Exercise 1

1. The flow direction should be downward because of the gravity. The fluid cannot slip over the solid surface, and since the shear of the ambient air is negligible, the velocity gradient should vanish on the fluid-air interface. Hence, the profile u_2 represents the flow.

2. Using the assumptions, the governing equations reduce to

$$\frac{dv}{dy} = 0\,, \tag{1a}$$

$$0 = \rho g \sin\alpha + \mu \frac{d^2u}{dy^2}\,, \tag{1b}$$

$$0 = -\rho g \cos\alpha - \frac{dp}{dy} + \mu \frac{d^2v}{dy^2}\,. \tag{1c}$$

At the wall, we must enforce that the fluid does not slip:

$$u(0) = v(0) = 0\,. \tag{2}$$

At the interface, we must enforce that no fluid crosses the interface, and that the constraint (tangential and normal to the interface) is continuous:

$$v(h) = 0\,, \qquad \mu\frac{du}{dy}(h) = 0\,, \qquad p(h) = p_a\,. \tag{3}$$

This yields $v = 0$ in the entire liquid layer. The pressure p is obtained by integration of ($1c$) as

$$p(y) = p_a - \rho g \cos\alpha(y - h)\,, \tag{4}$$

and the velocity u is ultimately determined by integration of ($1b$). We obtain the parabolic distribution

$$u(y) = \frac{\rho g \sin\alpha}{2\mu} y(2h - y)\,. \tag{5}$$

3. The volumetric flow rate is defined by

$$Q = \int_{y=0}^{h} u(y)dy = \frac{\rho g \sin\alpha h^3}{3\mu}\,. \tag{6}$$

Exercise 2

1. The flow being parallel to the (Oz) axis, we have $\boldsymbol{V} = W\boldsymbol{e_z}$ only with $W(r,z)$ and $P(r,z)$ in the most general case. The continuity equation reduces to

$$\frac{\partial W}{\partial z} = 0\,, \tag{7}$$

meaning that the velocity depends on the cross-stream position only. The momentum equations can now be rewritten as

$$0 = -\frac{\partial P}{\partial r}\,, \tag{8a}$$

$$0 = 0\,, \tag{8b}$$

$$0 = -\frac{\partial P}{\partial z} + \mu \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial W}{\partial r}\right)\,. \tag{8c}$$

From (8a), one sees that the pressure depends on z only. Since W depends on r only, (8c) can be satisfied only if both terms are constant. It is therefore possible to introduce the constant β such that

$$\frac{\partial P}{\partial z} = \beta \, . \tag{9}$$

2. The velocity W is such that

$$\mu \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial W}{\partial r} \right) = \beta \, . \tag{10}$$

It is straightforward to carry out the integration:

$$W = \frac{\beta}{4\mu} r^2 + A \ln r + B \, , \tag{11}$$

where A and B are constants to be determined. Since the solution must remain finite on the revolution axis, we have $A = 0$. B is determined from the no-slip boundary condition at the pipe wall, namely $W(R) = 0$. We obtain finally:

$$W = \frac{\beta}{4\mu}(r^2 - R^2) \, . \tag{12}$$

Since $r \leq R$, W is positive only for $\beta < 0$, i.e. the fluid flows in the direction opposite to that of the pressure gradient. For a small volume of fluid, this means that the pressure on the left must be larger than the pressure on the right, i.e. the pressure has to "push" the fluid. This is not surprising since the flow is caused here by the existence of the pressure gradient ($W = 0$ if $\beta = 0$).

3. We integrate the force along a portion of the pipe surface.

$$\bar{F} = \int \bar{\bar{\sigma}} \bar{n} dA \;\Rightarrow\; F_z = \int^{dL} \int_0^{2\pi} \mu \frac{\partial W}{\partial r} R d\theta dz = \pi \beta R^2 dL.$$

4. It can be seen from (8) that momentum is obviously conserved, since all inertial terms are zero. For the present flow configuration, it turns out that the pressure force exactly counterbalances the viscous friction force at the pipe wall, so that the pressure is allowed to vary whereas the velocity remains constant.

5. Here we investigate the effect of the viscous dissipation on the temperature of the fluid. Assuming the flow to be fully developed, the energy equation reduces to

$$0 = \mu \left(\frac{\partial W}{\partial r} \right)^2 + \kappa \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial T}{\partial r} \right) \, . \tag{13}$$

The integration yields

$$T(r) = T_w - \frac{\beta^2}{64\mu\kappa}(r^4 - R^4) \, . \tag{14}$$

The centerline temperature is therefore

$$T(0) = T_w + \frac{\beta^2}{64\mu\kappa} R^4 \, , \tag{15}$$

i.e. the temperature is larger at the center of the pipe than at the wall. Such a result may seem rather counterintuitive, as one may have thought that the temperature would have been larger at the wall where the viscous friction occurs. The higher temperature at the centerline is due to the heat diffusion in the fluid.

6. The pressure drop is simply $\Delta P = \beta L$. The volumetric flow rate reads

$$Q = 2\pi \int_{r=0}^{R} W(r) r dr = -\frac{\pi \beta R^4}{8\mu}\,, \tag{16}$$

so that the ΔP can be recast in terms of Q according to

$$\Delta P = -\frac{8\mu Q L}{\pi R^4}\,. \tag{17}$$

7. For the proposed conditions, we obtain $\Delta P \simeq 259\text{Pa}$.

8. The pressure gradient in each small pipe is noted β', so that the corresponding volumetric flow rate Q' is simply

$$Q' = -\frac{\pi \beta' a^4}{8\mu}\,. \tag{18}$$

The total flow rate must be such that $nQ' = Q$, i.e.

$$n\beta' a^4 = \beta R^4\,. \tag{19}$$

Finally, since the cross-sections are identical in both configurations, one has also $na^2 = R^2$. We obtain finally

$$\beta' = \beta \frac{R^2}{a^2}\,. \tag{20}$$

If the radius is divided by a factor λ, the pressure drop is therefore multiplied by a factor λ^2, which is highly detrimental to the application.

9. The reasoning used in question 1. holds for each fluid. The pressure is therefore such that

$$\frac{\partial p_1}{\partial z} = \beta_1''\,, \qquad \frac{\partial p_2}{\partial z} = \beta_2''\,, \tag{21}$$

the velocity reads

$$W_1 = \frac{\beta''}{4\mu} r^2 + A_1 \ln r + B_1 \,(\text{out}), \qquad W_2 = \frac{\beta''}{4\mu} r^2 + A_2 \ln r + B_2 \,(\text{in}). \tag{22}$$

Because the normal constraint is to be continuous at the interface, we must have $p_1(h) = p_2(h)$, which can be satisfied only if $\beta_1'' = \beta_2'' = \beta''$. Since the solution must remain finite on the revolution axis, $A_2 = 0$. To determine the remaining constants, we recall that the tangential constraint and the velocity must be continuous at the interface, and that a no-slip condition applies at the pipe wall:

$$W_1(h) = W_2(h)\,, \quad \mu_1 \frac{\partial W_1}{\partial r}(h) = \mu_2 \frac{\partial W_2}{\partial r}(h)\,, \quad W_1(R) = 0\,. \tag{23}$$

This yields

$$W_1(r) = \frac{\beta''}{4\mu_1}(r^2 - R^2)\,, \tag{24a}$$

$$W_2(r) = \frac{\beta''}{4\mu_2}(r^2 - R^2) + \frac{\beta''}{4}\left(\frac{1}{\mu_1} - \frac{1}{\mu_2}\right)(h^2 - R^2)\,. \tag{24b}$$

The flow rate for the inner and outer fluid can now be computed, which yields

$$Q_1 = -\frac{\pi \beta''}{8\mu_1}(R^2 - h^2)^2\,, \tag{25a}$$

$$Q_2 = -\frac{\pi \beta''}{8\mu_2} h^4 + \pi \frac{\beta''}{8\mu_1}(h^2 - R^2) h^2\,, \tag{25b}$$

with $Q_1 = \alpha Q_2$. We assume that Q_2 is identical to the flow rate of the fluid 2 alone, flowing through the whole pipe, so that

$$-\frac{\beta''}{8\mu_1}(R^2 - h^2)^2 = -\frac{\beta}{8\mu_2}R^4\alpha\,. \tag{26}$$

This allows to express the new pressure gradient as

$$\beta'' = \beta\frac{\mu_1}{\mu_2}\frac{R^4}{(R^2 - h^2)^2}\alpha\,. \tag{27}$$

Substitution in $(25b)$ now yields a formal relation between h and the problem parameters:

$$\frac{\mu_1}{\mu_2}h^4 + h^2(R^2 - h^2) = \frac{1}{\alpha}(R^2 - h^2)^2\,. \tag{28}$$

10. The viscous force is now simply

$$F'' = \pi\beta'' R^2 L\,. \tag{29}$$

Its magnitude compared to that of the only inner fluid therefore depends on the choice of the viscosities but also on the choice of the relative flow rates through the new pressure gradient β''.

Exercise 3

Look at matlab code: $exo1Full4.m$