Exercise 1

1. Owing to the problem assumptions, the continuity equation reads

$$\frac{1}{r}\frac{\partial}{\partial r}(rU) = 0\,, \tag{1}$$

which can be integrated into

$$U = \frac{A}{r}\,. \tag{2}$$

Since the fluid is viscous, we have no-slip conditions at the cylinder walls, namely $U(R_1) = U(R_2) = 0$, which yields $A = 0$, and finally, $U = 0$.

The momentum equations now read

$$-\rho\frac{V^2}{r} = -\frac{\partial P}{\partial r}\,, \tag{3a}$$

$$0 = \mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rV)\right) + \frac{\partial^2 V}{\partial z^2}\right)\,, \tag{3b}$$

$$0 = -\frac{\partial P}{\partial z}\,. \tag{3c}$$

From $(26a)$ and $(26c)$, it turns out that P and V are functions of r only.

2. Integration of $(26b)$ yields

$$V(r) = \frac{A}{2}r + \frac{B}{r}\,. \tag{4}$$

The no-slip conditions at the cylinder walls yield $V(R_1) = \Omega_1 R_1$ and $V(R_2) = \Omega_2 R_2$, so that we have

$$A = 2\frac{\Omega_2 R_2^2 - \Omega_1 R_1^2}{R_2^2 - R_1^2}\,, \quad B = (\Omega_1 - \Omega_2)\frac{R_2^2 R_1^2}{R_2^2 - R_1^2}\,. \tag{5}$$

Finally, the pressure field is obtained by integration of $(26a)$:

$$P(r) = P_0 + \rho\left(\frac{A^2}{8}r^2 + AB\ln r - \frac{B^2}{2r^2}\right)\,. \tag{6}$$

3. The velocity field is obtained by setting $\Omega_2 = 0$ in the preceding results.

4. The force on a ring shaped surface of length dL is calculated by integration of the viscous stresses:

$$\bar{F} = \int \bar{\bar{\sigma}}\bar{n}dA \;\Rightarrow\; F_\theta = \int^{dL}\int_0^{2\pi} \mu r\frac{\partial}{\partial r}\left(\frac{V}{r}\right) r d\theta dz = \frac{4\pi\mu B dL}{R_2}.$$

5. The torque on an cylinder portion of length dL is

$$C = R_2 F\,, \tag{7}$$

whose integration yields

$$C = 4\pi\mu B\,dL\,. \tag{8}$$

6. If C can be determined experimentally, then the viscosity reads simply

$$\mu = \frac{C}{4\pi B dL} = \frac{C(R_2^2 - R_1^2)}{4\pi R_2^2 R_1^2 \Omega_1 dL}\,. \tag{9}$$

7.-8. In the limit $R_2 \to \infty$, the velocity field obtained from (27) reads

$$V(r) = \frac{\Omega_1 R_1^2}{r}\,, \tag{10}$$

which satisfies rot$\boldsymbol{U} = 0$, i.e. the flow is irrotational.

9.-10. In the limit $R_1 \to 0$, the velocity field obtained from (27) now reads

$$V(r) = \Omega_2 r\,, \tag{11}$$

i.e. we have a solid rotation motion, for which rot$\boldsymbol{U} = 2\Omega_2 \boldsymbol{e_z} \neq 0$, i.e. the flow is rotational. It is thus possible to achieve a rotation motion that can be either irrotational (questions 7.-8.) or rotational (present questions), as the rotational is connected to the local rotation of a fluid particle (see figure 1).

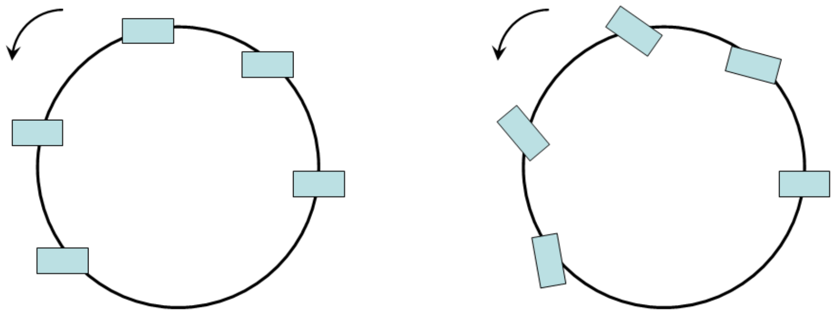

Figure 1. *Rotation motion: irrotational flow (left) or rotational flow (right).*

Exercise 2

A plate falling in a channel

1. We need information on the velocity field around the plate. We know that far from the plate the flow will be at rest. Near the top and bottom region of the plate will be transition regions where the flow adopts a constant profile.

   A smart selection of reference frame would simplify the formulation of problem. Looking from the frame of channel the flow is unsteady and as a consequence the time transient term of Navier-Stokes equations, $\frac{\partial u}{\partial t}$, could not be neglected. Now we consider a reference frame installed on the falling plate. For such frame, the plate is stationary and the walls moving at the speed of $v_w = U_0$ in z direction, creating a steady flow field with a uniform flow of the same velocity at the far field. Hence, we can eliminate time transient term of equations. We continue the solution considering the frame of the plate.

   Since the profile has a negligible thickness we infer that the streamlines remain parallel to the walls and are not perturbed by the flow. That means the velocity U (in the x-direction) is zero. From the continuity equation follows that V (in the z-direction) does not depend on z.

   From the momentum equation in x-direction remains:

$$\frac{\partial p}{\partial x} = 0.$$

Integrating the momentum equation in z-direction:

$$\mu\frac{\partial^2 v}{\partial x^2} - \frac{dp}{dz} = 0 \quad \Rightarrow \quad u = \frac{1}{2\mu}\frac{dp}{dz}(x^2 - xH) + U_0\frac{x}{H}.$$

The pressure gradient that we define contains also the hydrostatic pressure contribution from the gravitational forces. This can be written explicitly but that won't change the final result. The problem is symmetric and only one half space is solved, i.e. $x \geq 0$. The origin ($x = 0$) is located the plate and the outer wall at $x = H$. We see that the boundary conditions are fulfilled $v(0) = 0$ and $v(H) = U_0$ (for now we assume a positive velocity).

2. The velocity field equation contains two unknowns, the pressure gradient and the terminal velocity U_0. Since the channel is closed on the bottom we know that no flux is passing through any cross section of the channel as a consequence of incompressibility. In the frame of falling plate, the uniform inlet flux at far field should be preserved. That puts a further restriction on the velocity field, its net flow rate needs to be HU_0.

$$\dot{V} = HU_0 = \int v(x)dx, \quad \Rightarrow \quad \frac{dp}{dz} = -\frac{6\mu U_0}{H^2}, \quad \Rightarrow \quad v = U_0\left(-3\frac{x^2}{H^2} + 2\frac{x}{H}\right).$$

What actually happens here is that you drag liquid down the channel, this results in a pressure gradient which creates a recirculation near the outer walls. Therefore it would have been a mistake to discard the pressure gradient right away. The pressure gradient could only have been dicraded if the channel was open on the other side, than the pressure is equal on both sides and a net flux will be created.

3. The velocity is found by application of a force balance on the plate. There are two forces the buoyancy and friction.

On a plate that moves downward (our arbitrary assumption, walls move upward) the friction pulls the plate upwards. We get the friction force by integration of the stresses along the interface.

$$\mathbf{F}_\tau = \int \bar{\bar{\sigma}}\vec{n}dA.$$

For the viscous drag only the z-component is important, the x-component contains only the pressure and is balanced by symmetry.

$$F_\tau = -\mu\left(\frac{\partial v}{\partial x}\right)_{x=0} Ld = -4\frac{\mu U_0 Ld}{H}$$

Here d is the depth of the plate. The weight force $F_m = mg$. For consistency we take M to be the mass difference between plate and displaced liquid unit per depth d, so $m = M\,d$.

$$F_m = -Mgd.$$

If the plate reaches its terminal velocity the forces are at equilibrium:

$$F_m + 2F_\tau = 0, \quad \Rightarrow \quad U_0 = -\frac{M\,g\,H}{8\mu L}$$

4. If the thickness of the plate increases than the streamlines will be bend around the object and the flow is no longer parallel and our model would no longer be valid.