Exercise 1

Drag of rising bubbles

1. $\mathbf{F}_b$=$(\rho_{bubble} - \rho)Vg = -\rho\frac{4}{3}\pi a^3 g$, where g is the gravitational acceleration.
2. Drag force for law Reynolds number limit: $\mathbf{F}_d = 6\pi\rho\nu a U$, then

$$\mathbf{F}_b = \mathbf{F}_d, \tag{1}$$

$$6\pi\rho\nu a U = \rho\frac{4}{3}\pi a^3 g, \tag{2}$$

$$U = \frac{2}{9}\frac{a^2 g}{\nu}. \tag{3}$$

3. Viscous drag forces vs. surface tension forces.
   Dimensional analysis, $f(U, a, \nu, \gamma, \rho, g) = 0$.

$$U[LT^{-1}], a[L], \nu[L^2T^{-1}], \gamma[MT^{-2}], \rho[ML^{-3}], g[LT^{-2}] \tag{4}$$

   where L, T, M are dimensions of length, time and mass, respectively.
   Then one can form,

$$f\left(\frac{U\nu}{a^2 g}, Re, Ca\right) = 0 \tag{5}$$

$$\frac{U\nu}{a^2 g} = f(Re, Ca). \tag{6}$$

   Since $Ca \ll 1$,

$$\frac{U\nu}{a^2 g} = f(Re). \tag{7}$$

   The definition of Reynolds number is $Re = \frac{Ua}{\nu}$, then with $U = \frac{a^2 g}{\nu}$, $Re = \frac{a^2 g}{\nu}\frac{a}{\nu} = \frac{a^3 g}{\nu^2}$

4. $F_d = \frac{1}{2}\rho U^2 c_d (4\pi a^2)$.
   Again, balance with the buoyancy $\mathbf{F}_b$ one finds

$$U^2 = \frac{ag}{c_d}\frac{2}{3}\frac{\rho^* - \rho}{\rho} \propto ag, \tag{8}$$

   Now, taking eq.(1),

$$U^2 \propto ag = \frac{a^4 g^2}{\nu^2}(f(Re))^2, \tag{9}$$

   so

$$f(Re) \propto Re^{-1/2}. \tag{10}$$

5. Yes. it is surprising. For $Re \to \infty$, $f(Re) \to 0$ for a hard sphere settling case. The sphere case does not take into account the flow separation which breaks the pressure symmetry.

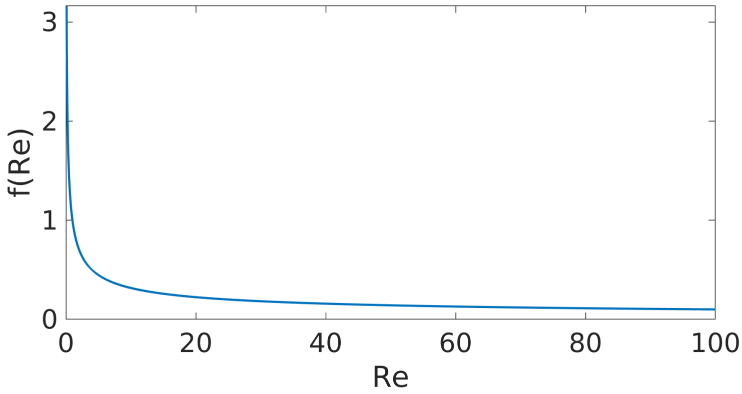

Figure 1. $f(Re) = Re^{-1/2}$

6. The drag force is
$$F = 4\pi\mu U a \frac{3\lambda + 2}{2(\lambda + 1)} \tag{11}$$
where $\lambda = \mu_{int}/\mu_{ext}$ is the viscosity ratio. For inviscid drop, where $\lambda \approx 0$, $F = 4\pi\mu U a$.

   The spherical shape satisfies the local normal stress balance on the interface, and it is the solution.

7. For Φ, the flow is irrotational (such that $\nabla \times \omega = 0$) and the flow is assumed inviscid (such that $Du/Dt = g - \nabla p/\rho$). For Ψ incompressibility such that $\nabla \cdot \mathbf{u} = 0$, 2D, steady.

8. The potential velocity Φ satisfies $\mathbf{u} = \nabla\Phi$. From the continuity equation, $\nabla \cdot \mathbf{u} = 0$, the equation to be solved is $\nabla^2\Phi = 0$.

   The boundary conditions are
$$u_r = \frac{\partial \Phi}{\partial r} = u_\theta = \frac{1}{r}\frac{\partial \Phi}{\partial \theta} = 0 \quad \text{at } r = \infty, \tag{12}$$
$$u_r = \frac{\partial \Phi}{\partial r} = U\cos\theta, \quad u_\theta = \frac{1}{r}\frac{\partial \Phi}{\partial \theta} = -U\sin\theta \quad \text{at } r = a. \tag{13}$$
   Thus, we search
$$\Phi = f(r)\cos\theta = \left(Ar + \frac{B}{r^2}\right)\cos\theta. \tag{14}$$
   From the first boundary condition $A = 0$, and from the second one ($\partial\Phi/\partial r|_{r=a} = U\cos\theta$), $B = -\frac{Ua^3}{2}$. Therefore,
$$\Phi = -\frac{Ua^3}{2r^2}\cos\theta. \tag{15}$$

9. Since $\mathbf{u} = \nabla\Phi$ and $\Phi = \frac{-Ua^3}{2r^2}\cos(\theta)$, $u_r = \frac{\partial\Phi}{\partial r} = \frac{Ua^3}{r^3}\cos(\theta)$ and $u_\theta = \frac{1}{r}\frac{\partial\Phi}{\partial\theta} = \frac{Ua^3}{2r^3}\sin(\theta)$.

10. For liquid-gas interface,
   $\mathbf{u} \cdot \mathbf{n} = U_{surface}$: no penetration (free slip)
   $\sigma \cdot \mathbf{n} = 0$: Continuity of stress
   The last condition can rewrite
$$(\sigma \cdot \mathbf{n}) \cdot \mathbf{n} = 0 \tag{16}$$
$$(\sigma \cdot \mathbf{n}) \cdot \mathbf{t} = 0 \tag{17}$$
   where $\mathbf{n} = [1\ 0\ 0]^T$, $\mathbf{t} = [0\ 1\ 0]^T$. The stress balance in tangential component satisfies if
$$\mu\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r}\right) = 0 \tag{18}$$

However, if we insert u_r and u_t obtained previous point 9, the condition $(\sigma \cdot \mathbf{n}) \cdot \mathbf{t} = 0$ is not hold.

11. Let us locate a new coordinate on the bubble interface which is adjusted for the boundary layer, $r = a + \delta y$ where $\delta \ll 1$. Since the boundary layer thickness is negligible compared to the bubble radius, this coordinate system is analogous to the Cartesian coordinates. In such analogy $(u_r, u_\theta) \sim (u_y, u_x)$ and $(e_x, e_y) = (e_\theta, e_r)$. Consider the two velocity scales V for u_r and U (as before) for u_θ, and two length scales a for x and δ for y. The dominant balance of the continuity equation results in

$$\frac{U}{a} = \frac{V}{\delta}.$$

Similarly, the dominant balance of the momentum equations results in

$$\frac{\rho V}{a} \mathbf{u} \cdot \nabla u_i = -\nabla p + \frac{\mu}{a^2}(\frac{\partial^2 u_i}{\partial x^2} + a^2 \delta^{-2} \frac{\partial^2 u_i}{\partial y^2}) \to \frac{\delta}{a} \sim (\frac{\rho U a}{\mu})^{-1/2} = Re^{-1/2}.$$

Shear free interface results that as $y \to \infty$, $\frac{\partial u_x}{\partial y} = 0$. This assumption forces the interface shear to be zero. As separation requires a change in the sign of the shear, our boundary layer prevents the flow separation.

12. For steady, inviscid, incompressible flow, from the Bernoulli's law,

$$p \sim \frac{\rho U^2}{2} \tag{19}$$

From the drag-buoyancy balance, $U \sim \frac{a^2 g}{\nu}$. Therefore,

$$p \sim \rho \left(\frac{a^2 g}{\nu} \right)^2. \tag{20}$$

13. Bubble stays spherical if the surface tension, γ / a, dominates over the normal stress, p. Hence

$$\rho \left(\frac{a^2 g}{\nu} \right)^2 \ll \frac{\gamma}{a},$$

which demonstrates $Ca \ll Re^{-1}$.

Exercise 2

Temperature profile in a heated channel

1. Recalling the second first exercise series of the course, the velocity profile, so called as Poiseuille velocity profile, writes

$$\tilde{u} = \frac{G}{4\mu}(R^2 - \tilde{r}^2), \tag{21}$$

and $U_{\max} = \frac{GR^2}{4\mu}$.

$$\tilde{U} = \frac{2}{R^2} \int_0^R \tilde{u} \tilde{r} d\tilde{r} = \frac{2}{R^2} \int_0^R \frac{G}{4\mu} \left(R^2 - \tilde{r}^2 \right) \tilde{r} d\tilde{r} = \frac{2G}{4R^2 \mu} (\frac{R^4}{2} - \frac{R^4}{4}) = \frac{G}{2\mu} \frac{R^2}{4} \tag{22}$$

Thus, $U_{\max} = 2\tilde{U}$.

2. Inserting the the Poiseuille velocity profile

$$\tilde{u}(z) = \frac{G}{4\mu}\left(\tilde{r}^2 - R^2\right) = 2\tilde{U}\left(1 - \frac{\tilde{r}^2}{R^2}\right) \tag{23}$$

in the energy equation and neglecting the viscous dissipation,

$$\frac{\partial T}{\partial t} + \mathbf{u}\cdot\nabla T = \kappa\nabla^2 T, \tag{24}$$

one deduces

$$2\tilde{U}\left(1 - \frac{\tilde{r}^2}{R^2}\right)\frac{\partial \tilde{T}}{\partial \tilde{z}} = \kappa\left(\frac{1}{\tilde{r}}\frac{\partial}{\partial \tilde{r}}\left(\tilde{r}\frac{\partial \tilde{T}}{\partial \tilde{r}}\right) + \frac{\partial^2 \tilde{T}}{\partial \tilde{z}^2}\right), \tag{25}$$

where $\kappa = \frac{k}{\rho C_p}$. The continuity of temperature sets $\tilde{T}(\tilde{r}, \tilde{z} < 0) = \tilde{T}_0$, and the heat conduction flux at the wall sets $k\frac{\partial \tilde{T}}{\partial \tilde{r}}(R, \tilde{z} > 0) = q$.

3. Conservation of energy implies that the net input heat of the system,$\int 2q\pi R d\tilde{z}$, equates the change in the internal energy of the fluid,$\int 2\rho C_p \Delta T \tilde{u}\pi\tilde{r}d\tilde{r}$.

4.

$$\tilde{r} = Rr, \quad \tilde{z} = Rz, \quad \tilde{T} = \frac{qR}{k}T + \tilde{T}_0 \tag{26}$$

And insert into the equation gives

$$2\tilde{U}\left(1 - \frac{(rR)^2}{R^2}\right)\frac{\partial(\frac{qR}{k}T - \tilde{T}_0)}{\partial Rz} = \kappa\left(\frac{1}{Rr}\frac{\partial}{\partial Rr}\left(Rr\frac{\partial(\frac{qR}{k}T - \tilde{T}_0)}{\partial Rr}\right) + \frac{\partial^2(\frac{qR}{k}T - \tilde{T}_0)}{\partial (Rz)^2}\right) \tag{27}$$

$$2\tilde{U}\frac{q}{k}\left(1 - r^2\right)\frac{\partial T}{\partial z} = \frac{\kappa}{R^2}\frac{qR}{k}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial z^2}\right) \tag{28}$$

This yields

$$2Pe\left(1 - r^2\right)\frac{\partial T}{\partial z} = \left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial z^2}\right), \tag{29}$$

where $Pe = \frac{\tilde{U}R}{\kappa}$. It is a dimensionless number which compares the strength of advection-to-conduction heat transfer.

5. Under the assumption of high Pe, the rule of dominant balance implies that $\partial T/\partial z = 0$ which results in $T = 0$. The flux boundary condition cannot be satisfied under such assumption.

6. Recalling the exercise of the thermal boundary layer, the boundary layer thickness scales as $Pe^{-1/3}$.

7. Rewrting the equation 10 (of the question sheet) in terms of $z = \hat{z}/\epsilon$ writes

$$2Pe\epsilon\left(1 - r^2\right)\frac{\partial T}{\partial \hat{z}} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \epsilon^2\frac{\partial^2 T}{\partial \hat{z}^2}. \tag{30}$$

If $\epsilon \sim Pe^{-1}$, we find the balance in the equation. So $L \sim Pe\, R$.

8. Using the above-obtained scale for L, one can neglect the ϵ^2 term. Since our assumption scales the $\tilde{z}$ to a large value, we are far from the physical inlet and $T(r, \hat{z} = 0) = 0$ is no more valid.

9. From eq(9) and point 3, we know that

$$2\pi qR\tilde{Z} = 4\pi\rho C_p\tilde{U}\int_0^R (\tilde{T}(\tilde{r}, \tilde{Z}) - \tilde{T}_0)(1 - \frac{\tilde{r}^2}{R^2})\tilde{r}d\tilde{r},$$

Using the non-dimensionalization,

$$2\pi qR^2\hat{z}/\epsilon = 4\pi\rho C_p\tilde{U}\int_0^1 (qR/k)T(1 - r^2)Rrd(Rr), \tag{31}$$

$$\hat{z}/\epsilon = 2Pe\int_0^1 T(1 - r^2)rdr, \tag{32}$$

Since $\epsilon \sim Pe^{-1}$,

$$\hat{z} = 2\int_0^1 T(1 - r^2)rdr, \tag{33}$$

10. We can rewrite the equation 13 (question sheet) as

$$2\left(1 - r^2\right)\frac{\partial T_1}{\partial \hat{z}} - \hat{z}\left(\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial T_1}{\partial r})\right) = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial T_2}{\partial r}). \tag{34}$$

By comparison of the two sides one can conclude that $\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial T_1}{\partial r}) = 0$, which results in $T_1 = C_0\ln(r) + C_1$. As T_1 should be a finite value at $r = 0$, $C_0 = 0$. Now we can solve equation 34 for T_2. The general solution is a polynomial of order 4. Applying $\frac{\partial T}{\partial r}(\hat{z} > 0) = 1$ results in $C_1 = 2$ and $T_2 = a_0 + r^2 - r^4/4$. The constant $a_0 = -7/24$ is obtained by applying the integral form of energy conservation.

11. Look at the figure 2.

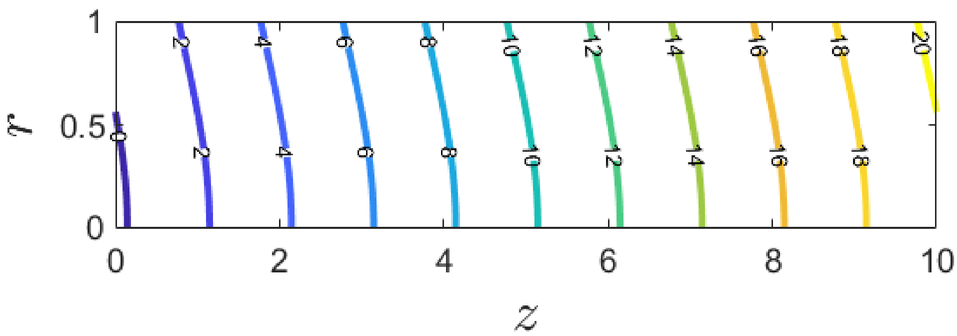

Figure 2. *Temperature contour.*