}{\omega}} \ll h$, then the equation simplify in

$$i\hat{u} = -1 \rightarrow u(y,t) = \Re\{u(y)\exp(i\omega t)\} = -\frac{A}{\rho\omega}\sin(\omega t). \tag{13}$$

This is a uniform flow pulsating in quadrature of phase with the pressure gradient. We clearly see that this flow cannot satisfies the no slip boundary conditions. We will thus define a

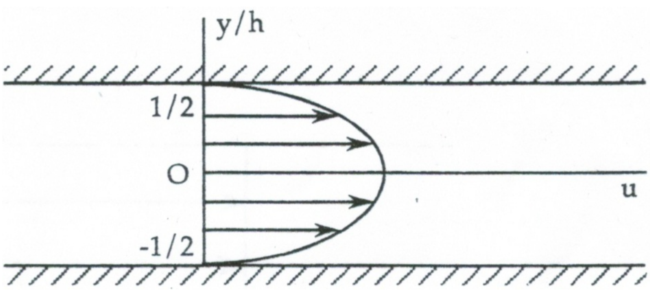

Figure 1. *Quasi-statically regime at low frequency. Poiseuille flow.*

region close to wall where the viscous effects (previously neglected) becomes important. For this purpose we introduce the change of variable $y = -\frac{h}{2} + \delta\tilde{y}$ where the tilde indicates the variable in the boundary layer, then

$$i\tilde{u} = \frac{\nu}{\omega h^2}\frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}\frac{h^2}{\delta^2} - 1, \tag{14}$$

in order to keep the viscous we set $\frac{\nu}{\omega\delta^2} = 1$ which set the thickness of the boundary layer as $\delta = \sqrt{\frac{\nu}{\omega}}$. The solution to the equation is

$$\tilde{u} = A\exp(\sqrt{i}\tilde{y}) + B\exp(-\sqrt{i}\tilde{y}) + C \tag{15}$$

with boundary conditions $\tilde{u}(0) = 0$ and $\tilde{u}(\tilde{y} \to \infty) = i = \hat{u}(0)$, considering that $\sqrt{i} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$ the condition at infinity gives $A = 0$ and $C = i$, while the condition at the wall gives $C = -B$, the solution for the boundary layer gives thus writes

$$\tilde{u} = i(1 - \exp(-\sqrt{i}\tilde{y})). \tag{16}$$

Note that this last solution is a solution to the same equation we solved for the general solution, it looks different because of the different boundary conditions. Finally the velocity profile looks like the one in figure 2.

5. It is interesting to check if the general solution that we found in question 2 becomes equal to the solutions for low and high frequency when we take the limit for $\frac{h}{\delta} \to 0$ and $\frac{h}{\delta} \to \infty$ respectively. In the first case one has to consider that

$$e^{\left((1+i)\frac{\hat{y}}{\delta}\right)} = 1 + (1+i)\frac{\hat{y}}{\delta} + O\left(\left(\frac{\hat{y}}{\delta}\right)^2\right). \tag{17}$$

and the velocity becomes

$$\hat{u} = i\left(1 - \left(1 + i\left(\frac{\hat{y}}{\delta}\right)^2\right)\right)\left(1 - i\left(\frac{1}{2\delta}\right)\right) \tag{18}$$

$$= i\left(\left(\frac{1}{2\delta}\right)^2 + \left(\frac{\hat{y}}{\delta}\right)^2\right) \tag{19}$$

$$= i\left(\left(\frac{h}{2}\right)^2 + \hat{y}^2\right) \tag{20}$$

which is a Poiseuille profile. In the second case the boundary layer is thinner $\frac{h}{\delta} \to \infty$, we notice that in the general solution the exponentials with negative exponent tends to zero while the one with positive exponent goes to infinity, the one at the numerator slower because has smaller exponent and thus tends finally to zero. This gives the desired uniform profiles far from the wall

$$\hat{u}(\hat{y}) = \lim_{\frac{h}{\delta}\to\infty} i\left(1 - \frac{e^{(1+i)\frac{\hat{y}}{\delta}} + e^{-(1+i)\frac{\hat{y}}{\delta}}}{e^{(1+i)\frac{h}{2\delta}} + e^{-(1+i)\frac{h}{2\delta}}}\right) = i. \tag{21}$$

Exercise 2

A thermal boundary layer: Solution

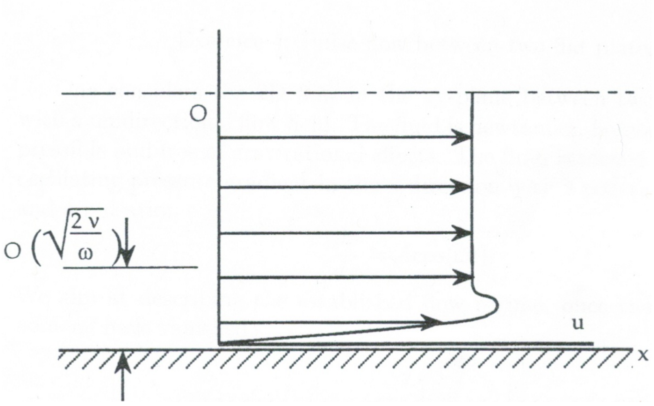

Figure 2. *Regime at high frequency. Stokes layer near the wall $y = -h/2$ and uniform flow at the center of the pipe.*

1. The solution was derived in a previous exercise.

$$U_r = U_\theta = 0, \quad U_z = -\frac{\Pi}{4\mu}\left(1 - \frac{r^2}{R^2}\right) = U_0\left(1 - \frac{r^2}{R^2}\right).$$

2. We use the solution of the velocity field in the heat equation and non-dimensionalise:

$$\frac{\varrho c_p U_0\, T}{R}\,\bar{u}\,\frac{\partial \bar{T}}{\partial \bar{z}} = \frac{\mu U_0^2}{R^2}\left(\frac{\partial \bar{u}}{\partial \bar{r}}\right)^2 + \frac{\kappa T}{R^2}\left(\frac{1}{\bar{r}}\frac{\partial}{\partial \bar{r}}\left(\bar{r}\frac{\partial \bar{T}}{\partial \bar{r}}\right) + \frac{\partial^2 \bar{T}}{\partial \bar{z}^2}\right).$$

We divide by $\kappa\, T$ and multiply by R^2:

$$\frac{\varrho U_0 R\, c_p}{\kappa}\,\bar{u}\,\frac{\partial \bar{T}}{\partial \bar{z}} = \frac{\mu U_0^2}{T\kappa}\left(\frac{\partial \bar{u}}{\partial \bar{r}}\right)^2 + \left(\frac{1}{\bar{r}}\frac{\partial}{\partial \bar{r}}\left(\bar{r}\frac{\partial \bar{T}}{\partial \bar{r}}\right) + \frac{\partial^2 \bar{T}}{\partial \bar{z}^2}\right).$$

Using the non-dimensional numbers

$$\mathrm{Re}\ \mathrm{Pr}\ \bar{u}\frac{\partial \bar{T}}{\partial \bar{z}} = \mathrm{Pr}\ \mathrm{E}\left(\frac{\partial \bar{u}}{\partial \bar{r}}\right)^2 + \left(\frac{1}{\bar{r}}\frac{\partial}{\partial \bar{r}}\left(\bar{r}\frac{\partial \bar{T}}{\partial \bar{r}}\right) + \frac{\partial^2 \bar{T}}{\partial \bar{z}^2}\right),$$

where boundary conditions read

$$\bar{T}(r = 1, z < 0) = 0, \quad \bar{T}(r = 1, z > 0) = 1, \quad \left(\frac{\partial \bar{T}}{\partial \bar{r}}\right)_{\bar{r}=0} = 0$$

The Prantl number (Pr) compares viscous diffusion to thermal diffusion. The Eckert number (E) compares thermal energy to kinetic energy of the moving liquid. We can interprete the equation as follows, the term to the left represents advection of the thermal field. The term in the middle represents thermal energy created by viscous dissipation and the right term represents thermal diffusion.

In the next steps we omit the overbar for the non-dimensional quantities.

3. For elevated Reynolds numbers the first term, the thermal advection becomes dominant. Simplifying the equation for (Pr Re $\to \infty$):

$$(1 - r^2)\frac{\partial T}{\partial z} = 0 \quad \Rightarrow \quad T = 0$$

We see for pure advection the boundary condition $T(r = 1, z > 0) = 1$ is not verified.

4. We now consider a boundary layer region near the wall. We see that close to the wall the scaling between the heat convection and heat diffusion is different. We apply the inner variable $r = 1 - \epsilon\tilde{r}$. Just recall that derivatives in r change in $\tilde{r}$.

$$\frac{\partial F}{\partial r} = \frac{\partial F}{\partial \tilde{r}}\frac{\partial \tilde{r}}{\partial r} = \frac{\partial F}{\partial \tilde{r}}\frac{\partial(1-r)}{\epsilon \partial r} = -\frac{\partial F}{\epsilon \partial \tilde{r}}$$

Be reminded that the viscous dissipation source term (with Eckert number) is still small. Again we omit the tildes in the equations below.

$$\text{Re Pr } (1 - (1 - 2r\epsilon + r^2\epsilon^2))\frac{\partial T}{\partial z} = \frac{1}{\epsilon^2(1-\epsilon r)}\frac{\partial}{\partial r}\left((1-\epsilon r)\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial z^2}.$$

5. We sum up and take only leading order terms on each side.

$$2\text{Re Pr } \epsilon r \frac{\partial T}{\partial z} = \frac{1}{\epsilon^2}\frac{\partial^2 T}{\partial r^2}.$$

We try now to solve the problem of balanced advection-diffusion in the boundary layer. We chose the scaling factor of the boundary layer $\epsilon = (2\text{Re Pr})^{-1/3}$ and end up with the equation:

$$r\frac{\partial T}{\partial z} = \frac{\partial^2 T}{\partial r^2}.$$

6. We look for a self-similar solution and therefore search for scale invariance for certain dilatation groups, R, Z and θ:

$$\frac{R\,\theta}{Z} : \frac{\theta}{R^2} \quad \text{PDE and} \quad \theta = 1 \quad \text{BC}.$$

We see that on a trajectory $R^3/Z = \text{constant}$, the temperature T remains 1.

The functional

$$F(r,z,T) = F(r, r^3/z, T) = 0 \quad \Rightarrow \quad T = f(r\,z^{-1/3}) = f(\eta).$$

Again, the way the self-similar variable is defined is not unique one could have chosen $r^3/z, z/r^3, ...$, here the motivation is that a term in r vanishes after derivation and does not create additional terms[1] after the second derivative.

We know that:

$$\frac{\partial F}{\partial r} = \frac{\partial F}{\partial \eta}\frac{\partial \eta}{\partial r} = \frac{\partial F}{\partial \eta}\frac{1}{z^{1/3}},$$

and

$$\frac{\partial F}{\partial z} = \frac{\partial F}{\partial \eta}\frac{\partial \eta}{\partial z} = -\frac{1}{3}\frac{\partial F}{\partial \eta}\frac{r}{z^{4/3}},$$

We substitute with the self-similar variable:

$$-\frac{1}{3}\frac{r^2}{z^{4/3}}f' = f''\frac{1}{z^{2/3}} \quad \Rightarrow \quad -\frac{1}{3}\eta^2 f' = f''.$$

[1] You can try the other two alternatives and see that you get always more then two terms in the end, which makes it quite hard to figure out an analytical solution

7. This is a non-linear ODE, a standard Ansatz like $f' = C\exp(a\,\eta)$ will not work. However we get an idea that a different kind of Ansatz could work, $f' = A\exp(a\,\eta^b)$.

   Pasting this into the equation gives:

$$-\frac{\eta^2}{3} = a\,b\,\eta^{b-1}\frac{f'}{f'} \quad \Rightarrow \quad b = 3,\ a = -1/9.$$

   So we get

$$T' = A\exp(-\eta^3/9).$$

   This equation needs to be integrated with the boundary conditions applied. The boundary conditions are:

$$T(r \neq 0, z = 0) = T(\eta = \infty) = 0 \quad \text{and} \quad T(r = 0, z > 0) = T(\eta = 0) = 1.$$

   Thus the constant A is obtained by integration from 0 to η

$$T(\eta = \infty) - T(\eta = 0) = \int_0^\infty A\exp(-\xi^3/9)d\xi,$$

   and using the other boundary conditions we find

$$T(\eta) = 1 - \frac{\int_0^\eta \exp(-\xi^3/9)d\xi}{\int_0^\infty \exp(-\xi^3/9)d\xi}.$$

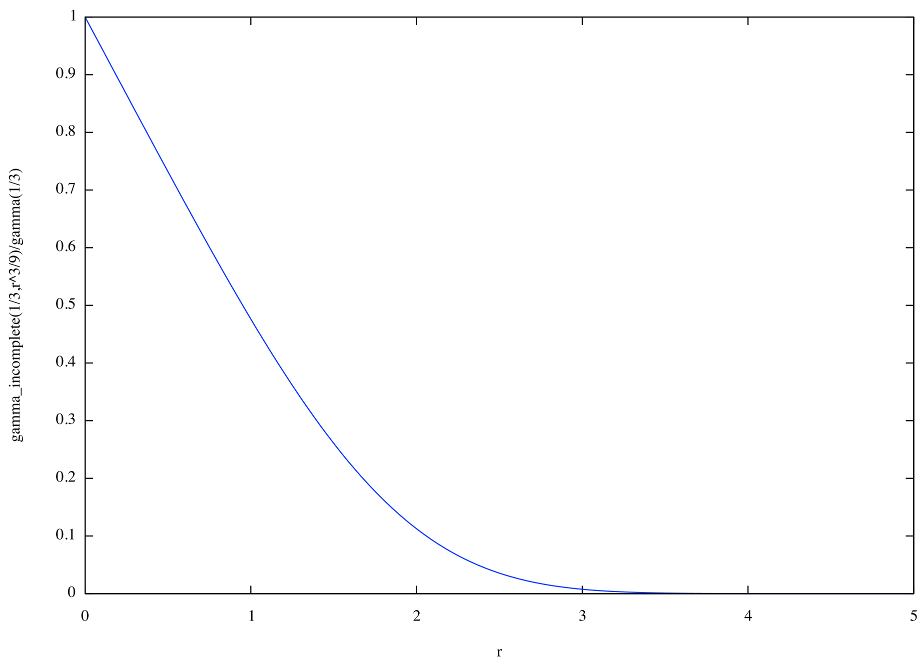

Figure 3. *Inner solution of the temperature near the wall.*

   This can be done numerically or with the help of a book of integral identities or simply with a computer algebra software like `maxima`. The result is expressed in forms of the "incomplete generalised Gamma function" and the Gamma function.

$$T = \frac{\Gamma(1/3, \eta^3/9)}{\Gamma(1/3)}$$

It remains to combine both solution into a composite form for inner and outer solution. The outer solution is T_0 everywhere, non-dimensional $T(R, z > 0) = 0$. The inner solution is $T(r \to \infty, z > 0) = 0$. So the composite form is easily obtained as the inner solution in terms of the initial variables.

We recall that the inner radial variable was defined as $\tilde{r} = (1 - r/R)\epsilon^{-1}$.

$$T = T_0 + (T_w - T_0)\frac{2\text{Re Pr}\,\Gamma\left(1/3, \frac{1}{9}(1 - r/R)^3 R/z\right)}{\Gamma(1/3)}$$

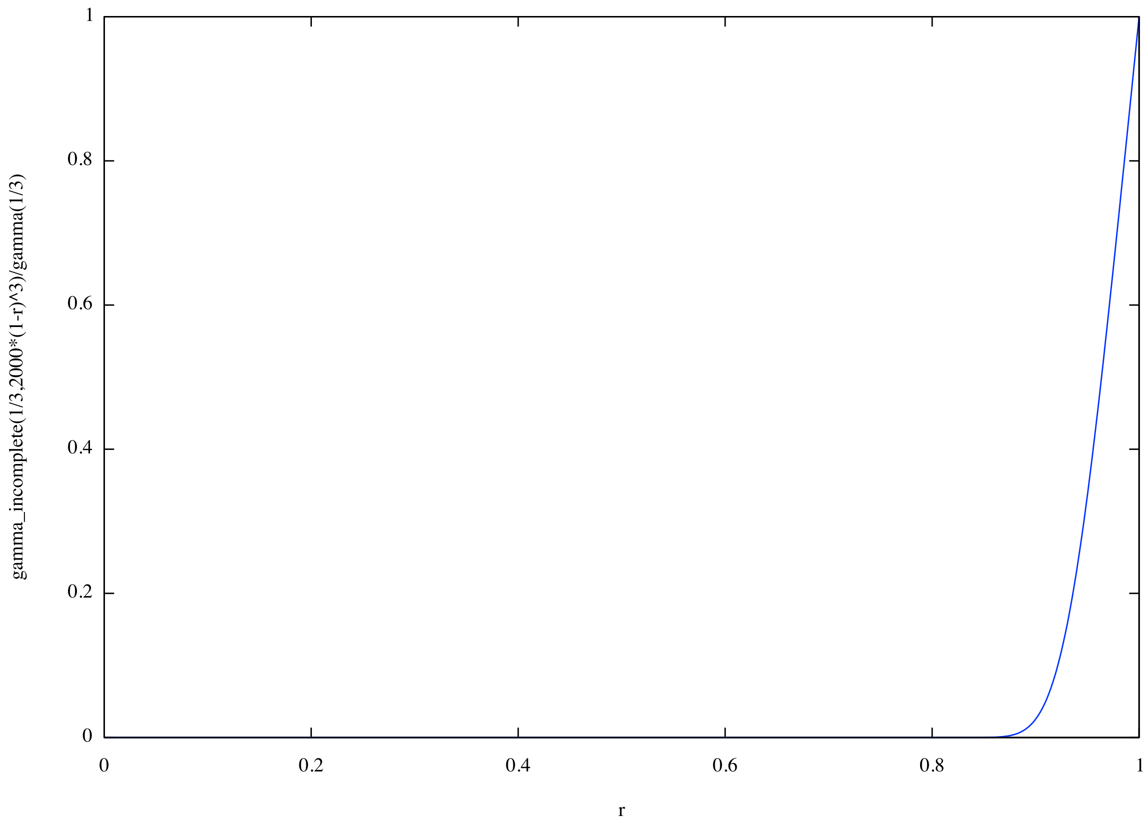

Figure 4. *Composite solution of the temperature for a pipe of radius radius $R = 1$, at $z = 1$ at $RePr = 1000$.*

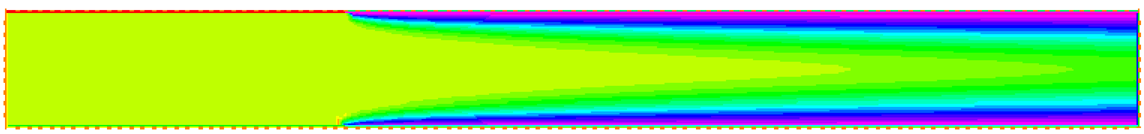

Figure 5. *Numerical solution of the temperature for a pipe of radius $R = 1$ at $RePr = 1000$.*